THE DUAL AND BIDUAL OF AN ECHELON KÖTHE SPACE

by

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Let $(E, \mathcal{A}, \mu)$ be a measure space. We study the class of echelon Köthe spaces $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$, and their α-duals $(\Lambda^p)^\alpha$. With a natural topology on Λ^p, if $p > 1$, $(\Lambda^p)' = (\Lambda^p)^\alpha$ if and only if $(E, \mathcal{A}, \mu)$ has the finite subset property, and if $p = 1$, $\Lambda' = \Lambda^\alpha$ if and only if $(E, \mathcal{A}, \mu)$ is localizable and has the finite subset property. We define the concept of K-isomorphism. We give also a representation theorem of certain Frechet vector lattices and we show that every $\Lambda^p(E, \mathcal{A}, \mu, g_k)$ is K-isomorphic to $\Gamma^p(T, \mathcal{M}, \nu, \varphi_k)$ with $(T, \mathcal{M}, \nu)$ localizable and with the finite subset property. We prove also that $[\Lambda'', \beta(\Lambda'', \Lambda^\alpha)]$ is isomorphic to an echelon Köthe space and we give an example which proves the incorrectness of an affirmation of Dieudonné in [1].

The vector spaces we use here are defined over the field R of the real numbers. Given a topological vector space $[E, \mathcal{T}]$ we denote by E' or $[E, \mathcal{T}]'$, and E'' the dual and bidual of E. If $\langle E, F \rangle$ is a dual pair, it is denoted by $\sigma(E, F)$ and $\beta(E, F)$ the weak and strong topologies on E respectively, and by $\langle x, y \rangle$ the canonical bilinear form on $E \times E'$. Given a measure space $(E, \mathcal{A}, \mu)$ we denote by $\Omega(E)$ the set of all real valued $\mathcal{A}$-measurable functions on E. We shall identify two functions f_1 and f_2 of $\Omega(E)$ if $f_1(x) = f_2(x)$ almost everywhere (a.e.) on E. The quotient set will be denoted by $\Omega_0(E)$. We shall use the same symbol to denote the elements of $\Omega(E)$ and their equivalence classes in $\Omega_0(E)$, when there is no risk of confusion. Given the function $f \in \Omega(E)$, we define the support of f as $S(f) = \{t \in E / f(t) \neq 0\}$. If $f \in \Omega_0(E)$ we define $S(f)$ as the support of any element of the class of f. Then $S(f)$ is a well defined set, except a set of zero measure. The characteristic function of a set A will be denoted by χ_A.

If $(E, \leqslant)$ is a vectorial lattice and $x \in E$, $y \in E$, we denote by $x \vee y$, $x \wedge y$, the sup $\{x, y\}$ and inf $\{x, y\}$ respectively. We define $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$. A set $B \subset E$ is called normal if $|x| \leqslant |y|$, $y \in B$ implies $x \in B$.

We say that a measure space $(E, \mathcal{A}, \mu)$ has the finite subset property if given $A \in \mathcal{A}$ with $\mu(A) > 0$, there is $B \in \mathcal{A}$ so that $B \subset A$ and $0 < \mu(B) < \infty$. For every $A \in \mathcal{A}$ with $\mu(A) < \infty$, let f_A be *a* measurable function with support in A. If $f \in \Omega(E)$ we denote by f^* its class in $\Omega_0(E)$. The family $\{f_A^\bullet, \mu(A) < \infty\}$ is called a cross section on E if for every A and B of finite measure, $(f_{A \cap B})^* = (f_A . \chi_{A \cap B})^* = (f_B . \chi_{A \cap B})^*$. This family will be denoted by $\langle f_A^\bullet \rangle$. The measure μ and the measure space are called localizable if, given a cross section $\langle f_A^\bullet \rangle$ on E, there is a measurable function f so that $f_A^\bullet = (f . \chi_A)^*$ for every A of finite measure. Then we say that $\langle f_A^\bullet \rangle$ is determined by f.

Let $(E, \mathcal{A}, \mu)$ be a measure space. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of $\mathcal{A}$-measurable functions so that $g_k(x) \geqslant 0$ for every $x \in E$, $k \in N$ (the set of natural numbers) and

$$\mu\left(\bigcap_{k=1}^{\infty} \{x \in E / g_k(x) = 0\} \right) = 0$$

If $p \in R$, $p \geqslant 1$, we define the echelon Köthe space of order p as

$$\Lambda^p = \Lambda^p(E, \mathcal{A}, \mu, g_k) = \left\{ f \in \Omega_0(E) / P_k(f) = \left(\int_E |f|^p g_k \, d\mu \right)^{1/p} < \infty \ \forall \ k \in N \right.$$

and its α-dual as

$$(\Lambda^p)^\alpha = \left\{ f \in \Omega_0(E) / \int_E |f| \, |h| \, d\mu < \infty \quad \forall \ h \in \Lambda^p \right\}$$

We shall write Λ and Λ^α instead of Λ^1 and $(\Lambda^1)^\alpha$. The formula

$$\langle f, h \rangle = \int_E f h \, d\mu \qquad f \in \Lambda^p, \ h \in (\Lambda^p)^\alpha \tag{1}$$

defines a canonical bilinear form on the cartesian product $\Lambda^p \times (\Lambda^p)^\alpha$. Then we have

PROPOSITION 1. *Let $\Lambda^p(E, A, \mu, g_h)$ be an echelon Köthe space, $p \geq 1$. $\langle \Lambda^p, (\Lambda^p)^\alpha \rangle$ is a dual pair in respect of the bilinear form* (1) *if and only if* $(E. A, \mu)$ *has the finite subset property.*

PROOF. Let us suppose that (E, A, μ) has the finite subset property. Let $h \in (\Lambda^p)^\alpha$, $h \neq 0$. Working with $-h$ if it is necessary, there is a set B of non zero measure so that $h(x) > 0$ if $x \in B$. By the finite subset property, we can suppose that $0 < \mu(B) < \infty$. We construct inductively a sequence of sets $\{B_n\}_{n=1}^\infty$ and an increasing sequence of natural numbers $\{n_k\}_{k=1}^\infty$, so that, if $B = B_0$, $0 < \mu(B_i)$, $B_i \subset B_{i-1}$, $\mu(B_{i-1} - B_i) < \frac{1}{2^{i+1}} \cdot \mu(B)$ and $g_i(x) \leqslant n_i$ if $x \in B_i$, $i \in N$. In fact, if B_k and n_k are defined,

$$B_k = \bigcup_{n=1}^{\infty} \{x \in B_k / g_{k+1}(x) \in [0, n]\}$$

Then, there is $n_{k+1} > n_k$ so that, if we define

$$B_{k+1} = \{x \in B_k / g_{k+1}(x) \in [0, n_{k+1}]\}$$

we have $0 < \mu(B_{k+1})$ and $\mu(B_k - B_{k+1}) < \frac{1}{2^{k+2}} \mu(B)$.

Let $M = \bigcap_{k=1}^{\infty} B_k$. It is easy to see that $\mu(B - M) \leqslant \frac{1}{2} \mu(B)$ and $\mu(M) > \frac{1}{2} \mu(B) > 0$. As $g_k(x) \leqslant n_k$ if $x \in M$ and $\mu(M) < \infty$, we have $\chi_M \in \Lambda^p$. Then

$$\int_E \chi_M \, h \, d\mu > 0$$

Now, let $f \in \Lambda^p$, $f \neq 0$. Then by the hypothesis on g_k and the finite subset property, working with $-f$ if it is necessary, there are $k \in N$ and a set B so that $0 < \mu(B) < \infty$, $f(x) > 0$ if $x \in B$ and $g_k(x) > 0$ if $x \in B$. If $p = 1$, it is clear that $\chi_B \cdot g_k \in \Lambda^\alpha$ and $\chi_B \cdot g_k \neq 0$ and

$$\int_E f \cdot \chi_B \, g_k \, d\mu > 0$$

If $p > 1$, and $h \in \Lambda^p$, by Hölder inequality

$$\int_E |h \chi_B g_k^{1/p}| \, d\mu \leqslant \left(\int_E |h|^p g_k \, d\mu \right)^{1/p} \left(\int_E \chi_B^q \, d\mu \right)^{1/q} < \infty \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Then we have $\chi_B \cdot g_k^{1/p} \in (\Lambda^p)^\alpha$. As $\chi_B \cdot g_k^{1/p} \neq 0$ and

$$\int_E f \cdot g_k^{1/p} \chi_B \, d\mu > 0 ,$$

$\langle \Lambda^p, (\Lambda^p)^\alpha \rangle$ is a dual pair.

Conversely: let $\langle \Lambda^p, (\Lambda^p)^\alpha \rangle$ be a dual pair. Let D be a set so that $\mu(D) = +\infty$ and so that, if $B \subset D$, $B \in \mathcal{A}$, we have $\mu(B) = 0$ or $\mu(B) = \infty$. By hypothesis on $\{g_k\}_{k=1}^\infty$ there are $k \in N$ and a set $M \subset D$ so that $g_k(x) > 0$ if $x \in M$ and $\mu(M) > 0$. Hence $\mu(M) = \infty$. Let $f \in \Lambda^p$. As

$$M \cap S(f) = \bigcup_{n=1}^{\infty} M_n = \bigcup_{n=1}^{\infty} \left\{ x \in M \cap S(f) \; / \; |f(x)|^p \cdot g_k(x) > \frac{1}{n} \right\} ,$$

$\mu(M_n)$ must be zero, because if it were not, $f \notin \Lambda^p$. Then $f \cdot \chi_M = 0$ and $\chi_M \in (\Lambda^p)^\alpha$. As $\chi_M \neq 0$ and there is no $f \in \Lambda^p$ so that

$$\int_E f \cdot \chi_M \, d\mu \neq 0$$

we conclude that $\langle \Lambda^p, (\Lambda^p)^\alpha \rangle$ is not a dual pair, which is a contradiction. Hence $(E, \mathcal{A}, \mu)$ has the finite subset property. q.e.d.

By Minkowski inequality, P_k is a seminorm on Λ^p. Then we shall consider always on Λ^p the topology $\mathcal{T}$ defined by the family of seminorms $\{P_k, \; k \in N\}$ except when otherwise is clearly stated. This topology is separated, because if $f \in \Lambda^p, f \neq 0$, we have $\mu(S(f)) > 0$ By hypothesis, there is $k \in N$ so that

$$\mu(A) = \mu\{x \in S(f) / g_k(x) > 0\} > 0$$

Then

$$P_k(f) = \left(\int_E |f|^p g_k \, d\mu \right)^{1/p} \geqslant \left(\int_A |f|^p g_k \, d\mu \right)^{1/p} > 0$$

and $\mathcal{T}$ is Hausdorff.

We observe that, given the echelon Köthe space $\Lambda^p(E, \mathcal{A}, \mu, g_k)$ if we define $g'_k = \sum_{i=1}^{k} g_i$ and

$$q_k(f) = \left(\int_E |f|^p \, g'_k \, d\mu \right)^{1/p}$$

we have $\Lambda^p(E, \mathcal{A}, \mu, g_k) = \Lambda^p(E, \mathcal{A}, \mu, g'_k)$ and the topologies on Λ^p defined by the family of seminorms $\{P_k, k \in N\}$ and $\{q_k, k \in N\}$ are the same. Then, we can always suppose that $g_k(x) \leqslant g_{k+1}(x)$ for every $x \in E$ and $k \in N$. Hence, we shall always suppose this condition except when otherwise is clearly stated.

