PSEUDO-DIFFERENTIAL OPERATORS ON V-MANIFOLDS AND FOLIATIONS

(Second part)

by

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This paper is a continuation of Part I of the same title which has appeared at the last issue of this journal.

CHAPTER 3

PSEUDO-DIFFERENTIAL OPERATORS ON V-MANIFOLDS

REVIEW OF PSEUDO-DIFFERENTIAL OPERATORS ON $\mathbf{R}^n$

This section summarizes material presented in detail in [4]. Given a multi-index $\alpha = (\alpha_1 \dots \alpha_n)$ we denote by

$$D^\alpha = (-i)^\alpha \quad \frac{\partial^{|\alpha|}}{x_1^{\alpha_1} \dots x_n^{\alpha_n}} .$$

We denote by dx the measure

$$dx = (1/\sqrt{2\pi})^n \, dm$$

where dm means the Lebesgue measure on $\mathbf{R}^n$.

We shall say that a function $p(x, \xi) \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a symbol of order $m \in \mathbf{R}$ if

(a) p is C^∞.

(b) p has compact x-support. (In other words, there exists a compact set $K \subset \mathbf{R}^n$ such that $p(x, \xi) = 0$ for any (x, ξ) such that $x \notin K$.)

(c) For all multi-indices α, β, there is a positive constant $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}.$$

We shall denote by S^m the space of symbols of order m. $S^{m'} \subset S^m$ if $m \geqslant m'$. Denote by D the space of C^∞ functions on $\mathbf{R}^n$ with compact support. Given $p(x, \xi) \in S^m$ we define its associate pseudo-differential operator (abbreviated in the following as p.d.o.) to be the mapping $P : \mathrm{D} \to \mathrm{D}$ given by

$$(3.1)\quad (Pu)(x) = \int_{\mathbf{R}^n} e^{ix.\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi = \int e^{i(x-y).\xi} p(x, \xi)\, .\, u(y)\, dy\, d\xi\,,$$

where $x.\xi$ means $x^1 \xi^1 + \ldots + x^n \xi^n$.

We shall say that two symbols a and b are equivalent and write $a \sim b$ if $a - b \in S^{-\infty} = \bigcap_{m \in \mathbf{R}} S^m$. We shall also say that its associate p.d.o. are equivalent.

We shall need the Kohn-Nirenberg theorem ([4], theorem on page 16) concerning the product of two p.d.o's.

We shall also need the following

Lemma. Let P be a p.d.o. that comes from a symbol $p(x, \xi)$ of order $-\infty$. Let $K \subset \mathbf{R}^n$ be a compact. There is a C^∞ function $k(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$, with compact support, such that, for any $u \in \mathrm{D}$ with support contained in K, one has

$$(Pu)(x) = \int_{\mathbf{R}^n} k(x, y)\, u(y)\, dy\,.$$

Sketch of Proof. Define $k'(x,y) = \int_{\mathbf{R}^n} e^{i(x-y).\xi} p(x, \xi)\, d\xi$. Since $p \in S^{-\infty}$, this integral is convergent and k' is well defined. Let $\Psi \in \mathrm{D}$ such that $\Psi = 1$ on K and put $k(x, y) = k'(x, y).\, \Psi(y)$. Then, k is the function looked for.

Condition (b) in the definition of a symbol simplifies many proofs, although it complicates some other ones. In the case of the existence of parametrices for elliptic differential operators, the proof is more complicate. The exposition in [4] has some lacks in that point. Because of (*b*), an elliptic differential operator is not a p.d.o. We need to introduce some definitions in order to enounce correctly the theorem on page 29 of [4]. Given a differential operator

$$P = \sum_{|\alpha| \leq m} a_\alpha(x)\, D^\alpha$$

we shall say that P is elliptic if its leading order symbol

$$\sigma_L(P) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha$$

only vanishes for $\xi = 0$.

Given an elliptic differential operator P, a p.d.o. Q and a compact K in $\mathbf{R}^n$ we shall say that $PQ \sim I$ ($I =$ identity) (res. $QP \sim I$) over the functions $u \in \mathcal{D}(K)$ if there exist p.d.o's. P' and I' such that $P'Q \sim I'$ (resp. $QP' \sim I'$) over the functions $u \in \mathcal{D}(K)$ and such that $P'Q\,u = PQu$ (resp. $QP'u = QPu$) and $I'u = Iu$ for any $u \in \mathcal{D}(K)$. One can prove the following

Theorem 3.1. *Let P be an elliptic differential operator. Given a compact K in $\mathbf{R}^n$, there exists a p.d.o. Q such that $QP \sim PQ \sim I$ over the functions u with support contained in K.*

A p.d.o. can be extended to the Sobolev spaces as we are going to see. Let $p(x, \xi) \in S^m$. Let P be its associate p.d.o. Let $\|\ \|_s$ be the Sobolev norm on $\mathcal{D}(\mathbf{R}^n)$ (see [4] page 3). Let us denote by $(\mathcal{D}, \|\ \|_s)$ the space $\mathcal{D}(\mathbf{R}^n)$ endowed with the norm $\|\ \|_s$. One has the following

Theorem 3.2. ([4] *pag.* 11-13.) *For any $s \in \mathbf{R}$ the mapping P: $(\mathcal{D}, \|\ \|_s) \to (\mathcal{D}, \|\ \|_{s-m})$ is continuous.*

By virtue of this theorem P can be extended to a mapping $P : H_s(\mathbf{R}^n) \to H_{s-m}(\mathbf{R}^n)$ in a natural way.

We also need the theorem of invariance of p.d.o.'s. under change of coordinates. In order to enounce it we shall give the following definition. Let V and $\widetilde{V}$ be two open sets in $\mathbf{R}^n$. Let $f : \widetilde{V} \to V$ be a C^∞ diffeomorphism. Let P be a p.d.o. acting on the C^∞ functions u with support contained in a compact $K \subset V$ by the expression (3.1), where $p(x, \xi)$ has x-support contained in V. Let $\widetilde{K} = f^{-1}(K)$. Given a C^∞ function $\widetilde{u}$ on $\widetilde{V}$ with support contained in $\widetilde{K}$, we define $(\widetilde{P}\widetilde{u})\,(\widetilde{x}) = (Pu)\,(x)$, where $f(\widetilde{x}) = x$ and $\widetilde{u} = u \circ f$. One has the following

Theorem 3.3. *(Change of coordinates.) In the situation above,* $\widetilde{P}$ *is a p.d.o.*

The proof is sketched in [4].

The notion of a pseudo-differential operator on a V-manifold

Let $(B, \mathcal{A})$ be a V-manifold of dimension n. We shall denote by $\mathcal{E}(B)$ the space of C^∞ complex valued functions on B and by $\mathcal{D}(B)$ the space of C^∞ complex valued functions on B with compact support. We shall denote by $\mathcal{H}$ the family of open sets U in B for which there exists a l.u.s. $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$.

Definition 3.1. We shall say that a linear mapping $P: \mathcal{D}(B) \to \mathcal{E}(B)$ is a p.d.o. on B of order k if for any $x_0 \in B$ there exists $U \in \mathcal{H}$ with $x_0 \in U$ satisfying the following condition:
(C). Given,

(i) a l.u.s. $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ corresponding to U,

(ii) a compact $\widetilde{K} \subset \widetilde{U}$, and

(iii) a C^∞ function $F: \widetilde{U} \to \mathbf{R}$ with compact support,

there exists a symbol of order k, $p(x, \xi)$, on $\widetilde{U}$, such that for any $f \in \mathcal{D}(B)$ with support contained in $K = \varphi(\widetilde{K})$ one has

$$F(x) \cdot (Pf)_{\widetilde{U}}(x) = \int e^{i(x-y)\cdot\xi}\, p(x, \xi)\, f_{\widetilde{U}}(y)\, dy\, d\xi\,.$$

Definition 3.2. Given a p.d.o. P on B, we shall denote by $\mathcal{H}_P$ the subset of $\mathcal{H}$ consisting of those $U \in \mathcal{H}$ satisfying condition (C) of definition 3.1.

Proposition 3.1. *Let P be a p.d.o.* on *B. If $U \in \mathcal{H}_P$ and $U' \in \mathcal{H}$ is contained in U, then $U' \in \mathcal{H}_P$.*

Proof. Let $\{\widetilde{U}', G', \varphi'\}$, $\{\widetilde{U}, G, \varphi\}$ be l.u.s.'s of $\mathcal{A}$ corresponding to U' and U respectively. Let $\widetilde{K}' \subset \widetilde{U}'$ be a compact and F' a C^∞ function $\widetilde{U}' \to \mathbf{R}$ with compact support. Let λ be an injection $\{\widetilde{U}', G', \varphi'\} \to \{\widetilde{U}, G, \varphi\}$ and $\eta: G' \to G$ its associated homomorphism. Put

$\widetilde{K} = \lambda(\widetilde{K}')$ and consider the function F on $\widetilde{U}$ defined by $F = F' \circ \lambda^{-1}$ on $\lambda(\widetilde{U}')$ and $F = 0$ outside. F is a C^∞ function with compact support. By definition 3.3 there is a symbol $p(x, \xi)$ on $\widetilde{U}$ such that, for any $f \in D(B)$ with support contained in $K = \varphi(\widetilde{K}) = \varphi'(\widetilde{K}')$ one has

$$F(x) \cdot (Pf)_{\widetilde{U}}(x) = \int e^{i(x-y).\xi} p(x, \xi) f_{\widetilde{U}}(y) dy\, d\xi .$$

Observe that, if $\Psi : \widetilde{U} \to \mathbf{R}$ is a C^∞ function with compact support contained in $\lambda(\widetilde{U}')$ and such that $\Psi = 1$ on sup F, then,

$$F(x)\, (Pf)_{\widetilde{U}}(x) = \int e^{i(x-y).\xi}\, \Psi(x) \cdot p(x, \xi) f_{\widetilde{U}}(y)\, dy\, d\xi .$$

Hence we can take the symbol $p(x, \xi)$ with the x-support contained in $\lambda(\widetilde{U}')$.

Given $f \in D(B)$ with support contained in K, define a C^∞ function $\tilde{f}$ on $\widetilde{U}$ such that $f = \tilde{f}_{\widetilde{U}'} \circ \lambda^{-1}$ on $\lambda(\widetilde{U}')$ and $\tilde{f} = 0$ outside. Then,

$$f_{\widetilde{U}} = \sum_{i=1}^{m} \tilde{f} \circ \sigma_i^{-1}$$

where $\sigma_1 = I, \sigma_2, \ldots, \sigma_m$ are representatives of each one of the classes of $G/\eta(G')$. Using this decomposition of $f_{\widetilde{U}}$ one has $F(x)\, (Pf)_{\widetilde{U}}(x) = \sum I_i(x)$, $i = 1 \ldots m$, where

$$I_i(x) = \int e^{i(x-y).\xi} p(x, \xi)\, \tilde{f}(\sigma_i^{-1}(y))\, dy\, d\xi .$$

Fix $i \neq 1$. Let $\varphi_i \in D(\widetilde{U})$ with compact support contained in $\sigma_i(\lambda(\widetilde{U}'))$ and such that $\varphi_i = 1$ on $\sigma_i(\widetilde{K})$. Then,

$$I_i(x) = \int e^{i(x-y).\xi} \varphi_i(y)\, p(x, \xi)\, \tilde{f}(\sigma_i^{-1}(y))\, dy\, d\xi .$$

Consider the operator Q_i, on $\widetilde{U}$, acting on the C^∞ functions with compact support contained in $\sigma_i(\widetilde{K})$, defined by

$$(Q_i g)(x) = \int e^{i(x-y).\xi} \varphi_i(y)\, p(x,\xi)\, g(y)\, dy\, d\xi\,.$$

By virtue of technical lemma ([4] pag 17), by setting $r(x, \xi, y) = \varphi_i(y)\, p(x, \xi)$, Q_i is a p.d.o. with symbol $q_i(x, \xi)$ such that

$$q_i(x,\xi) \sim \sum_\alpha \frac{\partial_\xi^\alpha D_y^\alpha(\varphi_i(y)\cdot p(x,\xi))|_{y=x}}{\alpha!} = 0$$

(the equality yields since the x-support of $p(x, \xi)$ and the support of φ_i are disjoint). In other words, Q_i comes from a symbol of order $-\infty$. Then, by virtue of the lemma in the first section of this chapter, there exists a C^∞ function, $k'_i(x, y)$, on $\widetilde{U} \times \widetilde{U}$, with compact support, such that, for any $g \in D(\widetilde{U})$ with support contained in $\sigma_i(\widetilde{K})$, one has

$$(Q_i g)(x) = \int k'_i(x,y)\, g(y)\, dy\,.$$

It is clear that we can take k'_i with the x-support contained in $\lambda(\widetilde{U}')$ and the y-support contained in $\sigma_i(\lambda(\widetilde{U}'))$. Hence,

$$I_i(x) = (Q_i(\widetilde{f} \circ \sigma_i^{-1}))(x) = \int k'_i(x,y)\, \widetilde{f}(\sigma_i^{-1}(y))\, dy =$$

$$= \int k'_i(x, \sigma_i(y))\, \widetilde{f}(y)\, J(\sigma_i)\, dy = \int k_i(x,y)\, \widetilde{f}(y)\, dy\,,$$

where $k_i(x, y) = J(\sigma_i)\, k'_i(x, \sigma_i(y))$ and $J(\sigma_i)$ denotes the Jacobian of σ_i. Observe that k_i has the x-support and the y-support contained in $\lambda(U')$. Put $k = \sum_{i=2}^{m} k_i$. We shall have

$$F(x)\cdot(Pf)_{\widetilde{U}}(x) = \int e^{i(x-y).\xi} p(x,\xi)\, \widetilde{f}(y)\, dy\, d\xi + \int k(x,y)\, \widetilde{f}(y)\, dy\,.$$

Both integrals define p.d.o's that we can think acting on $\lambda(\widetilde{U}')$. Now, theorem 3.3 applied to the open sets $\widetilde{U}'$ and $\lambda(\widetilde{U}')$ and to the diffeomorphism $\lambda : \widetilde{U}' \to \lambda(\widetilde{U}')$ completes the proof.

Corollary. If P_1 and P_2 are p.d.o's. on B, then so is $P_1 + P_2$.

PROOF. By virtue of proposition above, for each $p \in B$ there exists a neighborhood U of p such that $U \in \mathcal{H}_{P_1}$ and $U \in \mathcal{H}_{P_2}$. Obviously $U \in \mathcal{H}_{P_1+P_2}$. Then each $p \in B$ has a neighborhood $U \in \mathcal{H}$ satisfying condition (C) for $P_1 + P_2$.

Example 3.1. Consider the natural V-manifold structure on $\mathbf{R}^n$. Each p.d.o. on $\mathbf{R}^n$ in the sense of the preceding section is a p.d.o. in the sense of definition 3.1. Let $p(x, \xi)$ be a symbol on $\mathbf{R}^n$ satisfying conditions (a) and (c) of the definition of the preceding section, but not necessarily (b). The mapping $P : \mathcal{D}(\mathbf{R}^n) \to \mathcal{E}(\mathbf{R}^n)$ defined by (3.1) constitutes a p.d.o. in the sense of definition 3.1.

Composition of pseudo-differential operators

Let P and Q be two p.d.o's. on B. Let $f \in \mathcal{D}(B)$. Consider the operator $PfQ : \mathcal{D}(B) \to \mathcal{E}(B)$ defined by $(PfQ)(u) = P(f \cdot Q(u))$. If B is compact we can take $f \equiv 1$. Then PfQ is precisely the composition $P \circ Q$. (Observe that $P \circ Q$ is not defined if B is not compact since for $u \in \mathcal{D}(B), Q(u) \in \mathcal{E}(B)$ and P acts only on $\mathcal{D}(B)$).

Theorem 3.4. *PfQ is a p.d.o. on B.*

PROOF. Let $\{U_\alpha\}$ be a locally finite open cover of B such that each $U_\alpha \in \mathcal{H}_P$, $U_\alpha \in \mathcal{H}_Q$. There is a finite number of U_α such that $U_\alpha \cap \sup f \neq \phi$. Let us denote by $U_1 \dots U_k$ these U_α. Let $\{g_\alpha\}$ be a C^∞ partition of unity subordinate to the cover $\{U_\alpha\}$ (th. 2.1). Let $g_1 \dots g_k$ be the g_α's corresponding to $U_1 \dots U_k$. We shall have

$$(PfQ)(u) = P(f \cdot Q(u)) = P((\sum_\alpha g_\alpha f) \cdot Q(u)) .$$

Observe that $g_\alpha f = 0$ if g_α is different from $g_1 \dots g_k$. Hence

$$(PfQ)(u) = \sum_{i=1}^{k} (Pg_i fQ)(u) .$$

Let us prove that each $Pg_i fQ$ is a p.d.o. on B. We want to prove that the U_α's satisfy condition (C) of definition 3.1 for the operator $Pg_i fQ$. If $U_\alpha \cap U_i = \phi$ it is clear that U_α satisfies condition (C). Let U_j be such that $U_j \cap U_i \neq \phi$. Let $\{\tilde{U}_j, G_j, \varphi_j\}$ be a l.u.s. of $\mathcal{A}$ corresponding to U_j. Let $\tilde{K}_j$ be a compact contained in $\tilde{U}_j$. Let F_j

be a C^∞ function $\widetilde{U}_j \to \mathbf{R}$ with compact support. We want to show that $F_j(Pg_i fQu)_{\widetilde{U}_j}$ comes from a symbol for any $u \in \mathcal{D}(B)$ with support contained in K_j. Since P is a p.d.o. and $U_j \in \mathcal{H}_P$ there exists a symbol $p(x, \xi)$ on $\widetilde{U}_j$ such that

$$F_j(x)\,(Pg_i fQu)_{\widetilde{U}_j}(x) = \int e^{i(x-y).\xi}\, p(x, \xi)\,(g_i fQu)_{\widetilde{U}_j}(y)\, dy\, d\xi\,.$$

But $(g_i fQu)_{\widetilde{U}_j} = (g_i f)_{\widetilde{U}_j}(Qu)_{\widetilde{U}_j}$ and since $U_j \in \mathcal{H}_Q$ there exists a symbol $q(x, \xi)$ on U_j such that

$$(g_i f)_{\widetilde{U}_j}(x)\,(Qu)_{\widetilde{U}_j}(x) = \int e^{i(x-y).\xi}\, q(x, \xi) u_{\widetilde{U}_j}(y)\, dy\, d\xi\,.$$

By virtue of the theorem of Kohn-Nirenberg ([4] pag 16) there exists a symbol $s(x, \xi)$ such that

$$F_j(x) \cdot (Pg_i fQu)_{\widetilde{U}_j}(x) = \int e^{i(x-y).\xi}\, s(x, \xi)\, u_{\widetilde{U}_j}\, dy\, d\xi\,.$$

Elliptic differential operators on V-manifolds. Existence of parametrices

Definition 3.3. We shall say that a linear mapping $D: \mathcal{E}(B) \to \mathcal{E}(B)$ is a differential operator (abbreviated in the following as d.o.) of order k if for any $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ there exists a C^∞ d.o. of order k on $\widetilde{U} \subset \mathbf{R}^n$,

$$D_{\widetilde{U}} = \sum_{|\alpha| \leq k} a_\alpha(x)\, D^\alpha$$

in such a way that for any $u \in \mathcal{E}(B)$ one has $(Du)_{\widetilde{U}} = D_{\widetilde{U}}(u_{\widetilde{U}})$.

Remark. A d.o. is always a p.d.o. In particular the identity I is a p.d.o.

Definition 3.4. Given a d.o. D on B, we shall say that D is elliptic if the d.o.'s. $\sum a_\alpha(x)\, D^\alpha$ induced on each $\widetilde{U}$ are elliptic.

Proposition 3.3. *Let* $D : \mathcal{E}(B) \to \mathcal{E}(B)$ *be a d.o. Let* $P : \mathcal{D}(B) \to \mathcal{E}(B)$ *be a p.d.o. The composite operators* $D \circ P$ *and* $P \circ D$ *are defined and are p.d.o's.*

PROOF. If B is compact, the proposition is an immediate consequence of theorem 3.4. Let us give an independent proof for the non-compact case.

Observe first that $D \circ P$ is defined. In fact, P maps $\mathcal{D}(B)$ into $\mathcal{E}(B)$ and D maps $\mathcal{D}(B)$ into $\mathcal{D}(B)$. Hence $D \circ P$ maps $\mathcal{D}(B)$ into $\mathcal{E}(B)$. In an analogous way $P \circ D$ is defined as a mapping from $\mathcal{D}(B)$ into $\mathcal{E}(B)$. Let us prove that $D \circ P$ is a p.d.o. Let $\{U_\alpha\}$ be a locally finite open cover of B with $U_\alpha \in \mathcal{H}_P$. We want to show these U_α's satisfy condition (C) of definition 3.1 for the operator $D \circ P$. In fact, choose a $\{\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha\} \in \mathcal{A}$ corresponding to U_α. Choose a compact $\widetilde{K}_\alpha \subset \widetilde{U}_\alpha$ and a C^∞ function $F_\alpha : \widetilde{U}_\alpha \to \mathbf{R}$ with compact support. Let Φ_α be a C^∞ function on $\widetilde{U}_\alpha$ with compact support such that $\Phi_\alpha \equiv 1$ on sup F_α. For any $u \in \mathcal{D}(B)$ with support contained in $K_\alpha = \varphi_\alpha(\widetilde{K}_\alpha)$ we shall have

$$F_\alpha \cdot (DPu)_{\widetilde{U}_\alpha} = F_\alpha \cdot D_{\widetilde{U}_\alpha} (Pu)_{\widetilde{U}_\alpha} = F_\alpha \cdot D_{\widetilde{U}_\alpha} (\Phi_\alpha (Pu)_{\widetilde{U}_\alpha}) \,.$$

Since $U_\alpha \in \mathcal{H}_P$, $\Phi_\alpha \, (Pu)_{\widetilde{U}_\alpha}$ comes from a symbol on $\widetilde{U}_\alpha$, so does $F_\alpha \cdot D_{\widetilde{U}_\alpha} (\Phi_\alpha (Pu))_{\widetilde{U}_\alpha}$, by virtue of the theorem of Kohn-Nirenberg. Let us prove that the U_α's satisfy also condition (C) of definition 3.1 for the operator $P \circ D$. Choose $\{\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha\} \in \mathcal{A}$ corresponding to U_α and the compact $\widetilde{K}_\alpha \subset \widetilde{U}_\alpha$ as well as the function F_α as above. For any C^∞ function u on B with support contained in $K_\alpha = \varphi_\alpha(\widetilde{K}_\alpha)$ we shall have

$$F_\alpha(x) \, (PDu)_{\widetilde{U}_\alpha}(x) = \int e^{i(x-y).\xi} \, p(x, \xi) \, (Du)_{\widetilde{U}_\alpha}(y) \, dy d\xi$$

since $U_\alpha \in \mathcal{H}_P$. By virtue of the theorem of Kohn-Nirenberg, $F_\alpha (PDu)_{\widetilde{U}_\alpha}$ comes from a symbol on $\widetilde{U}_\alpha$.