We define on $\Omega_0(E)$ the order relation $f \leqslant h$ if and only if $f(x) \leqslant g(x)$ a.e. on E. This order induces an order on Λ^p and $(\Lambda^p)^\alpha$. It is clear that with this order, Λ^p and $(\Lambda^p)^\alpha$ are normal vector lattices and $[\Lambda^p, \mathfrak{I}, \leqslant]$ is a topological vector lattice.

Given the echelon Köthe space $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, we shall consider the measure spaces $(S(g_k), \mathcal{A}_k, \mu_k)$ where $\mathcal{A}_k$ and μ_k are the σ-algebra $\mathcal{A}$ restricted to $S(g_k)$ and the measure μ restricted to $\mathcal{A}_k$. This measure will be denoted by μ again. Then we consider the spaces

$$\Lambda_k^p = \Lambda_k^p(S(g_k)) = \left\{ f \in \Omega_0(S(g_k)) \, / \, P_k(f) = \left(\int_{S(g_k)} |f|^p \, g_k \, d\mu \right)^{1/p} < \infty \right\}$$

provided with the order induced by $\Omega_0(S(g_k))$ and with the topology $\mathfrak{I}_k$ defined by the norm P_k, because now P_k is a norm. Λ_k^p can be considered as an echelon Köthe space with all the echelons equal to g_k restricted to $S(g_k)$. Its α-dual is

$$(\Lambda_k^p)^\alpha = \left\{ f \in \Omega_0(S(g_k)) \, / \int_{S(g_k)} |f| \, |h| \, d\mu < \infty \ \forall \ h \in \Lambda_k^p \right\}$$

We note that for every $k \in N$, $S(g_k) \subset S(g_{k+1})$ because we suppose $g_k \leqslant g_{k+1}$. Then we define a map $I_k : \Lambda^p \to \Lambda_k^p$ so that, if $f \in \Lambda^p$, $I_k(f)$ is the restriction to $S(g_k)$ of f. For every $n < m$, we define a map $I_{nm} : \Lambda_m^p \to \Lambda_n^p$ so that if $f \in \Lambda_m^p$, $I_{nm}(f)$ is the restriction to $S(g_n)$ of f. Then, it is clear that $[\Lambda^p, \mathfrak{I}]$ is isomorphic to the projective limit $\underleftarrow{\lim} \, I_{nm}(\Lambda_m^p, \mathfrak{I}_m)$. Then we have the following theorem.

THEOREM 1. *An echelon Köthe space $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$, is a Frechet space. If $\{f_n\}_{n=1}^{\infty}$ is a $\mathfrak{I}$-convergent sequence in $[\Lambda^p, \mathfrak{I}]$ to the* *function f, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ so that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ a.e. on E.*

PROOF. If L^p is the Lebesgue space on $S(g_k)$, the map φ_k, $\Lambda_k^p \to L^p$ so that, if $f \in \Lambda_k^p$, $\varphi_k(f) = f \cdot g_k^{1/p}$ is clearly bijective and is an isometry from $[\Lambda_k^p, \mathcal{T}_k]$ onto L^p. Then, $[\Lambda_k^p, \mathcal{T}_k]$ is a Banach space. As $[\Lambda, \mathcal{T}]$ is isomorphic to the projective limit $\underleftarrow{\lim}\, I_{mn}\ (\Lambda_n^p, \mathcal{T}_n)$, $[\Lambda^p, \mathcal{T}]$ is complete. As $[\Lambda^p, \mathcal{T}]$ is metrizable and locally convex, $[\Lambda^p, \mathcal{T}]$ is a Frechet space.

Let $\{f_n\}_{n=1}^{\infty}$ be convergent to f on $[\Lambda^p, \mathcal{T}]$. Then, for every $k \in N$,

$$\lim_{n\to\infty} g_{k+1} \cdot I_{k+1}(f_n) = g_{k+1} \cdot I_{k+1}(f)$$

on $L^p(S(g_{k+1}))$. Then, by a well known theorem of L^p-spaces and by a diagonal procedure, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ so that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ a.e. on E. q.e.d.

Now, we define the notion of isomorphism for the structure of echelon Köthe space. An echelon Köthe space $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, is K-isomorphic to the echelon Köthe space $\Gamma^p(T, \mathcal{M}, \nu, \varphi_k)$ if there is a linear bijection ψ from Λ^p onto Γ^p so that

$$\int_T |\psi(f)|^p \varphi_k \, d\nu = \int_E |f|^p g_k \, d\mu \qquad \forall\, f \in \Lambda^p \ \ \forall\, k \in N \tag{1}$$

A subspace F of $\Lambda^p(E, \mathcal{A}, \mu, g_k)$ is K-isomorphic to $\Gamma^p\ (T, \mathcal{M}, \nu, \varphi_k)$ if there is a linear bijection ψ from F onto Γ^p so that (1) holds for every $f \in F$.

It is clear that a K-isomorphism is a topological isomorphism.

PROPOSITION 2. *Let $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geq 1$, be an echelon Köthe space. If $f \in \Lambda^p$, $S(f)$ is a σ-finite set.*

PROOF. Let

$$B_{nk} = \{x \in S(f) \ / \ |f(x)|^p \cdot g_k(x) \geq 1/n\}$$

Then $\mu(B_{nk}) < \infty$ because $f \in \Lambda^p$. But

$$A_k = \{x \in S(f) \ / \ g_k(x) \neq 0\} = \bigcup_{n=1}^{\infty} B_{nk}$$

and $\mu(S(f) - \bigcup_{k=1}^{\infty} A_k) = 0$, being Λ^p an echelon Köthe space. Then $S(f)$ is σ-finite. q.e.d.

PROPOSITION 3. *Let $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$ be an echelon Köthe space. Then : a) The set of simple functions of Λ^p is dense in $[\Lambda^p, \mathcal{J}]$ b) If $f \in \Lambda^p$, $f \geqslant 0$. there is a sequence $\{S_n\}_{n=1}^{\infty}$ of simple functions of Λ^p so that, for every $x \in E$ and every $n \in N$. $0 \leqslant S_n(x) \leqslant S_{n+1}(x) \leqslant \leqslant f(x)$ and $\lim_n S_n(x) = f(x)$ and $\lim_n S_n = f$ in $[\Lambda^p, \mathcal{J}]$. c) The conclusions a) and b) are true for $[\Lambda^p_k, \mathcal{J}_k]$.*

PROOF. If $f \in \Lambda^p$, $f = f^+ - f^-$. Then it is enough to prove *b*). Given $f \geqslant 0$, $f \in \Lambda^p$, there is a sequence $\{S_n\}_{n=1}^{\infty}$ of simple funtions so that, for every $x \in E$

$$0 \leqslant S_n(x) \leqslant S_{n+1}(x) \leqslant f(x) \qquad\qquad \lim_n S_n(x) = f(x)$$

As $|f - S|^p \cdot g_k \leqslant |f|^p \cdot g_k$ for every k, $n \in N$, we can apply the dominated convergence theorem of Lebesgue, and then $\lim_n S_n = f$ in $[\Lambda^p, \mathcal{J}]$. q.e.d.

PROPOSITION 4. *Let T be a compact topological space. Let μ be an inner regular measure on T. Let $\Lambda^p(T, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$, be an echelon Köthe space so that every g_k is integrable on T. Then the space $C(T)$ of continuous real functions on T, is dense in $[\Lambda^p, \mathcal{J}]$.*

PROOF. If $f \in C(T)$, f is bounded on T. As g_k is integrable over T, $C(T) \subset \Lambda^p$. Let $\chi_A \in \Lambda^p$ with $A \in \mathcal{A}$. Given $\varepsilon > 0$, there is $\delta > 0$, so that, if $\mu(M) < \delta$, we have

$$\int_M g_k \, d\mu \leqslant \frac{\varepsilon}{2^p}$$

By Lusin theorem, there is a closed set F so that $\mu(T - F) < \delta$ and the restriction f of χ_A to F is continuous on F. By Tietze extension theorem, f can be extended to a continuous function $\hat{f}$ on T and bounded by 1. Then

$$\int_T |\chi_A - \hat{f}|^p g_k \, d\mu = \int_{T-F} |\chi_A - \hat{f}|^p g_k \, d\mu \leqslant 2^p \int_{T-F} g_k \, d\mu \leqslant \varepsilon$$

Hence, $C(T)$ is dense in $[\Lambda^p, \mathcal{J}]$ by proposition 3. q.e.d.

Proposition 5. *Let $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$ be an echelon Köthe space. Then, for every $k \in N$, $I_k(\Lambda^p)$ is dense in $[\Lambda^p_k, \mathfrak{I}_k]$.*

Proof. As $f = f^+ - f^-$ if $f \in \Lambda^p_k$, we can suppose that $f \geqslant 0$. Further we can suppose that f is a characteristic function $\chi_A \in \Lambda^p_k$. Then A is a σ-finite set $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ and $\mu(A_n) < \infty$ for every $n \in N$. Let $\varepsilon > 0$ be given. As g_k is integrable on A, there is $\delta > 0$ so that $\mu(M) < \delta$, $A \supset M$, implies

$$\int_M g_k \, d\mu \leqslant \frac{\varepsilon}{2} \tag{1}$$

Further, there is $n_0 \in N$ so that, if $A_0 = \varnothing$, we have $\mu(A_{n_0}) > 0$ and

$$\int_{A - A_{n_0}} g_k \, d\mu = \sum_{n = n_0 + 1}^{\infty} \int_{A_n - A_{n-1}} g_k \, d\mu \leqslant \frac{\varepsilon}{2}$$

Now, we define inductively a contractive sequence $\{B_r\}_{r=1}^{\infty}$ of subsets of A_{n_0} with $0 < \mu(B_r) < \infty$ and $\mu(B_{r-1} - B_r) \leqslant \gamma/2^{r+1}$ where $\gamma = \inf(\delta, \mu(A_{n_0}))$. In fact, if $B_0 = A_{n_0}$ and B_{r-1} has been defined, we have $0 < \mu(B_{r-1}) \leqslant \mu(A_{n_0}) < \infty$ and

$$B_{r-1} = \bigcup_{n=1}^{\infty} B_{rn} = \bigcup_{n=1}^{\infty} \{x \in B_{r-1} / g_{k+r}(x) \leqslant n\}$$

Then, there is $B_r = B_{rn_r}$ so that $0 < \mu(B_r) < \infty$ and $\mu(B_{r-1} - B_r) \leqslant \leqslant \dfrac{\gamma}{2^{r+1}}$ and g_{k+r} is integrable on B_r.

Let $B = \bigcap_{r=1}^{\infty} B_r$. It is easy to see that $\mu(B) \geqslant \dfrac{1}{2}\mu(A_{n_0})$, $\mu(A_{n_0} - B) \leqslant \dfrac{\gamma}{2} \leqslant \dfrac{\delta}{2}$ and that $\chi_B \in \Lambda^p$. Then we have

$$\int_E |\chi_A - \chi_B|^p g_k \, d\mu = \int_{A-B} g_k \, d\mu =$$

$$= \int_{A - A_{n_0}} g_k \, d\mu + \int_{A_{n_0} - B} g_k \, d\mu \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and the proof is complete. q.e.d.

The first application of the latter result is to the problem of find the topological dual of Λ^p.

THEOREM 2. *If* $p > 1$, *the topological dual of an echelon Köthe space* $\Lambda^p(E, \mathcal{A}, \mu, g_k)$ *is* $(\Lambda^p)^\alpha$ *if and only if* $(E, \mathcal{A}, \mu)$ *has the finite subset property. If* $p = 1$, *the topological dual of* $\Lambda(E, \mathcal{A}, \mu, g_k)$ *is* Λ^α *if and only if* $(E, \mathcal{A}, \mu)$ *is localizable and has the finite subset property.*

PROOF. First, let us suppose $p \geqslant 1$ and that $(E, \mathcal{A}, \mu)$ has the finite subset property and that $(E, \mathcal{A}, \mu)$ is localizable if $p = 1$. Let $h \in (\Lambda^p)^\alpha$. Let us prove that the formula

$$\varphi(f) = \int_E f \cdot h \, d\mu \quad f \in \Lambda^p$$

defines a $\mathcal{J}$-continuous linear form on Λ^p. As $[\Lambda^p, \mathcal{J}]$ is bornological, it is enough to prove that φ is bounded in the bounded sets of $[\Lambda^p, \mathcal{J}]$. If it is not so, there is a bounded normal set $B \subset \Lambda^p$ and a sequence $\{f_n\}_{n=1}^\infty$ contained in B so that

$$\int_E |f_n \, h| \, d\mu \geqslant \left| \int_E f_n \, h \, d\mu \right| \geqslant n^3 \ \forall \ n \in N$$

There is also a sequence $\{M_k\}_{k=1}^\infty$, $M_k > 0$ so that

$$\left(\int_E |f_n|^p \, g_k \, d\mu \right)^{1/p} \leqslant M_k \quad \forall \, n \in N \quad k \in N$$

Then

$$\left\{ S_n = \sum_{i=1}^n \frac{|f_i|}{i^2} \right\}_{n=1}^\infty$$

is a Cauchy sequence in $[\Lambda^p, \mathcal{J}]$, because by Minkowski inequality

$$\left(\int_E |S_n - S_m|^p \, g_k \, d\mu \right)^{1/p} = \left(\int_E \left[\left(\sum_{i=m+1}^n \frac{|f_i|}{i^2} \right) g_k^{1/p} \right]^p d\mu \right)^{1/p} \leqslant$$

$$\leqslant \sum_{i=m+1}^n \left(\int_E \frac{|f_i|^p}{i^{2p}} \, g_k \, d\mu \right)^{1/p} \leqslant M_k \sum_{i=m+1}^n \frac{1}{i^2}$$

and this sum is arbitrarily small when m increases. Then there is

$\varphi = \lim_n S_n \in \Lambda^p$. By theorem 1, as $\{S_n\}_{n=1}^{\infty}$ is monotonous increasing, we have

$$\varphi(x) = \sum_{i=1}^{\infty} \frac{|f_i(x)|}{i^2}$$

almost everywhere on E. Then

$$\int_E |\varphi\, h|\, d\mu = \sum_{i=1}^{\infty} \int_E \frac{|f_i|\,|h|}{i^2}\, d\mu \geqslant \sum_{i=1}^{\infty} \frac{i^3}{i^2} = \infty$$

which is a contradiction with the fact that $\varphi \in \Lambda^p$. (We have used the Lebesgue monotonous convergence theorem.) Then $\varphi \in (\Lambda^p)'$.

Conversely, let $\varphi \in [\Lambda^p, \mathfrak{I}]'$. First, we suppose that $g_k(x) > 0$ for every $x \in E$ and $k \in N$. Then there are g_k and $\varepsilon > 0$ so that $|\varphi(f)| \leqslant 1$ when f belongs to the $\mathfrak{I}$-neighbourhood of zero

$$\left\{ f \in \Lambda^p \,/ \int_E |f|^p g_k\, d\mu < \varepsilon \right\}$$

As φ is bounded in the intersection of the unit ball of $[\Lambda_k^p, \mathfrak{I}_k]$ with Λ^p, φ is continuous on Λ^p with the topology induced by $\mathfrak{I}_k$. According to proposition 5, φ can be extended continuously to a continuous linear form on $[\Lambda_k^p, \mathfrak{I}_k]$ which we shall continue denoting by φ. As the map from L^p onto Λ_k^p which assigns to $f \in L^p(E, \mathcal{A}, \mu)$ the function $f \cdot g_k^{-1/p} \in \Lambda_k^p$, is an isometry from L^p onto $[\Lambda_k^p, \mathfrak{I}_k]$, we can define a continuous linear form on L^p by means of

$$\hat{\varphi}(r) = \varphi\left(\frac{r}{g_k{}^{1/p}}\right) \qquad r \in L^p$$

If $p > 1$, as $(E, \mathcal{A}, \mu)$ has the finite subset property, there is $h \in L^q$ with $1/p + 1/q = 1$, so that

$$\hat{\varphi}(r) = \int_E h \cdot r\, d\mu \qquad r \in L^p$$

If $p = 1$, as $(E, \mathcal{A}, \mu)$ is localizable and has the finite subset property, there is $h \in L^\infty$ so that

$$\hat{\varphi}(r) = \int_E h \cdot r \, d\mu \qquad r \in L^1$$

Then, if $\in \Lambda^p$ and $p \geqslant 1$,

$$\varphi(f) = \hat{\varphi}(f \cdot g_k^{1/p}) = \int_E f \cdot h \, g_k^{1/p} \, d\mu$$

Let us see that $h \cdot g_k^{1/p} \in (\Lambda^p)^\alpha$. Let $A = \{x \in E / h(x) g_k(x)^{1/p} \geqslant 0\}$ $B = \{x \in E / h(x) g_k(x)^{1/p} < 0\}$. If $f \in \Lambda_k^p$, $w = |f|\chi_A - |f|\chi_B$ also belongs to Λ_k^p. Then

$$\int_E |f \, h \, g_k^{1/p}| \, d\mu = \int_A |f| \, |h g_k^{1/p}| d\mu + \int_B |f| \, |h g_k^{1/p}| d\mu =$$

$$= \int_E h \, w \, g_k^{1/p} \, d\mu = \varphi(w) < \infty$$

and hence $h g_k^{1/p} \in (\Lambda_k^p)^\alpha \subset (\Lambda^p)^\alpha$. By proposition 1, we can identify $(\Lambda^p)'$ with $(\Lambda^p)^\alpha$.

In the general case, as Λ^p is an echelon Köthe space, the sequence of sets $\{S(g_k)\}_{k=1}^\infty$ is increasing and $\mu(E - \bigcup_{k=1}^\infty S(g_k)) = 0$. For every $k \in N$, we consider the subspace of Λ^p

$$A_k = \{f \cdot \chi_{S(g_k)}, \; f \in \Lambda^p\}$$

If we consider the echelon Köthe space $\Gamma_k^p(S(g_k), \mathcal{A}_k, \mu, \varphi_r)$ where $\mathcal{A}_k$ and μ are the σ-algebra and the measure induced by $\mathcal{A}$ and μ on $S(g_k)$ and $\varphi_r(x) = g_{k+r-1}(x)$ for every $x \in S(g_k)$ and $r \in N$, the mapping $i_k : \Gamma_k^p \to \Lambda^p$ so that if $f \in \Gamma_k^p$, $i_k(f)$ is the function zero on $E - S(g_k)$ and equal to f on $S(g_k)$, is a topological isomorphism from Γ_k^p, onto A_k with the topology induced by $\mathcal{J}$. Then the restriction of φ to A_k is continuous. By the previous result, there is $h_k \in (\Gamma_k^p)^\alpha$ so that

$$\varphi(i_k(f)) = \int_{S(g_k)} h_k \cdot f \, d\mu \qquad \forall f \in \Gamma_k^p$$

If $f \in \Gamma_k^p$, let $j(f) \in \Gamma_{k+1}^p$ be the function zero on $S(g_{k+1}) - S(g_k)$ and equal to f on $S(g_k)$. Then $i_k(f) = i_{k+1}(j(f))$. Hence we have $\varphi(i_k(f)) = \varphi(i_{k+1}(j(f)))$ and

$$\int_{S(g_k)} fh_k \, d\mu = \int_{S(g_{k+1})} j(f)\, h_{k+1} \, d\mu = \int_{S(g_k)} fh_{k+1} \, d\mu \qquad \forall f \in \Gamma_k^p$$

It is clear that h_{k+1} restricted to $S(g_k)$ is an element of $(\Gamma_k^p)^\alpha$. By proposition 1, we have $h_k = h_{k+1}$ a.e. on $S(g_k)$. Then we can define a function h on E by the rule $h(x) = h_k(x)$ if $x \in S(g_k)$ and $h(x) = 0$ if $x \in E - \bigcup_{k=1}^{\infty} S(g_k)$, changing the values of the h_k on a set of zero measure if it is necessary.

If $f \in \Lambda^p$, it is easy to see that $f = \lim_k f \cdot \chi_{S(g_k)}$ in $[\Lambda^p, \mathfrak{J}]$ because, given $r \in N$, if $k > r$, we have $S(g_k) \supset S(g_r)$.

Then, $h \in (\Lambda^p)^\alpha$ because if $f \in \Lambda^p$, the function f' so that $f'(x) = f(x)$ if $f(x) \cdot h(x) \geqslant 0$ and $f'(x) = -f(x)$ if $f(x) \cdot h(x) < 0$, belongs to Λ^p, $f' \cdot h \geqslant 0$ and

$$0 \leqslant \int_E |fh| \, d\mu = \int_E f' \cdot h \, d\mu = \lim_{k\to\infty} \int_{S(g_k)} f' \cdot h \, d\mu = \lim_{k\to\infty} \int_{S(g_k)} f' \cdot h_k \, d\mu =$$

$$= \lim_{k\to\infty} \varphi(f' \cdot \chi_{S(g_k)}) = \varphi(\lim_k f' \cdot \chi_{S(g_k)}) = \varphi(f') < +\infty .$$

Then, by the dominated convergence theorem, $\forall f \in \Lambda^p$

$$\varphi(f) = \lim_{k\to\infty} \varphi(f \chi_{S(g_k)}) = \lim_{k\to\infty} \int_{S(g_k)} f.\, h \, d\mu = \int_E f \cdot h \, d\mu .$$

Hence, by proposition 1, we can identify $(\Lambda^p)'$ with $(\Lambda^p)^\alpha$.

Conversely: let us suppose that $(\Lambda^p)^\alpha = (\Lambda^p)'$. By proposition 1, if $p \geqslant 1$, $(E, \mathcal{A}, \mu)$ has the finite subset property. Let us see that, if $p = 1$, $(E, \mathcal{A}, \mu)$ is localizable. Let $\langle f_A^* \rangle$ be a cross section on E. Let us see that $\langle f_A^* \rangle$ is determined by a measurable function f. We can suppose that $f_A \geqslant 0$ for every $A \in \mathcal{A}$, $\mu(A) < \infty$. First, we assume that there are $\alpha \in R$ and $k \in N$ so that $0 \leqslant f_A \leqslant \alpha g_k$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$. By proposition 2, if $h \in \Lambda$, $h \geqslant 0$, we have $S(h) = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$, $n \in N$ and $E_n \cap E_m = \varnothing$ if $n \neq m$. We define

$$\varphi(h) = \int_E h \cdot \left(\sum_{n=1}^{\infty} f_{E_n} \right) d\mu .$$

Then φ is a well defined map from Λ into R, because for any other decomposition $S(h) = \bigcup_{m=1}^{\infty} T_m$, $\mu(T_m) < \infty$ if $n \in N$, and $T_n \cap T_m = \emptyset$ if $n \neq m$, as $\langle f_\lambda^* \rangle$ is a cross section, we have

$$\int_E \left(\sum_{n=1}^{\infty} f_{E_n} \right) h \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} h f_{E_n} \, d\mu = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{E_n} h \cdot f_{E_n} \cdot \chi_{E_n \cap T_m} \, d\mu =$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{E_n} h \cdot f_{E_n} \chi_{E_n \cap T_m} \, d\mu = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{E_n} h \cdot f_{T_m} \chi_{E_n \cap T_m} \, d\mu =$$

$$= \sum_{m=1}^{\infty} \int_{T_m} h \cdot f_{T_m} \, d\mu = \int_E h \left(\sum_{m=1}^{\infty} f_{T_m} \right) d\mu .$$

having used the theorem of Lebesgue and the fact that every term of the series is positive.