Proposition 3.4. *Let* D: $\mathcal{E}(B) \to \mathcal{E}(B)$ *be an elliptic d.o. Let* $U \in \mathcal{H}$. *Let* $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ *corresponding to* U. *Let* $\widetilde{K}$ *be a G-invariant compact in* $\widetilde{U}$. *There exists a p.d.o. (in the sense of the first section of this chapter)* $Q_{\widetilde{U}} : \mathcal{D}(\widetilde{U}) \to \mathcal{D}(\widetilde{U})$ *such that* *(i)* $Q_{\widetilde{U}} \circ D_{\widetilde{U}} \sim D_{\widetilde{U}} \circ Q_{\widetilde{U}} \sim I$ *over the G-invariant functions with support contained in* $\widetilde{K}$ *(These equivalences in the sense of the first section of this chapter).*

(ii) $Q_{\widetilde{U}}$ *maps G-invariant functions into G-invariant functions.*

PROOF. By virtue of theorem 3.1, there exists $Q : \mathrm{D}(\widetilde{U}) \to \mathrm{D}(\widetilde{U})$ satisfying condition (i). One can see that there exist C^∞ functions $k_1(x, y)$, $k_2(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ with compact x and y-supports contained in $\widetilde{U}$ such that for any C^∞ function h on $\widetilde{U}$ with compact support contained in $\widetilde{K}$ one has

$$(D_{\widetilde{U}} Q h)(x) = h(x) + \int k_1(x, y)\, h(y)\, dy$$

$$(Q D_{\widetilde{U}} h)(x) = h(x) + \int k_2(x, y)\, h(y)\, dy\,.$$

Given $\sigma \in G$ let Q_σ be the operator $\mathrm{D}(\widetilde{U}) \to \mathrm{D}(\widetilde{U})$ defined by $(Q_\sigma(u))(x) = Q(u \circ \sigma^{-1})(\sigma(x))$. Theorem 3.3 asserts that Q_σ is a p. d.o. in the sense of the first section of this chapter. Let $n(G)$ be the order of G. Define

$$Q_{\widetilde{U}} = \frac{1}{n(G)} \sum_{\sigma \in G} Q_\sigma\,.$$

It is clear that this $Q_{\widetilde{U}}$ satisfies condition (ii). We want to show that it also satisfies condition (i). First observe that if $h : \widetilde{U} \to \mathbf{R}$ (or $\mathbf{C}$) is a G-invariant C^∞ function with compact support, so is $D_{\widetilde{U}}(h)$. In fact, it suffices to observe that there exists $u \in \mathrm{D}(B)$ such $h = u_{\widetilde{U}}$. Then, one has

$$D_{\widetilde{U}}(h) \circ \sigma = D_{\widetilde{U}}(u_{\widetilde{U}}) \circ \sigma = (Du)_{\widetilde{U}} \circ \sigma = (Du)_{\widetilde{U}} = D_{\widetilde{U}}(u_{\widetilde{U}}) = D_{\widetilde{U}}(h).$$

Let h be a G-invariant C^∞ function $\widetilde{U} \to \mathbf{C}$ with support contained in $\widetilde{K}$. We shall have

$$Q_\sigma(D_{\widetilde{U}}(h))(x) = Q(D_{\widetilde{U}}(h) \circ \sigma^{-1})(\sigma(x)) = Q(D_{\widetilde{U}}(h))(\sigma(x)) =$$

$$= h(\sigma(x)) + \int k_2(\sigma(x), y)\, h(y)\, dy = h(x) + \int k_2(\sigma(x), y)\, h(y)\, dy\,.$$

Hence

$$Q_{\widetilde{U}}(D_{\widetilde{U}}(h))(x) = \frac{1}{n(G)}\left(\sum_{\sigma\in G} h(x) + \sum_{\sigma\in G}\int k_2(\sigma(x), y)\, h(y)\, dy\right) =$$

$$= h(x) + \int \overline{k}_2(x, y)\, h(y)\, dy\,,$$

where $\overline{k}_2(x, y) = \frac{1}{n(G)} \sum_{\sigma\in G} k_2(\sigma(x), y)$. Hence one has

$Q_{\widetilde{U}} \circ D_{\widetilde{U}} \sim I$ over the G-invariant C^∞ functions with support contained in $\widetilde{K}$.

In order to prove $D_{\widetilde{U}} \circ Q_{\widetilde{U}} \sim I$ we shall need the following

Lemma. For any C^∞ function $h: \widetilde{U} \to \mathbf{C}$ with compact support and any $\sigma \in G$ we have $D_{\widetilde{U}}(h \circ \sigma) = (D_{\widetilde{U}} h) \circ \sigma$.

Proof of lemma. It suffices to prove the equality for the points $x \in \widetilde{U}$ whose isotropy group $G_x = \{I\}$ since, if the equality holds for these points, it holds, by continuity, for any point.

Let x_0 be a point of $\widetilde{U}$ such that $G_{x_0} = \{I\}$. Let $\sigma \in G$. We want to show that $D_{\widetilde{U}}(h \circ \sigma)(x_0) = (D_{\widetilde{U}} h)(\sigma(x_0))$. Let V_{x_0} be a small open neighborhood of x_0 such that for any $\sigma_1, \sigma_2 \in G$ with $\sigma_1 \neq \sigma_2$ one has $\sigma_1(V_{x_0}) \cap \sigma_2(V_{x_0}) = \phi$. (It is possible because G is finite and $G_{x_0} = \{I\}$). Let g be a C^∞ function on $\widetilde{U}$ with compact support contained in V_{x_0}, equal to $h \circ \sigma$ in a small neighborhood of x_0 contained in V_{x_0}. Let f be the function on $\widetilde{U}$ equal to $g \circ \tau^{-1}$ on each $\tau(V_{x_0})$, $\tau \in G$, and equal to zero outside the union of $\tau(V_{x_0})$ for any τ. f is a G-invariant C^∞ function with compact support, hence $D_{\widetilde{U}} f$ is G-invariant. In other words, $D_{\widetilde{U}} f = (D_{\widetilde{U}} f) \circ \tau$ for any $\tau \in G$. Since $h \circ \sigma$ is equal to g in a small neighborhood of x_0, one has $D_{\widetilde{U}}(h \circ \sigma)(x_0) = D_{\widetilde{U}}(g)(x_0) \cdot f \circ \sigma$ is equal to g on V_{x_0} since if $\sigma(x) \in \sigma(V_{x_0})$ one has $f(\sigma(x)) = g(\sigma^{-1}\sigma(x)) = g(x)$. Hence $D_{\widetilde{U}}(g)(x_0) = D_{\widetilde{U}}(f \circ \sigma)(x_0)$. Since f is G-invariant with compact support, so is $D_{\widetilde{U}} f$. Hence $D_{\widetilde{U}}(f \circ \sigma)(x_0) = (D_{\widetilde{U}} f)(x_0) = (D_{\widetilde{U}} f)(\sigma(x_0))$. We know that $f(\sigma(x)) = g(x) = h(\sigma(x))$ for any x in a small neighborhood of x_0. Hence f is equal to h in a small neighborhood of $\sigma(x_0)$. Hence $(D_{\widetilde{U}} f)(\sigma(x_0)) = D_{\widetilde{U}}(h)(\sigma(x_0))$. We have then proven that $D_{\widetilde{U}}(h \circ \sigma)(x_0) = (D_{\widetilde{U}} h)(\sigma(x_0))$.

End of the proof of proposition. Let us prove that $D_{\widetilde{U}} \circ Q_{\widetilde{U}} \sim I$ over the G-invariant C^∞ functions $h : \widetilde{U} \to \mathbf{C}$ with support contained in $\widetilde{K}$. We shall have

$$D_{\widetilde{U}}(Q_{\widetilde{U}} h) = \frac{1}{n(G)} \sum_{\sigma \in G} D_{\widetilde{U}}(Q h) .$$

Since h is G-invariant one has $(Q_\sigma h)\,(x) = Q(h)\ (\sigma(x))$. Hence

$$D_{\widetilde{U}}(Q_{\widetilde{U}} h) = \frac{1}{n(G)} \sum_\sigma D_{\widetilde{U}}(Q(h) \circ \sigma) = \text{(by virtue of lemma)} =$$

$$= \frac{1}{n(G)} \sum_\sigma D_{\widetilde{U}}(Q(h)) \circ \sigma = \frac{1}{n(G)} \sum_\sigma \{h(\sigma(x)) +$$

$$+ \int k_1(\sigma(x), y)\, h(y)\, dy\} = h(x) + \int \bar{k}_1(x, y)\, h(y)\, dy ,$$

where

$$\bar{k}_1(x, y) = \frac{1}{n(G)} \sum_\sigma k_1(\sigma(x), y) .$$

Hence $D_{\widetilde{U}} \circ Q_{\widetilde{U}} \sim I$ over these h' s.

Definition 3.5. Let P and P' be two p.d.o's. on B. We shall say that P is equivalent to $P'(P \sim P')$ if $P - P'$ is a p.d.o. of order $-\infty$ (that is, of order k for any k).

Theorem 3.5. *Let $D : \mathcal{E}(B) \to \mathcal{E}(B)$ be an elliptic d.o. There exists a p.d.o. $Q : \mathcal{D}(B) \to \mathcal{E}(B)$ such that $Q \circ D \sim D \circ Q \sim I$. The operator Q is called parametrix of D. (Observe that $Q \circ D$ and $D \circ Q$ are p.d.o's. by virtue of proposition* 3.3.)

The proof of this theorem is based on proposition 3.4 and on some ideas taken from [1].

Proof. Let $\{U_i\}$ be a locally finite open cover of B with $U_i \in \mathcal{H}$. Let $\{f_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Let g_i be a C^∞ function on B with compact support contained in U_i, equal to 1 on sup f_i. For any $\{\widetilde{U}_i, G_i, \varphi_i\} \in \mathcal{A}$ corresponding to U_i let $Q_{\widetilde{U}_i}$ be the operator satisfying conditions (i) and (ii) of proposition 3.4 for

$D_{\widetilde{U}_i}$. and the compact $\widetilde{K}_i = \varphi_i^{-1}$ (sup g_i). Consider on each U_i the induced V-manifold structure. Define on each U_i an operator $Q_i : D(U_i) \to D(U_i)$ in the following way. Let $U \in \mathcal{H}$, $U \subset U_i$. Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ corresponding to U. Let $f \in D(U_i)$. Take an injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}_i, G_i, \varphi_i\}$. We define $(Q_i f)_{\widetilde{U}}$ to be equal to $Q_{\widetilde{U}_i}(f_{\widetilde{U}_i})$ o λ. Since $Q_{\widetilde{U}_i}$ satisfies condition (ii) of proposition 3.4, $(Q_i f)_{\widetilde{U}}$ depends neither on the choice of $\{\widetilde{U}_i, G_i, \varphi_i\}$ nor on the choice of λ. By vitue of proposition 2.4 we can extend each element of $D(U_i)$ to a C^∞ function on B. Identify in that way $D(U_i)$ to a subspace of $D(B)$. The operator $f_i Q_i g_i$ that assigns $f_i Q_i (g_i f)$ to each $f \in D(B)$ is (by virtue of the inclusion $D(U_i) \subset D(B)$) an operator from $D(B)$ into $D(B)$. The operator $Q = \sum f_i Q_i g_i$ is also a well defined operator $D(B) \to D(B)$ since there is only a finite number of U_i such that $U_i \cap \sup f \neq \phi$, for a given $f \in D(B)$. Let us prove that Q is a p.d.o. Given $p \in B$ let I_p be the set of indices j such that $p \in U_j$. I_p is a finite set. Choose an open neighborhood U of p, $U \in \mathcal{H}$, contained in U_i for any $i \in I_p$. We want to show that U satisfies condition (C) of definition 3.1 for the operator Q. Let $\{\widetilde{U}, G, \varphi\}$ be a l.u.s. corresponding to U. Let $\widetilde{K}$ be a compact contained in $\widetilde{U}$. Let $F : \widetilde{U} \to \mathbf{R}$ be a C^∞ function with compact support. Let $\{\widetilde{U}_i, G_i, \varphi_i\}$ $\mathcal{A} \in$ corresponding to U_i for any $i \in I_p$. Given $f \in D(B)$ with support contained in $K = \varphi(\widetilde{K})$, then $f_i Q_i(g_i f) = 0$ if $i \notin I_p$. If $i \in I_p$ we shall have

$$(f_i Q_i(g_i f))_{\widetilde{U}} = (f_i)_{\widetilde{U}}(Q_i(g_i f))_{\widetilde{U}} = (f_i)_{\widetilde{U}}(Q_{\widetilde{U}_i}(g_i f)_{\widetilde{U}_i} \circ \lambda_i)$$

But there exist a symbol p_i on each $\widetilde{U}_i$ such that

$$(Q_{\widetilde{U}_i}(g_i f)_{\widetilde{U}_i})\,(\lambda_i(x)) = \int e^{i(\lambda_i(x)-y).\xi}\, p(\lambda_i(x), \xi)\,(g_i f)_{\widetilde{U}_i}(y)\, dy\, d\xi\,.$$

Observe that $(g_i f)_{\widetilde{U}_i}$ has support contained in $\lambda_i(K)$. By a reasoning similar to that in the proof of Proposition 3.1 one can prove that there is a symbol q_i on $\widetilde{U}$ such that

$$(Q_{\widetilde{U}_i}(g_i f)_{\widetilde{U}_i})\,(\lambda_i(x)) = \int e^{i(x-z).\eta}\, q_i(x, \eta)\,(g_i f)_{\widetilde{U}}\,(z)\, dz\, d\eta\,.$$

By virtue of technical lemma ([3] pag. 17), by setting $r(x, \eta, z) = q_i(x, \eta)\,(g_i)_{\widetilde{U}}\,(z)$, there exists a symbol q'_i such that

$$(Q_{\widetilde{U}_i}(g_i f)_{\widetilde{U}_i})\,(\lambda_i(x)) = \int e^{i(x-z)\cdot\eta}\, q'_i(x,\eta)\, f_{\widetilde{U}}(z)\, dz\, d\eta\,.$$

It is clear that

$$\sum_{i\in I_p} F(x)\,(f_i)_{\widetilde{U}}(x)\,(\quad Q_i(g_i f))_{\widetilde{U}}(x) =$$

$$= \sum_{i\in I_p} F(x)\,(f_i)_{\widetilde{U}}(x)\,(Q_{\widetilde{U}_i}(g_i f)_{\widetilde{U}_i})\,(\lambda_i(x))$$

has the required expression.