If $a \in R$, $a \neq 0$, $S(a \cdot h) = S(h)$ and hence $\varphi(a \cdot h) = a \cdot \varphi(h)$. If $h_i \in \Lambda$ $h_i \geqslant 0$ $i = 1,2$, we have $S(h_1 + h_2) \supset S(h_1) \cup S(h_2)$. If $S(h_1 + h_2) = \bigcup_{n=1}^{\infty} P_n$ with $\mu(P_n) < \infty$, $P_n \cap P_m = \emptyset$ for $n \neq m$, we have $S(h_i) = \bigcup_{n=1}^{\infty} (P_n \cap S(h_i))$. Then, by definition of φ

$$\int_E h_i \left(\sum_{n=1}^{\infty} f_{P_n} \right) d\mu = \sum_{n=1}^{\infty} \int_E h_i f_{P_n} \, d\mu = \sum_{n=1}^{\infty} \int_E h_i f_{P_n} \chi_{P_n \cap S(h_i)} \, d\mu =$$

$$= \sum_{n=1}^{\infty} \int_E h_i f_{P_n \cap S(h_i)} \, d\mu = \int_E h_i \left(\sum_{n=1}^{\infty} f_{P_n \cap S(h_i)} \right) d\mu = \varphi(h_i) \qquad i = 1,2$$

and hence $\varphi(h_1 + h_2) = \varphi(h_1) + \varphi(h_2)$. It is known that in this case, φ can be extended in a unique way to a linear form over the whole Λ, putting $\varphi(h) = \varphi(h_1) - \varphi(h_2)$, being $h = h_1 - h_2$ any decomposition of h in difference of two positive elements of Λ.

Further, φ is continuous on $[\Lambda, \mathfrak{J}]$ because

$$|\varphi(h)| = |\varphi(h^+ - h^-)| \leqslant |\varphi(h^+)| + |\varphi(h^-)| \leqslant$$

$$\leqslant \alpha \int_E h^+ g_k \, d\mu + \alpha \int_E h^- g_k \, d\mu = \alpha \int_E |h| \, g_k \, d\mu .$$

Then, by hypothesis, there is $f \in \Lambda^\alpha$ so that

$$\varphi(h) = \int_E h \cdot f \, d\mu \qquad h \in \Lambda \tag{1}$$

Let us see that $\langle f_A^* \rangle$ is determined by f. If $\mu(A) < \infty$ and $h \geqslant 0$, $h \in \Lambda$, we have $S(h \cdot \chi_A) \subset A$ and by (1) and the definition of φ

$$\varphi(h\chi_A) = \int_E h\chi_A \cdot f_{A \cap S(h\chi_A)} \, d\mu = \int_E h\chi_A f_A \, \chi_{A \cap S(h\chi_A)} \, d\mu =$$

$$= \int_E h\chi_A f_A \, d\mu = \int_E fh \, \chi_A \, d\mu .$$

Then, if h is arbitrary, we have the same formula for h. Hence, as $(E, \mathcal{A}, \mu)$ has the finite subset property, by proposition 1, we have $(f \cdot \chi_A)^* = (\chi_A \cdot f_A)^* = f_A^*$.

Now, we examine the general case. Given the positive cross section $\langle f_A^* \rangle$, we define $f_{n,A}(x) = \inf (f_A(x), n \cdot g_n(x))$. Then the cross section $\langle f_{n,A}^* \rangle$ is determined by a function $f_n \in \Lambda^\alpha$, $f_n \geqslant 0$. As $\langle f_{n,A}^* \rangle$ is increasing with n, we have $f_n \leqslant f_{n+1}$ a.e. on E for every $n \in N$. Then it is clear that $\langle f_A^* \rangle$ is determined by f so taht $f(x) = \lim_n f_n(x)$ a.e. on E. q.e.d.

Our next target is to prove that we can suppose without losing generality that $(E, \mathcal{A}, \mu)$ is always localizable and has the finite subset property. For this objective we develop our theory in the more general setting of certain vectorial lattices. Afterwards, we will apply these general results to obtain certain properties of the strong bidual Λ'' of Λ.

Let $(X, \mathcal{T})$ be a locally convex topological vector space and let $\{M_i, \, i \in I\}$ be a family of subspaces of X so that $M_i \cap M_j = \{0\}$ if $i \neq j$. We shall write

$$\sum_{i \in I} \oplus M_i$$

to represent the set of the elements $x \in X$ so that x is the sum of an absolutely summable family $\{x_i, \, i \in I\}$ with $x_i \in M_i$ for every $i \in I$. In a lattice $(X, \leqslant)$, if $A \subset X$, we define $A^\perp = \{x \in X / |x| \wedge \wedge |y| = 0, \forall \, y \in A\}$ and $A^{\perp\perp} = (A^\perp)^\perp$. If $A = \{u\}$, we write $A^\perp = u^\perp$ and $A^{\perp\perp} = u^{\perp\perp}$. We have now the following theorem:

THEOREM 3. *Let* $[E, \mathcal{T}]$ *be a Frechet topological vector lattice so that* $\mathcal{T}$ *is defined by an increasing family of lattice seminorms*

$\{\|\cdot\|_k, k \in N\}$, *so that there is* $p \geqslant 1$ *so that* $\|x+y\|_k^p = \|x\|_k^p + \|y\|_k^p$ *if* $x \wedge y = 0$, $k \in N$.

Then, if M *is a closed sublattice of* X, $M \neq \{0\}$, *there is a family* $\{u_i, i \in I\}$ *of positive elements of* M *so that*

$$M = \sum_{i \in I} \oplus (u_i^{\perp\perp} \cap M).$$

PROOF. We note that a seminorm $\|\cdot\|_k$ its called a lattice seminorm if $|x| \leqslant |y|$ implies $\|x\|_k \leqslant \|y\|_k$. Let us see that, in the hypothesis of theorem 3, $[E, \mathcal{T}]$ is order complete. Let $A_k = \{x \in E / \|x\|_k = 0\}$. A_k is a closed ideal in $[E, \mathcal{T}]$. The completion of the quotient space $(E, \|\cdot\|_k)/A_k$ is an abstract L^p-space. By a result of Bernau (see [2] p. 133), we have $\|x + \|y\|_k^p \geqslant \|x\|_k^p + \|y\|_k^p$ if $x \geqslant 0, y \geqslant 0, x \in E, y \in E$. Then, it is known (see [2] p. 21) that $\|\cdot\|_k$ is order continuous. Hence, if $\{x_d, d \in D\}$ is an upwards directed set with respect to the order on E, which has an upper bound, $\{x_d, d \in D\}$ is a Cauchy net in $[E, \mathcal{T}]$. Then, there is $x = \lim_{d \in D} x_d \in E$. But we have $x = \sup_{d \in D} x_d$ because $[E, \mathcal{T}]$ is a topological vector lattice (see [4] p. 90). Hence, it is easy to see that $[E, \mathcal{T}]$ is order complete. It is also clear that if $\{x_n\}_{n=1}^{\infty}$ is an order bounded increasing sequence of positive elements of E, there is $x \in E$ so that $x = \lim_n x_n = \sup_n x_n$.

Now, let $M^+ = \{x \in M / x \geqslant 0\}$. By Zorn lemma, there is a maximal family $\{u_i, i \in I\}$ of non zero elements of M^+ pairwise disjoints. By Riesz theorem, if $u > 0$, there is a projection J_u from E onto the band $u^{\perp\perp}$ generated by u, given by the formula

$$J_u(x) = \sup_n (x^+ \wedge nu) - \sup_n (x^- \wedge nu) = \lim_n (x^+ \wedge nu) - \lim (x^- \wedge nu).$$

Let us prove that $J_{u_i+u_j} = J_{u_i} + J_{u_j}$ if $i, j \in I$, $i \neq j$. If $x \geqslant 0$

$$n(u_i + u_j) \wedge x = n(u_i \vee u_j + u_i \wedge u_j) \wedge x = n(u_i \vee u_j) \wedge x = (nu_i \wedge x) \vee (nu_j \wedge x) = (nu_i \wedge x) + (nu_j \wedge x) - (nu_i \wedge x) \wedge (nu_j \wedge x) = (nu_i \wedge x) + (nu_j \wedge x).$$

Taking limits when n increases, we have $J_{u_i+u_j}(x) = J_{u_i}(x) + J_{u_j}(x)$. If $x \in E$ is arbitrary, $x = x^+ - x^-$ and the same formula holds for x.

Then $J_{u_i+u_j} = J_{u_i} + J_{u_j}$. It is easy to see that given a finite number of different elements $u_1, u_2, \ldots, u_n$, we have the same property respecting $\sum_{i=1}^{n} J_{u_i}$.

As E is order complete, each band $u_i^{\perp\perp}$ is closed in $[E, \mathcal{T}]$. Then $J_{u_i}(M) = u_i^{\perp\perp} \cap M$. Now we shall see that if $x \geqslant 0$, $x \in M$ the family $\{J_{u_i}(x), i \in I\}$ is absolutely summable in $[E, \mathcal{T}]$, and that its sum is x. First we note that if $u_i \neq u_j$.

$$J_{u_i}(x) \wedge J_{u_j}(x) = \lim_n (x \wedge nu_i) \wedge J_{u_j}(x) = \lim_n (x \wedge nu_i \wedge J_{u_j}(x)) =$$
$$= \lim_n (\lim_m x \wedge nu_i \wedge x \wedge mu_j) = 0 .$$

Then, as for every finite set $F \subset I$ we have $\sum_{i \in F} J_{u_i}(x) = J_{\sum_{i \in F} u_i}(x) \leqslant x$ by hypothesis, we have

$$\sum_{i \in F} ||J_{u_i}(x)||_k^p = || \sum_{i \in F} J_{u_i}(x)||_k^p \leqslant ||x||_k^p \qquad \forall\, k \in N \qquad (1)$$

Hence, $\{J_{u_i}(x), i \in I\}$ is an absolutely summable family in $[E, \mathcal{T}]$ and the net $\{\sum_{i \in F} J_{u_i}(x), F \subset I, F \text{ finite}\}$ is convergent to an element $\sum_{i \in I} J_{u_i}(x) \in M$. Let us see that $x = \sum_{i \in I} J_{u_i}(x)$. As $[E, \mathcal{T}]$ is a metric space, there is an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of I, so that

$$\lim_n \sum_{i \in F_n} J_{u_i}(x) = \sup_n \sum_{j \in F_n} J_{u_i}(x) = \sum_{i \in I} J_{u_i}(x) \leqslant x .$$

Let $z = x - \sum_{i \in I} J_{u_i}(x) \geqslant 0$. If $z \neq 0$, by the maximality of the set $\{u_i, i \in I\}$, there is $i \in I$ so that $z \wedge u_i \neq 0$. But

$$0 \leqslant z \wedge u_i = (x - \sum_{j \in I} J_{u_j}(x)) \wedge u_i \leqslant$$
$$\leqslant (x - J_{u_i}(x)) \wedge u_i \leqslant J_{u_i}(x - J_{u_i}(x)) = 0$$

because J_{u_i} is a projection. But this is a contradiction with the choice of u_i. Then $z = 0$ and $x = \sum_{j \in I} J_{u_i}(x)$. If $x \in M$ is arbitrary, $x^+ \in M$ and $x^- \in M$ because M is a lattice. Then $x = \sum_{i \in I} J_{u_i}(x^+) -$

$-\sum_{i\in I} J_{u_i}(x^-) = \sum_{i\in I} J_{u_i}(x)$ is the sum of an absolutely summable family of elements x_i with $x_i \in u_i^{\perp\perp}$, $i \in I$.

Let us see that this decomposition is unique. First we note that

$$|J_{u_i}(x)| \leqslant |J_{u_i}(x^+)| + |J_{u_i}(x^-)| = J_{u_i}(x^+) + J_{u_i}(x^-) \leqslant x^+ + x^- = |x|$$

Then, the projections J_{u_i} are continuous from $[E, \mathfrak{J}]$ onto $u_i^{\perp\perp}$, being provided this space with the induced topology. Hence, if we have

$0 = \sum_{i\in I} x_i$ with $x_i \in u_i^{\perp\perp}$ for every $i \in I$, we have also for every $j \in I$

$$0 = \sum_{i\in I} J_{u_j}(x_i) = \sum_{i\in I} J_{u_j}(J_{u_i}(x)) =$$

$$= \sum_{i\in I} J_{u_j}(\lim x_i^+ \wedge nu_i - \lim x_i^- \wedge nu_i) =$$

$$= \sum_{i\in I} (\lim_m \lim_n x_i^+ \wedge nu_i \wedge mu_j - \lim_m \lim_n x_i^- \wedge nu_i \wedge mu_j) =$$

$$= \lim_m x_j^+ \wedge mu_j - \lim_m x_j^- \wedge mu_j = J_{u_j}(x_j) = x_j \tag{2}$$

Then, the decomposition is unique.

Conversely, every absolutely summable fammily $\{x_i,\ i \in I\}$ with $x_i \in M \cap u_i^{\perp\perp}$ for every $i \in I$, defines an element $x \in M$, M being closed. Then

$$M = \sum_{i\in I} \oplus (M \cap u_i^{\perp\perp}).$$

It is important to note that if $0 \leqslant x = \sum_{i\in I} J_{u_i}(x)$, all the terms $J_{u_i}(x)$ are zero except a numerable set. In fact, from (1) we obtain that $\{||J_{u_i}(x)||_k^p,\ i \in I\}$ is summable for every $k \in N$. Then it is known, that for every $k \in N$, all the terms $||J_{u_i}(x)||_k$ are zero except a numerable set. Then except for a numerable set of $i \in I$, we have $||J_{u_i}(x)||_k = 0$ for every $k \in N$. As $[E, \mathfrak{J}]$ is separated, $J_{u_i}(x) = 0$ except a numerable set of $i \in I$. It is clear that the same is true for an arbitrary $x \in M$, because the decomposition is unique. q.e.d.

From the first part of theorem 3. we obtain the following corollary.

COROLLARY. Let Λ^p $(E, \mathcal{A}, \mu, g_k)$ $p \geqslant 1$ *be an echelon Köthe space.* Λ^p *is order complete. If* $\{f_n\}_{n=1}^{\infty}$ *is an order bounded monotonous in-*

creasing sequence of positive functions of Λ^p, *then there is* $\sup_n f_n = \lim_n f_n$ *in* $[\Lambda^p, \mathfrak{J}]$.

PROOF. Obvious.

We now prove the following main theorem.

THEOREM 4. *Let* $[E, \mathfrak{J}_0]$ *be a Frechet topological vector lattice so that* $\mathfrak{J}_0$ *is defined by an increasing family of lattice seminorms* $\{\|\cdot\|_k, k \in N\}$, *so that there is* $p \geqslant 1$ *so that* $\|x+y\|_k^p = \|x\|_k^p + \|y\|_k^p$ *if* $x \wedge y = 0$, $k \in N$. *Then* $[E, \mathfrak{J}_0]$ *is isomorphic to an echelon Köthe space* $\Lambda^p(T, \mathcal{M}, \mu, \varphi_k)$ *so that* $(T, \mathcal{M}, \mu)$ *is localizable and has the finite subset property, Further. the lattices E and* Λ^p *are order isomorphic.*

PROOF. By theorem 3, applied to the space E, there is a family $\{u_i, i \in I\}$ of positive elements of E, pairwise disjoints, so that

$$E = \sum_{i \in I} \oplus (u_i^{\perp\perp} \cap E) = \sum_{i \in I} \oplus u_i^{\perp\perp}.$$

a) Let us show that every $u_i^{\perp\perp}$ with the topology induced by $\mathfrak{J}_0$ is isomorphic to an echelon Köthe space $\Gamma_i^p(T_i, \mathcal{A}_i, \mu_i, \varphi_{ki})$ with $\mu_i(T_i) < \infty$. We write u instead of u_i. Let

$$\mathcal{A} = \{x \in u^{\perp\perp} / x \wedge (u - x) = 0\}.$$

We define the join and the meet of two elements x, y of $\mathcal{A}$ as $x \vee y$ and $x \wedge y$ respectively, and we define the complement of x as $u - x$. With these operations, $\mathcal{A}$ is a boolean algebra because

$$(x \vee y) \wedge (u - (x \vee y)) = (2(x \vee y) \wedge u) - (x \vee y) = (2x \wedge u) \vee (2y \wedge u) - (x \vee y) = (x \wedge (u - x) + x) \vee (y \wedge (u - y) + y) - (x \vee y) = x \vee y - x \vee y = 0$$

and hence $x \vee y \in \mathcal{A}$. Analogously $x \wedge y \in \mathcal{A}$. Given $x \in \mathcal{A}$,

$$(u - x) \wedge (u - (u - x)) = (u - x) \wedge x = 0$$

and then $u - x \in \mathcal{A}$.

By the Stone representation theorem, there is a compact topological space T, totally disconnected, so that there is an isomorphism Φ of boolean algebras from $\mathcal{A}$ onto the boolean algebra $\mathcal{T}$ of open-closed sets of T. Let $C(T)$ the space of real valued continuous functions on T, with the topology defined by the norm $\|f\| = \sup \{|f(t)|,$

$t \in T\}$. It is clear that, if $A \in \mathcal{T}$, we have $\chi_A \in C(T)$. Let S be the set of simple functions $f = \sum_{i=1}^{n} \alpha_i \chi_{M_i}$ so that $M_i \in \mathcal{T}$, $i = 1,2, \dots n$; $M_i \cap M_j = \varnothing$ if $i \neq j$, and $\bigcup_{i=1}^{n} M_i = T$. Then $S \subset C(T)$. Clearly S is a subalgebra of $C(T)$ which contains the constants. Given two points $t_1 \neq t_2$ of T, as T is a totally disconnected compact space, there is $M \in \mathcal{T}$ so that $t_1 \in M$, $t_2 \notin M$. Then $\chi_M \in S$ and $\chi_M(t_1) = 1$ and $\chi_M(t_2) = 0$. Then S separates points. By the Stone-Weiertrass theorem, S is dense in $C(T)$.

Now, we shall define for every $k \in N$, a positive continuous linear form on $C(T)$. If $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \in S$ with the A_i pairwise disjoints and $\bigcup_{i=1}^{n} A_i = T$, we define

$$\varphi_k(f) = \sum_{i=1}^{n} \alpha_i \, ||\Phi^{-1}(A_i)||_k^p .$$

It is easy to see that φ_k is well defined and linear on S. using the isomorphism Φ and the hypothesis of the theorem. Further φ_k is continuous on S with the induced topology by $C(T)$, because, if $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ with the A_i pairwise disjoints and $\bigcup_{i=1}^{n} A_i = T$, we have

$$p_k(f)| \leqslant \sum_{i=1}^{n} |\alpha_i| \, ||\Phi^{-1}(A_i)||_k^p \leqslant \underset{i=1,2,..n}{SUP} \{|\alpha_i|\} \cdot \sum_{i=1}^{n} ||\Phi^{-1}(A_i)||_k^p = ||f|| \cdot ||\Phi^{-1}(T)||_k^p .$$

Hence φ_k can be continuously extended to the closure of S in $C(T)$. But S is dense in $C(T)$. Then φ_k is defined on $C(T)$. Now let us see that φ_k is positive on $C(T)$. If $f \in C(T), f \geqslant 0$, given $\varepsilon > 0$, there is $s = \sum_{i=1}^{n} \alpha_i \chi_{M_i} \in S$ so that

$$\sup_{t \in T} |f(t) - s(t)| \leqslant \varepsilon .$$

We define $s' = \sum_{i=1}^{n} \beta_i \chi_{M_i}$; where $\beta_i = \alpha_i$ if $\alpha_i \geqslant 0$ and $\beta_i = 0$ if $\alpha_i < 0$. Then $s' \in S$, $s' \geqslant 0$ and

$$\sup_{t \in T} |f(t) - s'(t)| \leqslant \varepsilon . \tag{1}$$

Hence f is the limit of a sequence of positive function $s' \in S$. As $\varphi_k(s') \geqslant 0$, we have also $\varphi_k(f) \geqslant 0$.

Then, by the Riesz representation theorem, there is a σ-algebra $\mathcal{A}_k$ containing the Borel sets of T, and a regular Borel measure ν_k defined on $\mathcal{A}_k$ so that

$$\varphi_k(f) = \int_T f\, d\nu_k \qquad \forall\, f \in C(T)$$

$$\varphi_k(\chi_A) = \int_T \chi_A\, d\nu_k = \nu_k(A) = \|\Phi^{-1}(A)\|_k^p \qquad \forall\, A \in \mathcal{T}$$

Then, as $T \in \mathcal{T}$, $\varphi_k(\chi_T) = \nu_k(T) = \|\Phi^{-1}(T)\|_k^p < \infty$ and ν_k is finite.

Let $\mathcal{B}$ be the family of Borel sets of T. Clearly $\mathcal{B} \subset \bigcap_{k=1}^{\infty} \mathcal{A}_k$ and the set function

$$\mu(B) = \sum_{\substack{i=1 \\ \nu_i(T) \neq 0}}^{\infty} \frac{\nu_i(B)}{2^i\, \nu_i(T)} \qquad B \in \mathcal{B}$$

is a measure on $(T, \mathcal{B})$. Let us see that μ is inner regular. If $A \in \mathcal{B}$ and $\varepsilon > 0$, there is n_0 so that

$$\sum_{n > n_0} \frac{1}{2^n} < \frac{\varepsilon}{2}$$

As every ν_i is regular, there are compact sets F_i, $i = 1,2 \dots n_0$ so that $F_i \subset A$ and

$$\nu_i(A - F_i) \leqslant \frac{\varepsilon}{2\, n_0} \cdot 2^i\, \nu_i(T)$$

Then, $F = \bigcup_{i=1}^{n_0} F_i$ is compact, $F \subset A$ and it is easy to see that $\mu(A - F) < \varepsilon$.

It is also obvious that every ν_k is absolutely continuous respecting μ. Then by Radon-Nikodym theorem, there is a $\mathcal{B}$-measurable function $\hat{g}_k \geqslant 0$ so that

$$\nu_k(B) = \int_B \hat{g}_k\, d\mu \qquad \forall\, B \in \mathcal{B}$$

It is clear that $\mu\left(\bigcap_{k=1}^{\infty}\{t\in T/\hat{g}_k(t)=0\}\right)=0$. Then, let us consider the echelon Köthe space $\Sigma^p(T, \mathcal{B}, \mu, \hat{g}_k)$. We shall see that Σ^p and $u^{\perp\perp}$ are isomorphic and order isomorphic.