Observe that our calculations also prove that $(Q_i)_{\widetilde{U}}$ is a p.d.o. on $\widetilde{U}$ in the sense of the first section of this chapter.

In order to prove that $Q \circ D \sim D \circ Q \sim I$ we need the following.

Lemma. Let $U \in \mathcal{H}$, $U \subset U_i \cap U_j$. *Let* $K_i = sup\, g_i$ *and* $K_j = sup\, g_j$. *Let* $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ *corresponding to* U. *Over the functions* $f \in \mathcal{D}(B)$ *with support contained in* $K_i \cap K_j \cap U$, *the operator* $(Q_i)_{\widetilde{U}} : f_{\widetilde{U}} \to (Q_i)_{\widetilde{U}} f_{\widetilde{U}}$ *and the operator* $(Q_j)_{\widetilde{U}}$, *defined in an analogous way, are equivalent in the sense of the first section of this chapter.*

Proof of lemma. We have $D_{\widetilde{U}_i} \circ Q_{\widetilde{U}_i} \sim I_{\widetilde{U}_i}$ over the functions $f_{\widetilde{U}_i}$ such that sup $f \subset K_i$, where $I_{\widetilde{U}_i}$ means the identity on $\widetilde{U}_i$. We also have $Q_{\widetilde{U}_j} \circ D_{\widetilde{U}_j} \sim I_{\widetilde{U}_j}$ over the functions $f_{\widetilde{U}_j}$ such that sup $f \subset K_j$. Since we have assumed that sup $f \subset K_i \cap K_j$, the two preceding relations hold. From the first one we deduce $D_{\widetilde{U}} \circ (Q_i)_{\widetilde{U}} \sim I_{\widetilde{U}}$. From the second one we deduce $(Q_j)_{\widetilde{U}} \circ D_{\widetilde{U}} \sim I_{\widetilde{U}}$ (In the sense of the first section of this chapter). Hence

$$(Q_i)_{\widetilde{U}} = I_{\widetilde{U}}(Q_i)_{\widetilde{U}} \sim (Q_j)_{\widetilde{U}} \circ D_{\widetilde{U}} \circ (Q_i)_{\widetilde{U}} \sim (Q_j)_{\widetilde{U}} \circ I_{\widetilde{U}} = (Q_j)_{\widetilde{U}}\,.$$

End of the proof of theorem. Let us prove $D \circ Q \sim I$. For each $p \in B$ take an open neighborhood U of p, $U \in \mathcal{H}$ such that $U \subset U_i$ for any $i \in I_p$. Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ corresponding to U. Take a compact $\widetilde{K} \subset \widetilde{U}$ and a C^∞ function $F : \widetilde{U} \to \mathbf{R}$ with compact support. We have to prove that $F \cdot (D \circ Q)_{\widetilde{U}} \sim F \cdot I_{\widetilde{U}}$ over the $f_{\widetilde{U}}$'s such that $f \in \mathcal{D}(B)$ with support contained in $\widetilde{K}$. We have

$$F \cdot (D \circ Q)_{\widetilde{U}}(f_{\widetilde{U}}) = \sum_{i\in I_p} F \cdot D_{\widetilde{U}}(f_i Q_i g_i)_{\widetilde{U}}(f_{\widetilde{U}}) =$$

$$= \sum_{i \in I_p} F \cdot D_{\widetilde{U}} ((f_i)_{\widetilde{U}} (Q_i)_{\widetilde{U}} ((g_i)_{\widetilde{U}} f_{\widetilde{U}}) =$$

$$\sum_{j \in I_p} \sum_{i \in I_p} F \cdot D_{\widetilde{U}} ((f_i)_{\widetilde{U}} (Q_i)_{\widetilde{U}} (g_i f_j)_{\widetilde{U}} f_{\widetilde{U}})$$

(since $\sum (f_j)_{\widetilde{U}} = 1$). But $g_i f_j f$ has support contained in $K_i \cap K_j$. By virtue of lemma, we have

$$F \cdot (D \circ Q)_{\widetilde{U}} (f_{\widetilde{U}}) \sim \sum_{j \in I_p} \sum_{i \in I_p} F \cdot D_{\widetilde{U}}((f_i)_{\widetilde{U}} (Q_j)_{\widetilde{U}} (g_i f_j f)_{\widetilde{U}}) =$$

$$= \sum_{i,j} F \cdot D_{\widetilde{U}}((f_i (Q_j)_{\widetilde{U}} (f_j f)_{\widetilde{U}}) +$$

$$+ \sum_{i,j} F \cdot D_{\widetilde{U}}((f_i)_{\widetilde{U}} (Q_j)_{\widetilde{U}} ((g_i)_{\widetilde{U}} - 1) (f_j f)_{\widetilde{U}})$$

Since f_i and $g_i - 1$ have disjoint supports, it is easy to prove that $(f_i)_{\widetilde{U}} (Q_j)_{\widetilde{U}} ((g_i)_{\widetilde{U}} - 1)$ is a p.d.o. of order $-\infty$ on U. Hence,

$$F \cdot (D \circ Q)_{\widetilde{U}} (f_{\widetilde{U}}) \sim \sum_{i,j} F \cdot D_{\widetilde{U}}((f_i)_{\widetilde{U}} (Q_j)_{\widetilde{U}} (f_j f)_U) =$$

$$= \sum_j F \cdot D_{\widetilde{U}}((Q_j)_{\widetilde{U}} (f_j f)_{\widetilde{U}}) .$$

From $D_{\widetilde{U}_j} \circ Q_{\widetilde{U}_j} \sim I_{\widetilde{U}_j}$ over the $f_{\widetilde{U}_j}$. such that sup $f \subset K_j$, we deduce $D_{\widetilde{U}} \circ (Q_j)_{\widetilde{U}} \sim I_{\widetilde{U}}$. Hence, $F \cdot (D \circ Q)_{\widetilde{U}} (f_{\widetilde{U}}) \sim \sum_j F \cdot (f_j f)_{\widetilde{U}} = F \cdot f_{\widetilde{U}}$.

The equivalence $Q \circ D \sim I$ can be proven in an analogous way.

PSEUDO-DIFFERENTIAL OPERATORS ON V-VECTOR BUNDLES.

Let $\pi : E \to B$ be a V-vector bundle with fibre $\mathbf{C}^m$ on a V- manifold B (definition 1.8). Let $\mathcal{A}$ and $\mathcal{A}^*$ be the defining families of B and E respectively, satisfying the conditions of definition 1.8. Let $\mathcal{H}$ be the family of open sets in B for which there exists a l.u.s. $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. We shall denote by $\mathcal{E}(E)$ the space of C^∞ cross sections of E (definition 1.9) and by $\mathcal{D}(E)$ the space of C^∞ cross sections of E with compact support.

Given a cross section $s : B \to E$, $s = \{s_{\widetilde{U}}\}$, we know that each $s_{\widetilde{U}}$ is a section of the trivial bundle $\widetilde{U} \times \mathbf{C}^m \to \widetilde{U}$. We shall denote by $s_{\widetilde{U}}^A$ the composition $\pi_A \circ s_{\widetilde{U}}$, where π_A is the mapping $\widetilde{U} \times \mathbf{C}^m \to \mathbf{C}$ that assigns to each $(x, c) \in \widetilde{U} \times \mathbf{C}^m$ the A-coordinate of c.

Definition 3.6. Given a linear mapping $P : \mathcal{D}(E) \to \mathcal{E}(E)$, we shall say that P is a p.d.o. of order k if for any $x_0 \in B$ there exists $U \in \mathcal{H}$ with $x_0 \in U$ such that the following condition (C) holds: Given

(i) A l.u.s. $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ corresponding to U,

(ii) A compact $\widetilde{K} \subset \widetilde{U}$,

(iii) A C^∞ function $F : \widetilde{U} \to \mathbf{R}$ with compact support,

there exists a (mxm)-matrix of symbols of order k on $\widetilde{U}$, $p_B{}^A(x, \xi)$, such that for any $s \in \mathcal{D}(E)$ with support contained in $K = \varphi(\widetilde{K})$, one has

$$F(x) \cdot (Ps)_{\widetilde{U}}{}^B(x) = \sum_A \int e^{i(x-y).\xi} p_A{}^B(x, \xi)\, s_{\widetilde{U}}{}^A(y)\, dy\, d\xi .$$

Definition 3.7. Given a linear mapping $D : \mathcal{E}(E) \to \mathcal{E}(E)$, we shall say that D is a d.o. of order k if for any $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ there exists a (mxm)-matrix of C^∞ d.o's. of order k on $\widetilde{U} \subset \mathbf{R}^n$,

$$(D_A{}^B)_{\widetilde{U}} = \sum_{|\alpha| \leq k} (a_A{}^B)_\alpha (x)\, D^\alpha ,$$

such that for any $s \in \mathcal{E}(E)$ one has

$$(Ds)_{\widetilde{U}}{}^B = \sum_A (D_A{}^B)_{\widetilde{U}} (s_{\widetilde{U}}{}^A) .$$

Definitition 3.8. Let D: $\mathcal{E}(E) \to \mathcal{E}(E)$ be a d.o. For each $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ consider the matrix

$$(D_A{}^B)_{\widetilde{U}} = \sum_{|\alpha| \leq k} (a_A{}^B) (x)\, D^\alpha .$$

Set

$$q_{\widetilde{U}}(x, \xi)_A{}^B = \sum_{|\alpha| = k} (a_A{}^B) (x)\, \xi^\alpha .$$

We shall say that D is elliptic if for any $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ the matrix $(q_{\widetilde{U}}(x, \xi)_A{}^B)$ is only singular for $\xi = 0$.

We can now adapt the definitions and proofs of the preceding sections by substituting each symbol by a matrix of symbols. One proves in this way that an elliptic operator (acting on $\mathcal{E}(E)$) has a parametrix.

Sobolev spaces and the decomposition theorem of self-adjoint elliptic operators acting on a V-vector bundle

Let B an oriented compact Riemannian V-manifold. Let $\pi : E \to B$ be a V-vector bundle with fibre $\mathbf{C}^m$, endowed with a Hermitian metric h. We define the following Hermitian product on $\mathcal{D}(E)$:

$$\langle s_1, s_2 \rangle = \int_B h(s_1, s_2)\, \eta\,,$$

where η is the volume element corresponding to the Riemannian metric. Let $D : \mathcal{D}(E) \to \mathcal{D}(E)$ be a self-adjoint elliptic differential operator. (Self-adjoint means that $\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle$ for any s_1, s_2.)

Theorem 3.6. *$\mathcal{D}(E) = \ker D \oplus \operatorname{Im} D$ (othogonal direct sum). Moreover, $\ker D$ has finite dimension.*

Proof. We have now almost all the elements in order to adapt the proof in [4, pag. 43] to our situation. The main ingredient of that proof is the existence of parametrices. The unique element that we have not yet in order to copy the proof in [4] is the definition of Sobolev spaces for V-manifolds.

Consider the V-vector bundle $\pi : E \to B$. Let $\mathcal{A}$ and $\mathcal{A}^*$ be the defining families of B and E respectively, satisfying the conditions of definition 3.8. Let $\mathcal{H}$ be the family of open sets U of B for which there is $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. Choose a finite cover $\{U_i\}$ of B with $U_i \in \mathcal{H}$ and a C^∞ partition of unity subordinate to this cover. Choose a $\{\widetilde{U}_i, G_i, \varphi_i\} \in \mathcal{A}$ for each U_i. For any $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ let $\{U^*, G^*, \varphi^*\}$ be its corresponding element in $\mathcal{A}^*$. Given a section u on E, $u = \{u_{\widetilde{U}}\}$, define

$$E(u)_{\widetilde{U}} = \sqrt{\sum u_{\widetilde{U}}{}^A\, \overline{u_{\widetilde{U}}{}^A}}\,,$$

where the $u_{\widetilde{U}}{}^A$ have been defined in the preceding section. For any $s \in \mathbf{R}$ we define

$$||u||_s^2 = \sum_i \frac{1}{n(G_i)} ||E((f_i u)\widetilde{v}_i)||_s^2 .$$

One can prove that if we employ another finite cover $\{U_i\}$ and another partition of unity $\{f_i\}$ in the definition of $|| \ ||_s$, the two norms obtained are equivalent. We define the Sobolev space $H_s(E)$ to be the completion of $\mathcal{D}(E)$ in the norm $|| \ ||_s$. One can prove that the injection $H_t \hookrightarrow H_s$ for $t > s$ is compact (Reillich's lemma). By virtue of theorem 3.2, if $P: \mathcal{D}(E) \to \mathcal{D}(E)$ is a p.d.o. of order k, P can be extended to $P: H_s(E) \to H_{s-k}(E)$ for any $s \in \mathbf{R}$.