First we want to define a mapping $\psi:\Sigma^p\to u^{\perp\perp}$. It is clear that $S\subset\Sigma^p$. If $M\in\mathcal{T}$, $\chi_M\in\Sigma^p$ and we put

$$\psi(\chi_M)=\Phi^{-1}(M).$$

ψ is extended by linearity to the whole of S. It is easy to see that ψ is well defined an linear on S. Given the zero-neighbourhood in $u^{\perp\perp}$, $V=\{x\in u^{\perp\perp}/\|x\|_k^p\leqslant\varepsilon\}$, $\varepsilon>0$, we consider the zero-neighbourhood induced by Σ^p on S

$$W=\left\{s\in S\Big/\int_T|s|^p\,\hat{g}_k\,d\mu\leqslant\varepsilon\right\}$$

Let $s=\sum_{i=1}^{n}\alpha_i\chi_{M_i}\in W$, with the M_i pairwise disjoints and $\bigcup_{i=1}^{n}M_i=T$. We note that

$$\left|\sum_{i=1}^{n}\alpha_i\,\Phi^{-1}(M_i)\right|\leqslant\sum_{i=1}^{n}|\alpha_i|\,|\Phi^{-1}(M_i)|=$$
$$=\sum_{i=1}^{n}|\alpha_i|\,\Phi^{-1}(M_i)=\left|\sum_{i=1}^{n}|\alpha_i|\,\Phi^{-1}(M_i)\right|$$

and that $|\alpha_i|\,\Phi^{-1}(M_i)\wedge|\alpha_j|\,\Phi^{-1}(M_j)|=0$ if $i\neq j$, because Φ is an isomorphism of boolean algebras and $M_i\cap M_j=\varnothing$. As $\|\cdot\|_k$ is a lattice seminorm, we have using the hypothesis of the theorem,

$$\|\psi(S)\|_k^p=\|\sum_{i=1}^{n}\alpha_i\,\Phi^{-1}(M_i)\|_k^p\leqslant\|\sum_{i=1}^{n}|\alpha_i|\,\Phi^{-1}(M_i)\|_k^p=\sum_{i=1}^{n}|\alpha_i|^p\,\|\Phi^{-1}(M_i)\|_k^p=$$
$$=\sum_{i=1}^{n}|\alpha_i|^p\,\nu_k(M_i)=\sum_{i=1}^{n}|\alpha_i|^p\int_{M_i}\hat{g}_k\,d\mu=\int_T|s|^p\,\hat{g}_k\,d\mu\leqslant\varepsilon.$$

Hence $\psi(W)\subset V$ and ψ is continuous from S, with the induced topology by Σ^p, into $u^{\perp\perp}$. Then ψ can be extended by continuity to a linear mapping, again denoted by ψ, from the closure of S in Σ^p, into $u^{\perp\perp}$, because $u^{\perp\perp}$ is complete, being closed in $[E,\mathfrak{J}_0]$. Let us see that S is dense in Σ^p. If $f \in C(T)$, f is bounded because T is compact. Then $C(T) \subset \Sigma^p$. Given $f \in C(T)$ and $\varepsilon > 0$, there is $s \in S$ so that $\sup \{|f(t) - s(t)|, t \in T\} \leqslant \varepsilon$. Then

$$\int_T |f - s|^p \hat{g}_k \, d\mu \leqslant \varepsilon^p \int_T \hat{g}_k \, d\mu = \varepsilon^p \nu_k(T)$$

and S is dense in $C(T)$ with the topology induced by $\mathcal{J}_0$. But by proposition 4, $C(T)$ is dense in $[\Sigma^p, \mathcal{J}]$. Then S is dense in Σ^p and ψ is defined on Σ^p.

Now, let us see that, for every $k \in N$, if $f \in \Sigma^p$

$$\int_T |f|^p \hat{g}_k \, d\mu = ||\psi(f)||_k^p .$$

If $f \in \Sigma^p$, there is a sequence $\{S_n\}_{n=1}^{\infty}$ of functions of S

$$S_n = \sum_{i=1}^{p_n} \alpha_{in} \chi_{A_{in}}$$

with $A_{in} \cap A_{jn} = \varnothing$ if $i \neq j$; $A_{in} \in \mathcal{J}$ for every i and n, and so that $f = \lim_n S_n$ in the topology of Σ^p. Then $|f| = \lim_n |S_n|$ because

$$\int_T ||f| - |S_n||^p \hat{g}_k \, d\mu \leqslant \int_T |f - S_n|^p \hat{g}_k \, d\mu .$$

Then by the continuity of seminorms, as Φ is an isomorphism of boolean algebras and noting that $|\alpha_{in}|\psi(\chi_{A_{in}}) = |\alpha_{in}| \Phi^{-1}(A_{in}) \geqslant 0$, we have

$$\int_T |f|^p \hat{g}_k \, d\mu = \lim_{n\to\infty} \int_T |S_n|^p \hat{g}_k \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{p_n} |\alpha_{in}|^p \nu_k(A_{in}) =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{p_n} |\alpha_{in}|^p \, ||\Phi^{-1}(A_{in})||_k^p = \lim_{n\to\infty} \sum_{i=1}^{p_n} |\alpha_{in}|^p \, ||\psi(\chi_{A_{in}})||_k^p =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{p_n} || \, |\alpha_{in}|\psi(\chi_{A_{in}})||_k^p = \lim_{n\to\infty} \left\| \sum_{i=1}^{p_n} |\alpha_{in}| \, \psi(\chi_{A_{in}}) \right\|_k^p =$$

$$= \lim_{n\to\infty} \left\| \sum_{\alpha_{in} > 0} |\alpha_{in}| \, \psi(\chi_{A_{in}}) + \sum_{\alpha_{in} < 0} |\alpha_{in}| \, \psi(\chi_{A_{in}}) \right\|_k^p \qquad (2)$$

But

$$0 \leqslant \Big(\sum_{\alpha_{in} \geq 0} |\alpha_{in}| \, \psi(\chi_{A_{in}})\Big) \wedge \Big(\sum_{\alpha_{in} < 0} |\alpha_{in}| \, \psi(\chi_{A_{in}})\Big) \leqslant$$

$$\leqslant (\underset{1 \leq i \leq p_n}{MAX} \{|\alpha_{in}|\}) \cdot \Big[\Big(\sum_{\alpha_{in} \geq 0} \psi(\chi_{A_{in}})\Big) \wedge \Big(\sum_{\alpha_{in} < 0} \psi(\chi_{A_{in}})\Big)\Big] =$$

$$= (\underset{1 \leq i \leq p_n}{MAX} \{|\alpha_{in}|\}) \, [\psi(\chi_{\underset{\alpha_{in} \geq 0}{\cup} A_{in}}) \wedge \psi(\chi_{\underset{\alpha_{ni} < 0}{\cup} A_{in}})] =$$

$$= (\underset{1 \leq i \leq p_n}{MAX} \{|\alpha_{in}|\}) \, (\Phi^{-1}(\underset{\alpha_{in} \geq 0}{\cup} A_{in}) \wedge \Phi^{-1}(\underset{\alpha_{in} < 0}{\cup} A_{in})) = 0 .$$

Then, as in a lattice we have $|x + y| = |x - y|$ if $x \wedge y = 0$, the formula (2) gives

$$= \lim_{n \to \infty} \Big\| \sum_{\alpha_{in} \geq 0} |\alpha_{in}| \, \psi(\chi_{A_{in}}) - \sum_{\alpha_{in} < 0} |\alpha_{in}| \, \psi(A_{in}) \Big\|_k^p =$$

$$= \lim_{n \to \infty} \Big\| \sum_{i=1}^{p_n} \alpha_{in} \, \psi(\chi_{A_{in}}) \Big\|_k^p = \lim_{n \to \infty} \|\psi(S_n)\|_k^p = \|\lim_{n \to \infty} \psi(S_n)\|_k^p = \|\psi(f)\|_k^p \quad (3)$$

Then, the mapping ψ is injective, because if $\psi(f) = 0$, for every $k \in N$ we have

$$\|\psi(f)\|_k^p = \int_T |f|^p \hat{g}_k \, d\mu = 0$$

and hence f must be 0, Σ^p being separated. Let us see that $\psi: \Sigma^p \to u^{\perp\perp}$ is surjective. By (3), ψ is an open mapping. As Σ^p and $u^{\perp\perp}$ are metrizable complete spaces, by the Banach-Schauder theorem, $\psi(\Sigma^p)$ is closed. If we can prove that $\psi(\Sigma^p)$ is dense in $u^{\perp\perp}$, we will have $\psi(\Sigma^p) = u^{\perp\perp}$. Let us see that $\psi(\Sigma^p)$ is dense in $u^{\perp\perp}$. First, let $x \in u^{\perp\perp}$ be so that there is $m \in N$ so that $0 \leqslant x \leqslant mu$. It is known by the general theory of vectorial lattices that, given $\varepsilon > 0$, there are elements $v_i \in u^{\perp\perp}$, $i = 1,2, .. n$, and real numbers $a_i \in [0, 1]$, $i = 1,2,.. n$ so that

$$\sum_{i=1}^{n} a_i v_i \leqslant \frac{x}{m} \leqslant \sum_{i=1}^{n} a_i v_i + \varepsilon u$$

$$v_j \wedge (u - v_j) = 0 \qquad v_i \wedge v_j = 0 \qquad i, j = 1,2 .. n \qquad i \neq j$$

because $u^{\perp\perp}$ is a band and hence is order complete. As the seminorms are monotonous

$$\left\| \frac{x}{m} - \sum_{i=1}^{n} a_i v_i \right\|_k \leqslant \left\| \sum_{i=1}^{n} a_i v_i + \varepsilon u - \sum_{i=1}^{n} a_i v_i \right\|_k = \varepsilon \|u\|_k .$$

But $v_i \in \mathcal{A}$. Then, x can be approximated by elements of the type $\sum_{i=1}^{n} a_i v_i$ which belong to $\psi(\Sigma^p)$. Let x be now an arbitrary positive element of $u^{\perp\perp}$. Then, with the notation of theorem 3, by hypothesis we have

$$x = J_u(x) = \sup_n x \wedge nu = \lim_n x \wedge nu .$$

Then x can be approximated by elements of $\psi(\Sigma^p)$. If $x \in u^{\perp\perp}$ is arbitrary, $x = x^+ - x^-$ and the same conclusion holds. Hence $u^{\perp\perp} = \psi(\Sigma^p)$.

ψ is an order isomorphism: It is clear that if $f \in S$, $f \geqslant 0$, then $\psi(f) \geqslant 0$. If $f \in \Sigma^p$, $f \geqslant 0$, f is the limit of a sequence of positive functions of S, because if S_n converges to f, then $|S_n|$ converges to $|f| = f$. As the positive cone of $u^{\perp\perp}$ is closed, $\psi(f) \geqslant 0$. With ψ^{-1} the argument is the same because a positive element $x \in u^{\perp\perp}$ is the limit of a sequence of positive elements of the form $\sum_{i=1}^{n} a_i v_i$, $v_i \in \mathcal{A}$, $i = 1,2.. n$.

Now, let us see that changing the definition of $\hat{g}_k$, $k \in N$, on a set of zero measure, if it is neccesary, we have $\hat{g}_k \leqslant \hat{g}_{k+1}$ $k \in N$. Let $M_k = \{t \in T / \hat{g}_k(t) > \hat{g}_{k+1}(t)\}$. If $\mu(M_k) > 0$, arguing as in the proposition 1, there is a set $A \subset M_k$ so that $\mu(A) > 0$ and $\chi_A \in \Sigma^p$. Then

$$\|\psi(\chi_A)\|_k^p = \int_T \chi_A \hat{g}_k \, d\mu \geqslant \int_T \chi_A \hat{g}_{k+1} \, d\mu = \left\| \psi(\chi_A) \right\|_{k+1}^p .$$

But by hypothesis

$$\|\psi(\chi_A)\|_k \leqslant \|\psi(\chi_A)\|_{k+1}$$

and hence

$$\int_A \hat{g}_k \, d\mu = \int_A \hat{g}_{k+1} \, d\mu .$$

But $\hat{g}_k - \hat{g}_{k+1} \geqslant 0$ on A. Then $\hat{g}_k = \hat{g}_{k+1}$ a.e. on A, which is a contradiction. Then $\mu(M_k) = 0$. Then, changing the definition of the $\hat{g}_k$ on $\bigcup_{k=1}^{\infty} M_k$, we get $\hat{g}_k(t) \leqslant \hat{g}_{k+1}(t)$ for every $t \in T$. Hence, $u^{\perp\perp}$ is isomorphic and order isomorphic to the echelon Köthe space $\sum^p$.

b) For every $i \in I$, let ψ_i be an isomorphism and order isomorphism from $u_i^{\perp\perp}$ onto the echelon Köthe space $\sum_i^p (T_i, B_i, \mu_i, \hat{g}_{ki})$ so that

$$||x||_k^p = \int_{T_i} |\psi_i(x)|^p \hat{g}_{ki}\, d\mu_i \qquad \forall\, x \in u_i^{\perp\perp}.$$

Let T be the disjoint union of the sets T_i, $i \in I$. Let $\mathcal{M}$ be the family of sets

$$\mathcal{M} = \{M / M = \bigcup_{i \in I} A_i,\ A_i \in B_i\}$$

It is easy to see that $\mathcal{M}$ is a σ-algebra on T. For every $k \in N$ we define the function g_k on T so that $g_k(t) = \hat{g}_{ki}(t)$ if $t \in T_i$. It is clear that g_k is $\mathcal{M}$-measurable and that the set function

$$\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu_i(A_i)$$

is a measure on $(T, \mathcal{M})$. $\sum_{i \in I} \mu_i(A_i)$ is the sum, finite or not, of the family of real numbers $\mu_i(A_i)$, in the sense of the summable families As $\mu_i(T_i) < \infty$ by a), we have that μ is localizable and has the finite subset property.

Now we consider the echelon Köthe space Λ^p $(T, \mathcal{M}, \mu, g_k)$. We define $\psi : E \to \Lambda^p$ by the rule

$$\psi(x) = \sum_{n=1}^{\infty} \psi_{i_n}(x_{i_n})$$

where $\{i_n\}_{n=1}^{\infty}$ is the sequence of natural numbers so that, in the unique decomposition $x = \sum_{i \in I} x_i$, $x_i \in u_i^{\perp\perp}$, we have $x_{i_n} \neq 0$. Each function $\psi_{i_n}(x_{i_n})$ is considered as an element of Λ^p zero on $T - T_{i_n}$. Then, ψ is well defined, linear and, by continuity of seminorms, we have $\psi(x) \in \Lambda^p$ because

$$\int_T |\psi(x)|^p g_k \, d\mu = \sum_{n=1}^{\infty} \int_{T_{in}} |\psi_{i_n}(x_{i_n})|^p \, \hat{g}_{k i_n} \, d\mu_{i_n} = \sum_{n=1}^{\infty} ||x_{i_n}||_k^p =$$
$$= ||x||_k^p < \infty \qquad \forall \, k \in N$$

This equality shows that ψ preserves the seminorms of elements. Then ψ is injective. ψ is also surjective, because if $f \in \Lambda^p$ by proposition 2, $S(f)$ is σ-finite. Then f is zero on every T_i except in a numerable set $\{T_{i_n}\}_{n=1}^{\infty}$. It is clear that the pre-image of f is

$$\sum_{n=1}^{\infty} \psi_{i_n}^{-1}(f \cdot \chi_{T_{in}})$$

and then ψ is an isomorphism. As the ψ_i are order isomorphisms, ψ is also an order isomorphism. q.e.d.

From this theorem we can obtain important results. The first theorem is the following:

THEOREM 5. *Every echelon Köthe space $\Lambda^p(E, \mathcal{A}, \mu, g_k)$, $p \geqslant 1$, is K-isomorphic and order isomorphic to an echelon Köthe space $\Gamma^p(T, \mathcal{M}, \nu, \varphi_k)$ with $(T, \mathcal{M}, \nu)$ localizable and with the finite subset property.*

PROOF. It is an immediate application of the theorem 4.
For this reason, we shall always suppose that the measure spaces $(E, \mathcal{A}, \mu)$ are localizable and with the finite subset property.

Now we study some properties of the dual $(\Lambda^p)^\alpha$. In $(\Lambda^p)^\alpha$ we have the relation of order $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ a.e. on E, in the same manner as on Λ^p. But $(\Lambda^p)^\alpha$ is the dual of $[\Lambda^p, \mathcal{J}]$ Then we can also consider the order between $f \in (\Lambda^p)^\alpha$ and $g \in (\Lambda^p)^\alpha$:

$$f \leqslant g \text{ if and only if} \int_E f\, h \, d\mu \leqslant \int_E g\, h \, d\mu \qquad \forall \, h \in \Lambda^p, \quad h \geqslant 0$$

We note that the two definitions are the same : in fact, if $f \geqslant 0$ in the latter sense but $\mu(A) = \mu\,\{x \in E / f(x) < 0\} > 0$, as $(E, \mathcal{A}, \mu)$ has the finite subset property, there is a set $M \subset A$ of finite measure different from zero. Arguing as in the proposition 1, we get a set $D \subset M$ so that $\chi_D \in \Lambda^p$ and $\mu(D) > 0$. Then

$$\int_E \chi_D f \, d\mu < 0$$

which contradicts the assumption about f. Then the order of $(\Lambda^p)^\alpha$ is the puntual order of functions.

By theorem 2, $(\Lambda_k^p)' = (\Lambda_k^p)^\alpha$. As $I_k : \Lambda^p \to \Lambda_k^p$ is continuous, its transposed map I_k' maps $(\Lambda_k^p)^\alpha$ into $(\Lambda^p)^\alpha$. Let us see that, if $f \in (\Lambda_k^p)^\alpha$, $I_k'(f)$ is the function $\hat{f}$ equal to f on $S(g_k)$ and equal to zero on $E - S(g_k)$. If $h \in \Lambda^p$

$$\langle I_k'(f), h\rangle = \int_E I_k'(f) \cdot h \, d\mu = \langle I_k(h), f\rangle = \int_{S(g_k)} h f \, d\mu = \int_E h \hat{f} \, d\mu .$$

Then, by proposition 1, $I'_k(f) = \hat{f}$. Now, we have the following characterization of equicontinuous sets of $(\Lambda^p)^\alpha$.

THEOREM 6. *Let M be a subset of $(\Lambda^p)^\alpha$. If $p = 1$, the following are equivalent:* 1) *M is $\mathfrak{I}$-equicontinuous.* 2) *There are $k \in N$ and an equicontinuous set M' of $(\Lambda_k)^\alpha$ so that $M = I'_k(M')$.* 3) *There are $k \in N$ and $C > 0$ so tha $|f| \leqslant C g_k$ for every $f \in M$.*

If $p > 1$, the following are equivalent: 1) *M is $\mathfrak{I}$-equicontinuous.* 2) *There are $k \in N$ and an equicontinuous set M' of $(\Lambda_k^p)^\alpha$ so that $M = I'_k(M')$.* 3) *There are $k \in N$ $\alpha > 0$ so that if $1/p + 1/q = 1$*

$$\underset{f \in M}{SUP} \left(\int_{S(g_k)} |f|^q \, g_k^{-q/p} \, d\mu \right)^{1/q} = \alpha < \infty .$$

PROOF. 1) $\Rightarrow$ 2). Let $p \geqslant 1$. By 1) there are $\varepsilon > 0$ and $k \in N$ so that $M \subset V^0$ where

$$V = \left\{ f \in \Lambda^p / \int_E |f|^p \, g_k \, d\mu < \varepsilon \right\} .$$

Hence we can suppose that M is normal. If $h \in M$, and $A = E - S(g_k)$, we have $h \cdot \chi_A = 0$. In fact, if h is not zero a.e. on A, arguing as in proposition 1, there is $D \subset A$ so that $\mu(D) > 0$, $\chi_D \cdot h \neq 0$, $\chi_D \in \Lambda^p$ Hence, for every $n \in N$, $n \cdot \chi_D \in V$. As $|h| \in V^0$, $\chi_A \cdot h$ must be 0, contradiction. Now, it is enough to show that $M' \subset W^0$, where M' is the set of restrictions on $S(g_k)$ of the elements of M, and

$$W = \left\{ f \in \Lambda_k^p / \int_{S(g_k)} |f|^p \, g_k \, d\mu < \frac{\varepsilon}{2} \right\} .$$

Let $h \in M'$, $h \geqslant 0$ and let $f \in W$, $f \geqslant 0$. By proposition 5 and theorem 1, there is a sequence $\{f_n\}_{n=1}^{\infty} \subset \Lambda^p$ of positive functions so that $\lim\limits_n I_k(f_n)(x) = f(x)$ a.e. on $S(g_k)$ and $\lim\limits_n I_k(f_n) = f$ in $[\Lambda_k^p, \mathfrak{I}_k]$. Then there is n_0 so that if $n \geqslant n_0$, by Minkowski inequality

$$\left(\int_{S(g_k)} |f_n|^p g_k \, d\mu\right)^{1/p} \leqslant \left(\int_{S(g_k)} |f_n - f|^p g_k \, d\mu\right)^{1/p} + \left(\int_{S(g_k)} |f|^p g_k \, d\mu\right)^{1/p} \leqslant$$

$$\leqslant \frac{\varepsilon}{2} - \left(\int_{S(g_k)} |f|^p g_k \, d\mu\right)^{1/p} + \left(\int_{S(g_k)} |f|^p g_k \, d\mu\right)^{1/p} = \frac{\varepsilon}{2} \qquad (1)$$

Hence $f_n \in \frac{1}{2} V$. By Fatou lemma, as $M \subset V^0$, we have

$$\int_{S(g_k)} f \cdot h \, d\mu = \int_{S(g_k)} \underline{\lim} f_n \, h \, d\mu \leqslant \underline{\lim} \int_{S(g_k)} f_n \, h \, d\mu \leqslant \frac{1}{2}.$$

Then, if $f \in W$, as f^+ and f^- also belong to W, $|\langle f, h\rangle| \leqslant 1$. Hence $h \in (\Lambda_k^p)^\alpha$ and $h \in W^0$, because every $r \in \Lambda_k^p$ is absorbed by W, Then, as M' and W^0 are normal, we have $M' \subset W^0$.

2) $\Rightarrow$ 3)If $h \in M$, h is zero on $E - S(g_k)$, and there are $k \in N$ and $\varepsilon > 0$ so that $M' \subset W^0$, where

$$W = \left\{ f \in \Lambda_k^p / \left(\int_{S(g_k)} |f|^p \cdot g_k \, d\mu\right)^{1/p} \leqslant \varepsilon \right\}$$

for every $p \geqslant 1$.

Let $p = 1$. If $f \in \Lambda_k$, $f \geqslant 0$, $f \neq 0$, the function

$$\frac{\varepsilon}{\int_{S(g_k)} f \cdot g_k \, d\mu} \cdot f$$

belongs to W. Hence, if $h \in M'$, $h \geqslant 0$.

$$\int_{S(g_k)} \varepsilon f \cdot h \, d\mu \leqslant \int_{S(g_k)} f \cdot g_k \, d\mu \quad \forall\, f \in \Lambda_k, \; f \geqslant 0, \; f \neq 0.$$

Then, by the remarks previous to the theorem, $h \leqslant \frac{1}{\varepsilon} \cdot g_k$ and hence $|h| \leqslant \frac{1}{\varepsilon} g_k$ for every $h \in M$, because M can be supposed normal.