With these ingredients we can now literally copy the proof in [4 pag 43].

CHAPTER 4

APPLICATIONS TO FOLIATE MANIFOLDS

HAUSDORFF FOLIATIONS

Example 4.1. Let D be the open unit ball in $\mathbf{R}^n$. Let G be a finite subgroup of $O(n)$. Let L be a compact manifold. Suppose that there is a free C^r action ($r \geqslant 1$) of G on L, on the right. Define an action of G on $L x D$ by $g(p, x) = (pg^{-1}, gx)$. We shall denote by $L x_G D$ the quotient of $L x D$ by this action, endowed with the quotient topology. If $(p, x) \in L x D$, we shall denote by $\widetilde{(p, x)}$ its class in $L x_G D$. $L x_G D$ is a differentiable manifold in a natural way. To see this, consider local charts V in L such that $Vg \cap V = \phi$ if $g \neq I$, $g \in G$. The mapping

$$V x D \to V x_G D$$

$$(p, x) \to \widetilde{(p, x)}$$

is then a homeomorphism. In fact, if $\widetilde{(p, x)} = \widetilde{(p', x')}$ with $p, p' \in V$ then, there is $g \in G$ such that $p' = pg^{-1}$ and $x' = gx$. But g must be the identity since p and pg^{-1} belong to V. Hence, $p = p'$ and $x = x'$. We take $Vx D$ as a local chart corresponding to the open set $Vx_G D$. We get in this way a natural differentiable manifold structure in $L x_G D$. We foliate now $L x D$ with leaves of the form $L x$ {point}.

This foliation is preserved by the action of G. So, we have a foliation induced on $L x_G D$.

This example is interesting by virtue of the following theorem due to Reeb, Ehresmann, Haefliger and Epstein [3] (See also Reinhart [9] for colsed metric foliations).

Theorem 4.1. *Let M be a manifold of dimension $n + m$, endowed with a C^r foliation $\mathcal{F}$ $(r \geqslant 1)$ of codimension n. Suppose that the quotient space $M/\mathcal{F}$ obtained by identifying each leaf to a point, with its quotient topology, is Hausdorff. Suppose moreover that all the leaves are compact. (Such foliations will be called «compact Hausdorff foliations» from now on). Then, there is a «generic» leaf L with the property that there is an open dense subset of M where the leaves are all diffeomorphic to L. Moreover, given a leaf L_0, there is*

(a) A finite subgroup G of $O(n)$,

(b) A free C^r action of G on L, on the right,

(c) An open neighborhood V of L_0,

(d) A C^r diffeomorphism $\Phi : L x_G D \to V$ which preserves leaves if one takes the foliation on $L x_G D$ introduced in the above example.

From theorem 4.1 we shall deduce the following.

Theorem 4.2. *Let M be a manifold of dimension $n + m$ endowed with a compact Hausdorff foliation $\mathcal{F}$ of codimension n. Let $B = M/\mathcal{F}$ be the quotient space endowed with its quotient topology. There is a V-manifold structure of dimension n on B, in a natural way.*

PROOF. Denote by φ the canonical projection $M \to B$. Given a leaf L_0 of M we take a finite subgroup G of $O(n)$, a free C^r action α of G on L, an open neighborhood V of L_0, and a C^r diffeomorphism $\Phi : L x_G D \to V$ such that the conditions of theorem 4.1 are fulfilled. Let $\mathfrak{a}$ be the family of such collections (V, G, α, Φ) corresponding to all leaves L_0 of M. Let D' be the open ball in $\mathbf{R}^n$ centered at the origin, of radius $1/2$. The canonical injection $D' \hookrightarrow D$ gives rise to an injection $L x_G D' \to L x_G D$. Let $\mathcal{H}$ be the family of those open sets in B of the form $\varphi\Phi(L x_G D')$ for any $(V, G, \alpha, \Phi) \in \mathfrak{a}$. Given $p \in L$ we shall denote by s_p the mapping $D \to L x_G D$ given by $s_p(x) = \widetilde{(p, x)}$. It is easy to prove that for any $U \in \mathcal{H}$ of the form $\varphi\Phi(L x_G D')$ and any $p \in L$ the collection $\{D', G, \varphi\Phi s_p\}$ is a l.u.s. corresponding to U. Let $\mathcal{A}$ be the family of such l.u.s's. To see that $\mathcal{A}$ defines a V-manifold structure on B we only have to prove the following

Proposition 4.1. *Let* $U, U' \in \mathcal{H}$. *Let* $\{D', G, \varphi\Phi s_p\}$ *and* $\{D', G', \varphi\Psi s_{p'}\}$ *be two l.u.s's of* $\mathcal{A}$ *corresponding to* U *and* U' *respectively. If* $U \subset U'$, *there exists a diffeomorphism* λ *from* D' *onto an open set in* D' *such that* $\varphi\Phi s_p = \varphi\Psi s_{p'}\lambda$.

PROOF. Take the mappings $Lx_G D' \xrightarrow{\Phi} \varphi^{-1}(U) \subset \varphi^{-1}(U') \xrightarrow{\Psi^{-1}} Lx_G D'$ that preserve leaves. We have then the following mappings induced in its respective quotients $D'/G \xrightarrow{\widetilde{\Phi}} U \subset U' \xrightarrow{\widetilde{\Psi}^{-1}} D'/G'$, where $\widetilde{\Phi}$ and $\widetilde{\Psi}^{-1}$ denote the mappings induced by Φ and Ψ^{-1} respectively. Let β be $\widetilde{\Psi}^{-1} \circ \widetilde{\Phi}$ from D'/G to D'/G'. It is clear that β is injective and open. Denote by π and π' the canonical projections $D' \to D'/G$ and $D' \to D'/G'$ respectively. To prove the proposition we need the following

Lemma. Let A *be a connected open set in* D'. *Suppose that we have two mappings* $\lambda : A \to D'$, $\mu : A \to D'$ *such that each of them is a diffeomorphism from* A *onto an open set in* D' *and that* $\pi' \circ \lambda = \beta \circ \pi$, $\pi' \circ \mu = \beta \circ \pi$. *Then, there is a unique* $g' \in G'$ *such that* $g'\lambda(x) = \mu(x)$ *for any* $x \in A$.

PROOF OF LEMMA. Since β is open and π and π' are both continuous and open then, λ and μ must be open. Let us prove the uniqueness of g'. If there were g'_1 and g'_2 such that $g'_1\lambda(x) = \mu(x)$ and $g'_2\lambda(x) = \mu(x)$, we would have $g'_1\lambda(x) = g'_2\lambda(x)$ for any $x \in A$. Choose x_0 such that the isotropy group $G'_{\lambda(x_0)}$ is the identity. (It is possible since $\lambda(A)$ is open). We have $g'_1 g'^{-1}_2 \in G'_{\lambda(x_0)}$, hence $g'_1 = g'_2$. Let us prove the existence. We know that D'/G' has a natural V-manifold structure (analogous to that in example 1.1). Given $x \in A$, let $U'_{\lambda(x)}$ be a small open ball centered at $\lambda(x)$, contained in $\lambda(A)$, such that $\{U'_{\lambda(x)}, G'_{\lambda(x)}, \pi'\}$ is a l.u.s. Fix $x \in A$. $\mu \circ \lambda^{-1}$ gives rise to a diffeomorphism from $U'_{\lambda(x)}$ onto a neighborhood of $\mu(x)$. Take $U'_{\lambda(x)}$ sufficiently small such that $\mu\lambda^{-1}(U'_{\lambda(x)}) \subset U'_{\mu(x)}$. Since $\pi' \circ \lambda = \beta \circ \pi = \pi' \circ \mu$ then $\pi' \circ \mu \circ \lambda^{-1} = \pi'$ on $U'_{\lambda(x)}$. Hence $\mu \circ \lambda^{-1}$ is an injection $\{U'_{\lambda(x)}, G'_{\lambda(x)}, \pi'\{ \to \{U'_{\mu(x)}, G'_{\mu(y)}, \pi'\}$. Let i be the canonical injection $\{U'_{\mu(x)}, G_{\mu(x)}, \pi'\} \to \{D', G', \pi'\}$. Then $i \circ \mu \circ \lambda^{-1}$ will be an injection $\{U'_{\lambda(x)}, G'_{\lambda(x)}, \pi'\} \to \{D', G', \pi'\}$. Let j be the canonical injection

$\{U'_{\lambda(x)}, G'_{\lambda,(x)}\ \pi'\} \to \{D', G', \pi'\}$. We know that there is a unique $\sigma' \in G'$ such that $i \circ \mu \circ \lambda^{-1} = \sigma' \circ j$. In other words, $\mu\lambda^{-1}(y) = \sigma'(y)$ for any $y \in U'_{\lambda(x)}$. That is, $\mu(z) = \sigma'\lambda(z)$ for any $z \in \lambda^{-1}(U'_{\lambda(x)})$. We can summarize this fact as follows. Given $x \in A$ there is a neighborhodd U_x of x contained in A and a unique $\sigma' \in G'$ such that $\mu(z) = \sigma'\lambda(z)$ $\forall\ z \in U_x$.

Fix $x_0 \in A$ and choose U_{x_0} and $g' \in G'$ such that $\mu = g' \circ \lambda$ on U_{x_0}. We want to see that $\mu = g' \circ \lambda$ on A. Let C be the subset of A consisting of those x such that there is an open neighborhood of x such that $\mu = g'\lambda$ on this neighborhood. C is obviously open. Let us prove that it is closed in A. Let $x \in A \cap \bar{C}$. Let U_x be a neighborhood of x as above such that there is a unique $\sigma' \in G'$ such that $\mu = \sigma' \circ \lambda$ on U_x. Let $y \in U_x \cap C$. There is a small neighborhood of y (that we can suppose contained in U_x) such that $\mu = g' \circ \lambda$ on this neighborhood. Let z be a point of this neighborhood such that $G'_{\lambda(z)} = I$. We shall have $\sigma\lambda(z) = g'\lambda(z)$. Hence $\sigma' = g'$. Hence $\mu = g' \circ \lambda$ on U_x. Hence $x \in C$. Since C is closed, open, and not empty, then $C = A$.