If $p > 1$ let $h \in M'$. If $r \in L^p(S(g_k), A, \mu)$, $r \cdot g_k^{-1/p} \in \Lambda_k^p$. Hence $h \cdot g_k^{-1/p} \in (L^p)^\alpha = L^q$, where $1/p + 1/q = 1$, because $(S(g_k), A, \mu)$ has the finite subset property. As for every r of the unit ball of $L^p(S(g_k), A, \mu)$, $\varepsilon \cdot r \cdot g_k^{-1/p} \in W$ and $h \in M' \subset W^0$, we see that $\varepsilon h \cdot g_k^{-1/p}$ is an element of the unit ball of $L^q(S(g_k), A, \mu)$. Then

$$\left(\int_{S(g_k)} |h|^q g_k^{-q/p} \, d\mu \right)^{1/q} < \frac{1}{\varepsilon} \quad \forall \, h \in M'.$$

3) $\Rightarrow$ 1) If $p = 1$, it is clear that

$$M \subset \left\{ f \in \Lambda / \int_E |f g_k| \, d\mu \leqslant \frac{1}{\alpha} \right\}^0.$$

If $p > 1$, by Hölder inequality, if $h \in M$ and $f \in V$, where

$$V = \left\{ f \in \Lambda^p / \left(\int_E |f|^p g_k \, d\mu \right)^{1/p} \leqslant \frac{1}{\alpha} \right\},$$

we have

$$\left| \int_E h f \, d\mu \right| \leqslant \int_{S(g_k)} |h \, g_k^{-1/p} g_k^{1/p} f| \, d\mu \leqslant$$

$$\leqslant \left(\int_{S(g_k)} |h|^q g_k^{-q/p} \, d\mu \right)^{1/q} \left(\int_E |f|^p g_k \, d\mu \right)^{1/p} \leqslant 1.$$

Hence $M \subset V^0$. q.e.d.

Corollary. *If Λ^p (E, A, μ, g_k) is an echelon Köthe space*

$$(\Lambda^p)^\alpha = \bigcup_{k=1}^{\infty} \Gamma'_k \left((\Lambda_k^p)^\alpha \right).$$

Proof. It is immediate because $(\Lambda^p)^\alpha$ is the dual of $[\Lambda^p, \mathcal{T}]$.

Example. In [1], Dieudonné develops a theory of Köthe functions spaces similar to ours. Dieudonné considers a locally compact topological space T, a Radon measure on T and the space Ω_1 of the classes of locally integrable functions on T. Given a subspace $\Lambda_D \subset \Omega_1$, Dieudonné defines its α-dual as

$$\Lambda_D^\alpha = \left\{ f \in \Omega_1 / \int_T |f|\,|h|\,d\mu < \infty \qquad \forall\, h \in \Lambda_D \right\}.$$

In particular, he considers the echelon space

$$\Lambda_D = \left\{ f \in \Omega_1 / P_k(f) = \int_T |f|\,g_k\,d\mu < \infty \qquad \forall\, k \in N \right\}$$

for a given sequence $\{g_k\}_{k=1}^{\infty}$. The theory of Dieudonné has an inconvenience: the topological dual of Λ_D with the topology $\mathfrak{I}_D$ defined by the seminorms $\{P_k,\ k \in N\}$ can be different from Λ_D^α. We give an example which proves this assertion. This also proves the incorrection of an affirmation by Dieudonné. He says on pag. 113 that the topological dual of $[\Lambda_D, \mathfrak{I}_D]$ is Λ_D^α. This is not true:

Let $E = [0, \infty[$ with the measure of Lebesgue. Let g be the function

$$g(x) = \begin{cases} e^{-1/x} & \text{if} \quad x \in]0, 1] \\ 1/e & \text{if} \quad x \in [1, \infty[\cup \{0\}. \end{cases}$$

Let Λ_D be the space of Dieudonné

$$\Lambda_D = \left\{ f \text{ locally integrable on } E / \int_0^\infty |f|\,g\,d\mu < \infty \right\}$$

and its α-dual in the sense of Dieudonné

$$\Lambda_D^\alpha = \left\{ f \text{ locally integrable on } E / \int_0^\infty |f|\,|h|\,d\mu < \infty \qquad \forall\, h \in \Lambda_D \right\}$$

As every $f \in \Lambda_D$ is integrable on $[0,1]$, we have $\chi_{[0,1]} \in \Lambda_D^\alpha$. But $\chi_{[0,1]}$ is not a linear form over $[\Lambda_D, \mathfrak{I}_D]$: let $\{h_n\}_{n=1}^{\infty}$ be the sequence

$$h_n(x) = \begin{cases} n & \text{if} \quad x \in [0, 1/n] \\ 0 & \text{if} \quad x \in [1/n, \infty[. \end{cases}$$

As h_n is locally integrable on E and

$$\int_0^\infty |h_n g| \, d\mu = \int_0^{1/n} n \, e^{-1/x} \, dx \leqslant \frac{n \, e^{-n}}{n} = \frac{1}{e^n}$$

we have $h_n \in \Lambda_D$ and $\lim_n h_n = 0$ in $[\Lambda_D, \mathfrak{I}_D]$. But

$$\lim_{u \to \infty} \int_0^1 h_n \, d\mu = \lim_{n \to \infty} \int_0^{1/n} n \, d\mu = 1$$

and hence $\chi_{[0,1]}$ is not continuous.

We observe that for our space $\Lambda_1([0, \infty[, \mathcal{A}, \mu, g)$, $\chi_{[0,1]} \notin \Lambda_1{}^\alpha$ because, by theorem 6, if $\chi_{[0,1]} \in \Lambda_1{}^\alpha$ there is $C > 0$ so that $\chi_{[0,1]} \leqslant \leqslant C \, e^{-1/x}$ But this is impossible because $\lim_{x \to 0+} g(x) = 0$. q.e.d.

Finally, we apply the theorem 4 to the study of the strong bidual $[\Lambda'', \beta(\Lambda'', \Lambda^\alpha)]$ of an echelon Köthe space $\Lambda(E, \mathcal{A}, \mu, g_k)$. In general $\Lambda \neq \Lambda''$ because it is known that there are non reflexive echelon sequence spaces. Further, in [3] we characterize the reflexive echelon Köthe spaces. Now, we have the following important theorem.

THEOREM 7. *The strong bidual* $[\Lambda'', \beta(\Lambda'', \Lambda^\alpha)]$ *of an echelon Köthe space* $[\Lambda \; (E, \mathcal{A}, \mu, g_k), \mathfrak{I}]$ *is isomorphic and order isomorphic to an echelon Köthe space.*

PROOF. Λ'' as dual of the topological vector lattice $[\Lambda^\alpha, \beta(\Lambda^\alpha, \Lambda)]$ has a canonical order given by the rule:

$$\varphi_1 \in \Lambda'', \varphi_2 \in \Lambda'' \qquad \varphi_1 \leqslant \varphi_2 \Leftrightarrow \langle \varphi_1, f \rangle \leqslant \langle \varphi_2, f \rangle \qquad \forall f \in \Lambda^\alpha, \; f \geqslant 0$$

We use the following notation: let V_k the $\mathfrak{I}$-neighbourhood of zero

$$V_k = \left\{ f \in \Lambda / \int_E |f| \, g_k \, d\mu < \frac{1}{k} \right\}.$$

If $f \in V_k{}^0$, we write

$$||f||_k = \underset{h \in V_k}{SUP} |\langle h, f \rangle| = \underset{h \in V_k}{SUP} \left| \int_E f \, h \, d\mu \right| = \underset{h \in V_k}{SUP} \left| \int_{S(g_k)} f \, h \, d\mu \right|$$

because by theorem 6, every $f \in V_k{}^0$ is zero on $E - S(g_k)$. If $\varphi \in \Lambda''$ we write

$$||\varphi||_k = \sup_{r \in V_k^0} |\langle \varphi, r \rangle| \, .$$

Note that $||\varphi||_k < \infty$ because V_k^0 is $\sigma(\Lambda^\alpha, \Lambda)$ compact and hence $\beta(\Lambda^\alpha, \Lambda)$-bounded and φ is $\beta(\Lambda^\alpha, \Lambda)$ continuous. Clearly $||\varphi||_k \leqslant ||\varphi||_{k+1}$ because $V^0 \subset V^0_{k+1}$.

If

$$W_k = \left\{ f \in \Lambda_k / \int_{S(g_k)} |f| \, g_k \, d\mu < \frac{1}{k} \right\}$$

we have $I'_k(W_k^0) = V_k^0$ (taking the polar W_k^0 in $(\Lambda_k)^\alpha$), because putting $\varepsilon = 2/k$ in the formula (1) of theorem 6, we show that $I_k(V_k)$ is dense in W_k with the topology $\mathcal{T}_k$. Then if $f \in V_k^0$

$$||f||_k = \underset{h \in I_k(V_k)}{SUP} \left| \int_{S(g_k)} h \cdot I_k'^{-1}(f) \, d\mu \right| = \underset{h \in W_k}{SUP} \left| \int_{S(g_k)} h \cdot I_k'^{-1}(f) \, d\mu \right| .$$

As Λ_k is an abstract normed L-space, Λ_k^α is an abstract normed M-space. Then, if $f_1, f_2 \in V_k^0$, we have $||f_1 \vee f_2||_k = SUP(||f_1||_k, ||f_2||_k)$. Hence $(\Lambda_k)''$ is an abstract normed L-space and using the bitransposed I''_k, we show that, if $\varphi \in \Lambda''$

$$||\varphi||_k = \underset{h \in V_k^0}{SUP} |\langle \varphi, h \rangle| = \underset{h \in V_k^0}{SUP} |\langle I''_k(\varphi), I'_k{}^{-1}(h) \rangle| = \underset{h \in W_k^0}{SUP} |\langle I''_k(\varphi), h \rangle| \qquad (1$$

Hence, if $\varphi_1 \geqslant 0$, $\varphi_2 \geqslant 0$, $\varphi_1, \varphi_2 \in \Lambda''$, we have $||\varphi_1 + \varphi_2||_k = ||\varphi_1||_k + + ||\varphi_2||_k$.

It is clear that the topology $\beta(\Lambda'', \Lambda^\alpha)$ is determined by the seminorms $\{||\varphi||_k, \; k \in N, \; \varphi \in \Lambda''\}$. These seminorms are lattice seminorms, because if $\varphi \in \Lambda''$ and $h \in \Lambda_k^\alpha$, $h \geqslant 0$

$$\langle |I''_k(\varphi)|, h \rangle = SUP \; \{\langle I''_k(\varphi), y - z \rangle, \; y \geqslant 0, \; z \geqslant 0, \; y + z = h, \; y, z \in \Lambda_k^\alpha\} =$$

$$= SUP \; \{\langle \varphi, I_k'(y - z) \rangle, \; y \geqslant 0, \; z \geqslant 0, \; z + y = h, \; y, z \in \Lambda_k^\alpha\} =$$

$$= \langle |\varphi|, \; I_k'(h) \rangle = \langle I''(|\varphi|), h \rangle .$$

Hence, if $\varphi_1, \varphi_2 \in \Lambda''$, $|\varphi_1| \leqslant |\varphi_2|$, and $h \in \Lambda_k^\alpha$, $h \geqslant 0$, as $I'_k(h) \geqslant 0$, we have $|I''_k(\varphi_1)| \leqslant |I''_k(\varphi_2)|$. Then $||\varphi_1||_k \leqslant ||\varphi_2||_k$ because by (1),

$||\varphi||_k$ is equal to the norm of $I''_k(\varphi)$ in $(\Lambda)_k''$ multiplied by a fix constant, and this norm is a lattice seminorm. Then, by the theorem 4 with $p = 1$, $[\Lambda'', \beta(\Lambda'', \Lambda^\alpha)]$ is isomorphic and order isomorphic to an echelon Köthe space of order 1. q.e.d.

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