End of proof of proposition. We want to define $\lambda : D' \to D'$ such that λ is a diffeomorphism from D' onto an open set of D' and that $\varphi\Phi s_p = \varphi\Psi s_{p'}\lambda$. We know that $U = \varphi\Phi(Lx_G D') \subset U'$. Let D'' be an open ball in $\mathbf{R}^n$ centered at the origin, of radius $r, (1/2) < r < 1$, such that $\varphi\Phi(Lx_G D'') \subset U'$. We shall have $\Phi s_p(D'') \subset \Phi(Lx_G D'') = = \varphi^{-1}\varphi\ \Phi(Lx_G D'') \subset \varphi^{-1}(U')$. We shall have then $\Phi \circ s_p : D'' \to \to \varphi^{-1}(U')$. Take the composition $\Psi^{-1}\Phi s_p : D'' \to Lx_{G'} D'$. Let $\mathcal{B}$ be the family of local charts V of L such that $Vg' \cap V = \phi$ if $g' \neq I$, $g' \in G'$. Let $\mathcal{C}$ be the family of those open sets in $Lx_{G'} D'$ of the form $Vx_{G'} D'$ with $V \in \mathcal{B}$. For any $x_0 \in D''$ take a neighborhood W of $\Psi^{-1}\Phi s_p x_0$ such that $W \in \mathcal{C}$ and take a small open ball A_{x_0} centered at x_0, contained in D'', such that $\Psi^{-1}\Phi s_p(A) \subset W$. The family of all these balls $\{A_x\}_{x \in \bar{D}'}$ constitutes an open cover of $\bar{D}'$. Since $\bar{D}'$ is compact, we can choose a finite subcover. Denote by $A_1 \ldots A_r$ the balls of this subcover. One can rearrange $A_1 \ldots A_r$ in another way such that $B_i = \bigcup_{j=1}^{i} A_j$ is connected for any $i = 1 \ldots r$ and that the intersection $A_i \cap B_{i-1}$ is connected for any $i = 2 \ldots r$. At the end of the proof we shall justify this fact. Accept this possibility and continue the proof. For any A_i denote by W_i the $W \in \mathcal{C}$ such that $\Psi^{-1}\Phi s_p(A_i) \subset W_i$. W_i is of the form $V_i x_{G'} D'$ with $V_i \in \mathcal{B}$. For any $g' \in G'$ let $f_{ig'}$ be the mapping from W_i to D' defined by

$$V_i x_{G'} D' \longrightarrow D'$$
$$\widetilde{(v, x)} \longrightarrow g' x$$

$f_{ig'}$ is well defined. Let us define λ on $\bigcup_{j=1}^{r} A_j$. Let us begin by defining λ on A_1. Fix any $g' \in G'$ and define λ on A_1 by $\lambda = f_{1g'} \Psi^{-1} \Phi s_p \cdot \lambda$ is then a diffeomorphism from A_1 to an open set in D' and one has $\pi' \circ \lambda = \beta \circ \pi$. Let us extend λ to $\bigcup_{j=1}^{r} A_j$ by recurrence. Suppose that λ is already extended to $B_{i-1} = \bigcup_{j=1}^{i-1} A_j$ in such a way that λ is a diffeomorphism from B_{i-1} to an open set in D' and that $\pi' \circ \lambda = \beta \circ \pi$. Let us extend λ to B_i. Choose any $\tau' \in G'$ and take the mapping $\nu\colon A_i \to D'$ defined by $\nu = f_{i\tau'} \Psi^{-1} \Phi s_p$. ν is then a diffeomorphism from A_i to an open set in D' and one has $\pi' \circ \nu = \beta \circ \pi$. Take now $\lambda | (B_{i-1} \cap A_i)$ and $\nu | (B_{i-1} \cap A_i)$. Since $B_{i-1} \cap A_i$ is connected, there is (by lemma) a unique $\sigma' \in G'$ such that $\sigma' \nu(x) = \lambda(x)$ for any $x \in B_{i-1} \cap A_i$. Define λ on A_i to be equal to $\sigma' \circ \nu$. In this way we define λ on $\bigcup_{j=1}^{r} A_j$ and hence on D'. Condition $\pi' \circ \lambda = \beta \circ \pi$ imply $\varphi \Phi s_p = \varphi \Psi s_{p'} \lambda$. In fact, by virtue of the commutativity of the diagram

$$\begin{array}{ccccccc}
D' & \xrightarrow{\lambda} & D' & \xrightarrow{s_{p'}} & L x_{G'} D' & \xrightarrow{\Psi} & \varphi^{-1}(U') \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & \downarrow \\
D'/G & \xrightarrow{\beta} & D'/G' & \xleftrightarrow{I} & D'/G' & \xrightarrow{\widetilde{\Psi}} & U'
\end{array}$$

one has $\varphi \Psi s_{p'} \lambda = \widetilde{\Psi} \beta \pi$. But since $\beta = \widetilde{\Psi}^{-1} \widetilde{\Phi}$ one has $\varphi \Psi s_{p'} \lambda = \widetilde{\Phi} \circ \pi$. On the other hand, by virtue of the commutativity of the diagram

$$\begin{array}{ccccc}
D' & \xrightarrow{s_p} & L x_G D' & \xrightarrow{\Phi} & \varphi^{-1}(U) \\
 & {\scriptstyle \pi} \searrow & \downarrow & & \downarrow{\scriptstyle \varphi} \\
 & & D'/G & \xrightarrow{\widetilde{\Phi}} & U
\end{array}$$

one has $\widetilde{\Phi} \circ \pi = \varphi \, \Phi s_p$. Hence, $\varphi \, \Psi s_{p'} \, \lambda = \varphi \, \Phi s_p$. It remains to justify the possibility of rearranging $A_1 \ldots A_r$ in such a way that each B_i is connected and each $A_i \cap B_{i-1}$ is connected. Suppose, by recurrence, that we have already chosen $A_1 \ldots A_{i-1}$ with this property, and let us pick A_i. It is easy to see that given $\varepsilon > 0$ there is a neighborhood B_ε of $\bar{B}_{i-1}$ such that any ball A centered at $x \in B_\varepsilon$, of radius $> \varepsilon$, is such that $A \cap B_{i-1}$ is connected and $A \cup B_{i-1}$ is connected. Let $3\,\varepsilon$ be the Lebesgue number of the cover $A_1 \ldots A_r$ before rearranging. Take any $x \in B_\varepsilon - \bar{B}_{i-1}$. Take the ball centered at x of radius $2\,\varepsilon$. This ball will be contained in some A_j. Choose such a j. We have $j \notin \{1 \ldots i-1\}$ since $x \notin B_{i-1}$. Take A_j as a new A_i. Obviously, all the required properties will be fulfilled.

An important class of foliate vector bundles (allowable vector bundles)

Let M be a manifold of dimension $n + m$ endowed with a C^r foliation $\mathcal{F}$ $(r \geqslant 1)$ of codimension n. Each point has a neighborhood U and a coordinate system $(x^1 \ldots x^n, y^1 \ldots y^m)$ such that the leaves are given in U by $x^1 = \text{constant}, \ldots, x^n = \text{constant}$. We suppose, for U sufficiently small, that U is homeomorphic to a product $U_x \times U_y$ where U_x is a cubical neighborhood in $\mathbf{R}^n$ and U_y a cubical neighborhood in $\mathbf{R}^m$. A coordinate neighborhood $(U, x^1 \ldots x^n, y^1 \ldots y^m)$ satisfying all these conditions will be called *flat*. Let $E \to M$ be a vector bundle on M. We shall say that E is *foliate* if it is possible to find a cover $\mathcal{U} = \{W\}$ of M, by flat local charts, and trivializations of E on each W, such that the transition functions of E on each $W \cap W'$ $(W, W' \in \mathcal{U})$ only depend on the coordinates $x^1 \ldots x^n$. The transversal bundle is such.

When the foliation $\mathcal{F}$ is compact, Hausdorff, we are going to introduce a very important class of foliate vector bundles. The vector bundles in this class will be called *allowable*. Let us begin by an example.

Example 4.2. Let $L x_G D$ be the foliate manifold of example 4.1. Let $T(D)$ be the tangent bundle of D. The action of G on D gives rise to a natural action of G on $T(D)$. Define an action of G on $L x \, T(D)$ by $g(p, x) = (pg^{-1}, gx)$. Denote by $L x_G \, T(D)$ the quotient of $L x \, T(D)$ by this action. Take the canonical projection $L x_G \, T(D) \to L x_G D$.

It is easy to see (using the fact that the action of G on L is free) that $L x_G T(D) \to L x_G D$ is a vector bundle of class C^r on $L x_G D$ (We shall see this fact later, in general). This vector bundle is precisely the transversal bundle of the foliation in $L x_G D$ introduced in example 4.1. Remark that $T(D)$ is the trivial bundle $D x \mathbf{R}^n$. The action of G on $T(D)$ verifies the following property: If $X_x \in T_x(D)$, one has $g(X_x) \in T_{g(x)}(D)$ for any $g \in G$. If we think $T(D)$ as the product $D x \mathbf{R}^n$ then X_x will be a couple (x, v). $g(X_x)$ will be a couple $(g(x), w)$. Given $x \in D$ and $g \in G$, the mapping $v \to w$ is an element of $GL(n, \mathbf{R})$ depending on x and g. Denote by $\eta(g)(x)$ this element. The action of G on $T(D)$ will be of the form $g(x, v) = (g(x), \eta(g)(x)v)$. Take the action of G on $L x D x \mathbf{R}^n$ defined by $g(p, x, v) = (pg^{-1}, gx, \eta(g)(x)v)$. Denote by $(L x D x \mathbf{R}^n)/G$ the quotient space of $L x D x \mathbf{R}^n$ by this action. It is clear that we can think the transversal bundle $L x_G T(D)$ as $(L x D x \mathbf{R}^n)/G$.

Generalization of example 4.2. Let $L x_G D$ be the foliate manifold of example 4.1. Denote by $C^{n-1}(D, GL(k, \mathbf{R}))$ the space of C^{n-1} mappings from D to $GL(k, \mathbf{R})$. Suppose that (as in example 4.2) we have a mapping $\eta : G \to C^{n-1}(D, GL(k, \mathbf{R}))$ such that for any $x \in D$ and for any $\sigma, \tau \in G$ one has $\eta(\sigma\tau)(x) = \eta(\sigma)(\tau x) \circ \eta(\tau)(x)$. Take the action of G on $L x D x \mathbf{R}^k$ defined by $g(p, x, v) = (pg^{-1}, gx, \eta(g)(x)v)$. Denote by $(L x D x \mathbf{R}^k)/G$ the quotient by this action. If $(p, x, v) \in L x D x \mathbf{R}^k$, we denote by $\widetilde{(p, x, v)}$ its class in $(L x D x \mathbf{R}^k)/G$. The projection p of $(L x D x \mathbf{R}^k)/G$ to $L x_G D$ defined by $\widetilde{(p, x, v)} \to \widetilde{(p, x)}$ is well defined. We are going to see that $p : (L x D x \mathbf{R}^k)/G \to L x_G D$ is a vector bundle with fibre $\mathbf{R}^k$. In fact, given $u \in L x_G D$, let (p, x) be a representative of u. Take a local chart U in L such that $Ug \cap U = \phi$ for any $g \neq I$, $g \in G$. (It is possible since the action of G on L is free). Let V be the subset of $L x_G D$ consisting of those classes $\widetilde{(p', x)}$ with $p' \in U$ and $x \in D$. Define a trivialization of $p^{-1}(V)$, $f : p^{-1}(V) \to V x \mathbf{R}^k$ in the following way. Let $\alpha \in p^{-1}(V)$. There is a unique $(p', x) \in L x D$ such that $p(\alpha) = \widetilde{(p', x)}$. Then $\alpha = \widetilde{(p', x, v)}$ with v uniquely determined. Define $f\widetilde{(p', x, v)}$ to be $(\widetilde{(p', x)}, v)$.

Definition 4.1. Given a mapping $\eta : G \to C^{n-1}(D, GL(k, \mathbf{R}))$ such that for any $x \in D$ and for any $\sigma, \tau \in G$ one has $\eta(\sigma\tau)(x) = \eta(\sigma)(\tau x) \circ \eta(\tau)(x)$, take the action of G on $L x D x \mathbf{R}^k$ defined above and take the vector bundle $p : (L x D x \mathbf{R}^k)/G \to L x_G D$. Such a vector bundle on $L x_G D$ will be called *allowable*.

Definition 4.2. Suppose that M is endowed with a compact, Hausdorff foliation of class C^r, say $\mathcal{F}$. Let $E \to M$ be a vector bundle. Such a vector bundle will be called *allowable* if given any leaf L_0 there is a subgroup G of $O(n)$, a free action of G on L, a neighborhood U of L_0, and a diffeomorphism $\Phi : L x_G D \to U$ verifying the properties of theorem 4.1 such that the pull-back of $E|U$ by Φ is an *allowable* vector bundle on $L x_G D$.

We have the following

Proposition 4.2. *An allowable vector bundle is foliate.*

SKETCH OFP ROOF. Take an allowable vector bundle $p: (L x D x \mathbf{R}^k)/G \to \to L x_G D$. Take two connected charts V_1 and V_2 in L such that $g V_i \cap V_i = \phi$ for any $g \in G$, $g \neq I$, $i = 1, 2$. We know that $V_i x D$ is a local chart for $V_i x_G D$ and that the induced vector bundle over $V_i x_G D$ is trivial. Suppose $(V_1 x_G D) \cap (V_2 x_G D) \neq \phi$. Then, there is a unique $g \in G$ such that $V_1 g^{-1} \cap V_2 \neq \phi$. The trivialization taken on each $\mathrm{V}_i x_G D$ is given by $(\widetilde{p_i, x, v}) \to ((\widetilde{p_i, x}), v)$. So, the transition function on $(V_1 x_G D) \cap (V_2 x_G D)$ is given by $\eta(g)(x)$ (depending only on $x^1 \ldots x^n$).

Remark. The trivial bundle and the transversal bundle are both allowable. To give examples of allowable vector bundles the following considerations are useful. Let T be a differentiable functor in the category of finite dimensional vector spaces. It is well known that T can be extended to a functor T_M in the category of vector bundles on M. Suppose for example that T is a functor in $r + s$ variables, contravariant in the first ones and covariant in the other variables. Let T_M be its extension. Since the definition of allowable vector bundle is obviously functorial, if $E_1 \ldots E_r$ $F_1 \ldots F_s$ are allowable, then $T_M(E_1 \ldots E_r, F_1 \ldots F_s)$ is such. Hence. if E and F are allowable, then $E \oplus F . E \otimes F$, E^*, Hom (E, F), $\wedge^k E, \ldots$ etc, are allowable.

Definition 4.3. Let $E \to M$ be a foliate vector bundle. Let $\mathcal{U} = \{W\}$ be a cover of M by flat local charts, and trivializations of $E|W$ for each $W \in \mathcal{U}$ given by bases of sections $s_1 \ldots s_k$ of $E|W$ such that the transition functions only depend on $x^1 \ldots x^n$. We shall say that a cross section γ of E is *base-like* if one has an expression $\gamma = \sum \gamma^A s_A$ on each $W \in \mathcal{U}$, where the functions γ^A only depend on $x^1 \ldots x^n$.

From now on we shall denote by $\Gamma_b(E)$ the space of base-like cross sections of E.

Allowable vector bundles are important because of the following

Theorem 4.3. *Let M be a manifold with a compact Hausdorff C^r foliation $\mathcal{F}$ $(r \geqslant 1)$. Let B be the quotient space $M/\mathcal{F}$ with the structure of V-manifold given in theorem* 4.2. *Let $E \to M$ be an allowable vector bundle with fibre $\mathbf{R}^k$. Then there is a V-vector bundle on B with fibre $\mathbf{R}^k$ that we shall denote by $B(E) \to B$ such that if we call $\Gamma(B(E)$ the space of differentiable cross sections of $B(E)$ one has $\Gamma(B(E)) \simeq \Gamma_b(E)$.*

Sketch of Proof. Let $M \xrightarrow{\varphi} B$ be the canonical projection. Each leaf L_0 has an open neighborhood U with the properties required in Definition 4.2. Let $\mathcal{G}$ be the family of those open sets in B of the form $\varphi(U)$, where U is a neighborhood of a leaf with the properties required in Definition 4.2. Let $\mathcal{A}$ be the defining family (of the V-manifold structure) introduced in theorem 4.2. Let $\mathcal{A}'$ be the subfamily of those $\{D', G, \varphi\, \Phi s_p\} \in \mathcal{A}$ such that $\varphi\, \Phi s_p(D') \in \mathcal{G}$. $\mathcal{A}'$ is also a defining family. Let $\{D', G, \varphi\, \Phi s_p\}$ and $\{D', G', \varphi\, \Psi s_{p'}\} \in \mathcal{A}'$ such that $\varphi\, \Phi s_p(D') \subset \varphi\, \Psi s_{p'}(D')$. We have

$$L x_G D' \xrightarrow{\Phi} U \subset U' \xleftarrow{\Psi} L x_{G'} D' .$$

Let $(L x D' x \mathbf{R}^k)/G$ be the pull-back of $E|U$ by Φ and $(L x D' x \mathbf{R}^k)/G'$ the pull-back of $E|U'$ by Ψ. We know that the restriction of $(L x D' x \mathbf{R}^k)/G$ to $s_p(D')$ is trivial. The pull-back of $(L x D' x \mathbf{R}^k)/G$ by $s_p : D' \to L x_G D'$ will be trivial. Analogously, the pull-back of $(L x D' x \mathbf{R}^k)/G'$ by $s_{p'} : D' \to L x_{G'} D'$ will be trivial. Let λ be an injection $\{D', G' \varphi\, \Phi s_p\} \to \{D', G', \varphi\, \Psi s_{p'}\}$. We have the following commutative diagram

$$\begin{array}{ccc} L x_G D' & \xrightarrow{\Psi^{-1} \circ \phi} & L x_{G'} D' \\ s_p \uparrow & \circ & \uparrow s \\ D' & \xrightarrow{\lambda} & D' \end{array}$$

Let F and F' be the pull-backs of $(L x D' x \mathbf{R}^k)/G$ and $(L x D' x \mathbf{R}^k)/G'$ by s_p and $s_{p'}$ respectively. The pull-back of F' by λ is F. Hence we have a mapping $\bar{\lambda} : F' \to F$ such that

$$\begin{array}{ccc} F' & \xrightarrow{\bar{\lambda}} & F \\ \downarrow & & \downarrow \\ D' & \xrightarrow{\lambda} & D' \end{array}$$

is commutative. But F' and F are trivial and we have a canonical trivialization of each one of them since $(L x D' x \mathbf{R}^k)/G$ over $s_p(D')$ and $(L x D' x \mathbf{R}^k)/G'$ over $s_{p'}(D')$ are canonically trivial. The mapping $\bar{\lambda}$ will have the form

$$\begin{aligned} \bar{\lambda} : D' x \mathbf{R}^k &\to D' x \mathbf{R}^k \\ (x, v) &\to (\lambda(x), g_\lambda(x) v) \end{aligned}$$

where g_λ is a C^r map from D' to $GL(k, \mathbf{R})$. Suppose now that one has $\{D', G, \varphi\, \Phi s_p\}$, $\{D', G', \varphi\, \Psi s_{p'}\}$ and $\{D', G'', \varphi\, \varrho s_{p''}\} \in \mathcal{A}'$ with $\varphi\, \Phi s_p(D') \subset \varphi\, \Psi s_{p'}(D') \subset \varphi\, \varrho s_{p''}(D')$. Let λ be an injection $\{D', G', \varphi\, \Phi s_p\} \to \{D', G', \varphi\, \Psi s_{p'}\}$ and let μ be an injection $\{D', G', \varphi\, \Psi s_{p'}\} \to \{D', G'', \varphi\, \varrho s_{p''}\}$. As above we shall have mappings $\bar{\lambda} : D' x \mathbf{R}^k \to D' x \mathbf{R}^k$ and $\bar{\mu} : D' x \mathbf{R}^k \to D' x \mathbf{R}^k$. We shall have $\bar{\mu}\, \bar{\lambda}(x, v) = \bar{\mu}(\lambda(x), g_\lambda(x) v) = (\mu\, \lambda(x), g_\mu(\lambda(x))\ \mathrm{o}\ g_\lambda(x) v)$. Hence,

$$(*)\ g_{\mu \mathrm{o} \lambda}(x) = g_\mu(\lambda(x)\ \mathrm{o}\ g_\lambda(x)\,.$$

We have then a system of functions $\{g_\lambda(x)\}$ for any injection λ satisfying (*). By virtue of theorem 1 on page 472 of [11] these g_λ determine a V-vector bundle with fibre $\mathbf{R}^k$ on B that we call $B(E) \to B$. Let γ be an element of $\Gamma_b(E)$. Let $a = \{D', G, \varphi\, \Phi s_p\}$ be an element of $\mathcal{A}'$. We have $\Phi : L x_G D' \to U$. The restriction γ_U of γ to U gives, by Φ, a base-like section of $(L x D' x \mathbf{R}^k)/G$. Take the $p \in L$ that appears in s_p. Take V a small neighborhood in L such that there is a canonical trivialization of $(L x D' x \mathbf{R}^k)/G$ over $V x_G D'$. This base-like section restricted to $V x_G D'$ will give a function $D' \to \mathbf{R}^k$ and so, a section of the trivial bundle $D' x \mathbf{R}^k$ over D'. Call γ_a this section. The system $\{\gamma_a\}$ for any $a \in \mathcal{A}'$ is a differentiable cross section of $B(E)$ in the sense of definition 1.9. One can see that the correspondence $\gamma \to \{\gamma_a\}$ from $\Gamma_b(E)$ to $\Gamma(B(E))$ is the isomorphism looked for.

Remark. We can repeat all this section assuming that E is a vector bundle with fibre $\mathbf{C}^k$. We shall then get as a bundle $B(E)$ a V-vector bundle with fibre $\mathbf{C}^k$.

Complex analytic foliations with bundle-like metrics. Cohomology of base-like forms.

Let M be a compact complex manifold of complex dimension $n + m$, endowed with a complex analytic foliation of complex codimension n whose leaves are closed in M. Call $\mathcal{F}$ this foliation. We suppose that $\mathcal{F}$ is defined by an adapted atlas $\{(U_\alpha, z_\alpha^a, z_\alpha^u)\}$ (index convention: $a, b, \ldots = 1, \ldots n; u, v, \ldots = n + 1, \ldots n + m$) where $z^a =$ constant define the leaves. We suppose that M is endowed with a complete *bundle-like* Hermitian metric g. By a theorem of R. Hermann [6], the quotient space $B = M/\mathcal{F}$ is Hausdorff, so we can apply to this situation the results of previous sections.

We shall denote by $*$ the Hodge star operator corresponding to g and by $\tilde{*}$ the operator $\tilde{*}\,\varphi = \overline{*\,\varphi}$. We shall have a Hermitian scalar product defined on the space of differential forms by $\langle \varphi, \Psi \rangle = \int_M \varphi \wedge \tilde{*}\,\Psi$ and the operator $\delta = -\tilde{*}\, d\, \tilde{*}$ such that $\langle d\varphi, \Psi \rangle = \langle \varphi, \delta \Psi \rangle$. We shall have decompositions $d = d' + d''$, $\delta = \delta' + \delta''$ with respect to complex types.

Let $E \to M$ be a foliate complex analytic vector bundle with r-dimensional fibre (the transition functions only depend on z^a), endowed with a foliate Hermitian metric h (given locally by a Hermitian matrix (h_{AB}) depending only on z^a and $\bar{z}^a$) (index convention: $A, B \ldots = 1 \ldots r$). We shall use the associate Hermitian connexion locally given by $\omega_A{}^B = (d' h_{AC})\, h^{CB}$. We can define a Hermitian product on E-valued differential forms by $\langle \varphi, \Psi \rangle = \int_M h_{AB}\, \varphi^A \wedge \tilde{*}\, \Psi^B$ (If φ is an E-valued differential form then φ has the local expression $\varphi = \varphi^A \otimes s_A$ where φ^A are ordinary forms and $\{s_A\}$ is the basis of sections given by the taken trivialization).

We can define the operators d''_E and δ''_E on E-valued forms by the following local expressions:

$$(d''_E\, \varphi)^A = d''\, \varphi^A$$
$$(\delta''_E\, \varphi)^A = \delta''\, \varphi^A - \tilde{*}\, e(\theta)\, \tilde{*}\, \varphi^A\ ,$$

where $e(\theta)$ means the exterior product by the matrix $\theta = (\theta_A{}^B)$, $\theta_A{}^B = h^{BC}(d'' h_{CA})$.

If φ and Ψ are E-valued forms, one has

$$\langle d''_E \varphi, \Psi \rangle = \langle \varphi, \delta''_E \Psi \rangle .$$

Let Δ''_E be the Laplace operator (acting on E-valued forms) defined by $\Delta''_E = d''_E \delta''_E \delta''_E d_E$. We shall denote by $D^{p,q}(E)$ the space of C^∞ E-valued base-like (p, q)-forms, that is, the space of E-valued forms which have a local expression:

$$(4.1) \quad \varphi = \frac{1}{p!} \cdot \frac{1}{q!} \varphi^A{}_{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q} \, dz^{a_1} \wedge \ldots \wedge dz^{a_p} \wedge dz^{\bar{b}_1} \ldots dz^{\bar{b}_q} \otimes s_A$$

where the coefficients $\varphi^A{}_{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q}$ only depend on $z^1 \ldots z^n, \bar{z}^1 \ldots \bar{z}^n$. Let $H^{p,q}(M, E)$ be the space of those $\varphi \in D^{p,q}(E)$ such that $\Delta''_E \varphi = 0$. Observe that the operator d''_E maps $D^{p,q}(E)$ into $D^{p,q+1}(E)$.

Theorem 4.4. $\delta''_E(D^{p,q+1}(E)) \subset D^{p,q}(E)$ *and if E is an allowable complex analytic vector bundle (see def.* 4.1 *and* 4.2*) one has the following orthogonal decomposition with respect to* $\langle \ , \rangle$:

$$(4.2) \quad D^{p,q}(E) = H^{p,q}(M, E) \oplus d''_E(D^{p,q-1}(E)) \oplus \delta''_E(D^{p,q+1}(E)).$$

Moreover, the space $H^{p,q}(M, E)$ has finite dimension.

PROOF. Let $D^{p,q}$ be the space of ordinary base-like (p, q)-forms. At each point $x_0 \in M$ we can define algebraically a Hodge star operator $\tilde{*}_b : D_{x_0}{}^{p,q} \to D_{x_0}{}^{n-p, n-q}$ (using only the transversal part $(g_{a\bar{b}})$ of the metric g) in the following way. We can always suppose that the coordinates $z^1 \ldots z^n$ have been taken such that the matrix $(g_{a\bar{b}})$ at x_0 is the identity. (This can be got by a linear change $z^a = \alpha_b{}^a \overset{*}{z}{}^b$, where $\alpha_b{}^a$ are constants). Let $A_p = \{a_1 \ldots a_p\}$ be a set of indices $a_1 < \ldots < a_p$, $0 \leqslant a_i \leqslant n$. We denote by $A'_p = \{1 \ldots n\} - A_p$ ordered by $<$. We can define $\tilde{*}_b$ at x_0 by

$$\tilde{*}_b(f dz^{A_p} \wedge d\overline{z^{B_q}}) = (-1)^{q(n-p)} \varepsilon(A_p, A'_p) \, \varepsilon(B_q, B'_q) \, \bar{f} dz^{A'_p} \wedge d\overline{z^{B'_q}},$$

where f is any complex number. Define δ_b on $D^{p,q}$ by $\delta_b = - \tilde{*}_b \, d \, \tilde{*}_b$. We need the following

Lemma. If $\varphi \in D^{p,q}$ then $\delta_b \varphi = \delta \varphi$.

PROOF OF LEMMA. It suffices to prove this relation at a point x_0. Take an adapted coordinate system (U, z^a, z^u) with $x_0 \in U$ such that $(g_{a\bar{b}})$ and $(g_{u\bar{v}})$ are the identity at x_0. If φ has the local expression (4.1) we shall have

$$\tilde{*}\, d\, \tilde{*}\, \varphi = \frac{1}{p!} \cdot \frac{1}{q!}\, \partial_{\bar{c}}\, \varphi_{A_p \bar{B}_q}\, \tilde{*}\, (dz^c \wedge \tilde{*}\, (dz^{A_p} \wedge d\overline{z^{B_q}})) +$$

$$+ \frac{1}{p!} \cdot \frac{1}{q!}\, \partial_c\, \varphi_{A_p \bar{B}_q}\, \tilde{*}\, (d\overline{z^c} \wedge \tilde{*}\, (dz^{A_p} \wedge d\overline{z^{B_q}}))\, .$$

We shall have an analogous expression for $\tilde{*}_b\, d\, \tilde{*}_b$. In order to prove the lemma it suffices to prove that one has at x_0:

$$\tilde{*}\, (dz^c \wedge \tilde{*}\, (dz^{A_p} \wedge d\overline{z^{B_q}})) = \tilde{*}_b (dz^c \wedge \tilde{*}_b (dz^{A_p} \wedge d\overline{z^{B_q}})),$$

and the analogous expression for $d\overline{z^c}$. But these algebraic relations are obvious. This concludes the proof of lemma.

Define the operator $(\delta''_E)_b$ on $D^{p,q}(E)$ by $(\delta''_E)_b = \delta''_b - \tilde{*}\, e(\theta)\, \tilde{*}_b$. Since the metric h is foliate $(\delta''_E)_b$ maps $D^{p,q}(E)$ into $D^{p,q-1}(E)$. One can also prove (in the same way that in lemma) that $(\delta''_E)_b = \delta''_E$ on $D^{p,q}(E)$. On $D^{p,q}(E)$ we shall have $\Delta''_E = d''_E(\delta''_E)_b + (\delta''_E)_b\, d''_E$. Hence Δ''_E maps $D^{p,q}(E)$ into itself. The spaces $d''_E(D^{p,q-1}(E))$, $\delta''_E(D^{p,q+1}(E))$ and $H^{p,q}(M, E)$ are mutually orthogonal in $D^{p,q}(E)$ with respect to the Hermitian product $\langle\ ,\ \rangle$. The spaces $\Delta''_E(D^{p,q}(E))$ and $H^{p,q}(M, E)$ are also orthogonal. If we prove that if E is allowable one has

$$D^{p,q}(E) = H^{p,q}(M, E) \oplus \Delta''_E(D^{p,q}(E)) \tag{4.3}$$

then $\Delta''_E(D^{p,q}(E))$ will be the orthogonal complement of $H^{p,q}(M, E)$. Hence, $d''_E(D^{p,q-1}(E)) \oplus \delta''_E(D^{p,q+1}(E))$ (orthogonal to $H^{p,q}(M, E)$) will be contained in $\Delta''_E(D^{p,q}(E))$. But (by definition of Δ''_E) $\Delta''_E(D^{p,q}(E)) \subset d''_E(D^{p,q-1}(E)) \oplus \delta''_E(D^{p,q+1}(E))$. Then, we shall have (4.2). Therefore it suffices to prove (4.3) and that $H^{p,q}(M, E)$ has finite dimension in order to prove the theorem.

Let us denote by η_b the base-like volume element. η_b is the element of $D^{n,n}$ defined by

$$\eta_b = \sqrt{\det(g_{a\bar{c}})}\, dz^1 \wedge \ldots dz^n \wedge d\overline{z^1} \wedge \ldots d\overline{z^n}\, .$$

If $\varphi, \Psi \in D^{p,q}(E)$ we denote by $(\varphi, \Psi)_b$ the base-like function defined locally by

$$h_{AB}\, \varphi^A \wedge \tilde{*}_b\, \Psi^B = (\varphi, \Psi)_b\, \eta_b$$

Let $\varphi \in D^{p,q-1}(E)$ and $\Psi \in D^{p,q}(E)$. One can easily prove the following local expression

$$d(h_{AB}\varphi^A \wedge \tilde{*}_b\, \Psi^B) = d''(h_{AB}\, \varphi^A \wedge \tilde{*}_b\, \Psi^B) = h_{AB}\, d''\, \varphi^A \wedge \tilde{*}_b\, \Psi^B - $$
$$- h_{AB}\, \varphi^A \wedge \tilde{*}_b\, ((\delta''_E)_b\, \Psi)^B = (d''_E\, \varphi, \Psi)_b\, \eta_b - (\varphi, (\delta''_E)_b\, \Psi)_b\, \eta_b\,.$$

In other words, we have

$(d''_E\, \varphi, \Psi)_b\, \eta_b = (\varphi, (\delta''_E)_b\, \Psi)_b\, \eta_b +$ differential of a base-like $(2n-1)$-form.

Therefore, if $\varphi, \Psi \in D^{p,q}(E)$ we shall have

$(\Delta''_E\, \varphi, \Psi)_b\, \eta_b = (\varphi, \Delta''_E\, \Psi)_b\, \eta_b +$ differential of a base-like $(2n-1)$-form.

Let us denote by Q the transversal (orthogonal) bundle of complex type $(1, 0)$ corresponding to the foliation $\mathcal{F}$. Q is an allowable complex analytic vector bundle. Let us denote by $F = \wedge\, {}^pQ^* \otimes \otimes \wedge\, {}^q\overline{Q^*} \otimes E$. F is allowable. We shall have $\Gamma_b(F) = D^{p,q}(E)$. On each fibre F_{x_0} of F we shall have the Hermitian product induced by $(\,,\,)_b$. Hence, we shall have a Hermitian product in the vector bundle $F \to M$ also called $(\,,\,)_b$. This Hermitian product is foliate.

Consider now the quotient space $B = M/\mathcal{F}$ as a real $2n$-dimensional V-manifold. Let $B(F) \to B$ be the V-vector bundle (with complex fibres) described in the preceding section. Given $s \in \Gamma(B(F))$, there exists $\gamma \in \Gamma_b(F)$ such that $\{\gamma_a\} = s$, $a \in \mathcal{A}'$ (we use the notation of proof of th. 4.3). We define $\Delta''_E\, s$ by setting $(\Delta''_E\, s)_a = (\Delta''_E\, \gamma)_a$. We have then an elliptic operator $\Delta''_E : \Gamma(B(F)) \longrightarrow \Gamma(B(F))$. The foliate Hermitian product $(\,,\,)_b$ of F induces a Hermitian product in the V-vector bundle $B(F) \to B$ that we shall also denote by $(\,,\,)_b$. The bundle-like metric on M induces a Riemannian metric on B that we shall also denote by g. The V-manifold B is orientable. In fact, it suffices to observe that if $(U_\alpha\ z_\alpha^a\ z_\alpha^u)$, $(U_\beta\ z_\beta^a\ z_\beta^u)$ are two flat local charts of $\mathcal{F}$ in M, one has

$$\frac{\partial(z_\alpha{}^1 \ldots z_\alpha{}^n, \overline{z_\alpha{}^1} \ldots \overline{z_\alpha{}^n})}{\partial(z_\beta{}^1 \ldots z_\beta{}^n, \overline{z_\beta{}^1} \ldots \overline{z_\beta{}^n})} > 0$$

The volume element on B corresponding to g is precisely the $2n$-form defined by $\{(\eta_b)_{\widetilde{U}}\}$ that we shall also denote by η_b.

Define a Hermitian product $[\ ,\ ]$ on $\Gamma(B)(F)$ by

$$[\varphi, \Psi] = \int_B (\varphi, \Psi)_b \, \eta_b \, .$$

If $s = \{s_{\widetilde{U}}\}$ is an r-form over B, we define its exterior differential ds to be the $(r+1)$-form on B $ds = \{ds_{\widetilde{U}}\}$. If $\varphi, \Psi \in \Gamma(B(F))$ we shall have

$$(\Delta''_E \varphi \;\; \Psi)_b \, \eta_b = (\varphi \;\; \Delta''_E \Psi)_b \, \eta_b + \text{differential of a } (2n-1)\text{-form on } B.$$

By integrating over B we shall have

$$[\Delta''_E \varphi \;\; \Psi] = [\varphi . \Delta''_E \Psi] + \int_B \text{differential of a } (2n-1)\text{-form on } B.$$

But this integral vanishes since B is compact. In fact, if β is a $(2n-1)$-form over B, $\beta = \{\beta_{\widetilde{U}}\}$, we take a partition of unity subordinate to a finite cover $\{U_i\}$ of B with $\{\widetilde{U}_i, G_i, \varphi_i\} \in A$ and we shall have

$$\int_B d\beta = \int_B \Sigma\, d(f_i \beta) = \Sigma \frac{1}{n(G_i)} \int_{\widetilde{U}_i} d(f_i \beta)_{\widetilde{U}_i} = 0 \, .$$

We have then proven that Δ''_E is a self adjoint elliptic operator acting on $\Gamma(B(F))$. By virtue of theorem 3.6, $\Gamma(B(F)) = \mathit{ker}\ \Delta''_E \oplus$ $\oplus\ \mathit{Im}\ \Delta''_E$ and ker Δ''_E has finite dimension. From the isomorphism $\Gamma(B(F)) \simeq \Gamma_b(F)$ we obtain (4.3) and that $H^{p,q}(M, E)$ has finite dimension. This ends the proof.

Corollary. Let $H^q(D^{p,\cdot}(E), d''_E)$ *be the* q *cohomology space of the complex*

$$\ldots \longrightarrow D^{p,q}(E) \xrightarrow{d''_E} D^{p,q+1}(E) \longrightarrow \ldots$$

We have $H^q(D^{p,\cdot}(E), d''_E) \simeq H^{p,q}(M, E)$, *and these spaces have finite dimension. (We are assuming that E is allowable).*

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Added in proof

In theorem 3.5 we must also assume that the family $\mathcal{H}$ of open sets in B corresponding to l.u.s's is a basis of open sets in B. In fact, in order to prove that $Q = \sum f_i Q_i g_i$ is a p.d.o., given $p \in B$ we must take a neighborhood U of p, $U \in \mathcal{H}$, contained in U_i if $i \in I_p$ and such that $U \cap \sup g_i = \phi$ for any $i \notin I_p$. Such an U satisfies condition (C). (Note that it does not suffice to take $U \in \mathcal{H}$ contained in U_i for any $i \in I_p$ to assure that $f_i Q_i (g_i f) = 0$ if $i \notin I_p$).

Because of this we also must assume this condition on the base space B of the V-vector bundle $E \to B$ in the last two sections of chapter 3. Observe that the examples considered in chapter 4 fulfil this supplementary assumption.