UNCONDITIONAL BASIC SEQUENCES OF RANDOM VARIABLES IN $L^p(E)$

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ABSTRACT

Let E be a Banach space of type p, $p \in [1,2]$. We obtain for unconditional basic sequences in $L^p(E)$ inequalities of type

$E \|\sum_{i=1}^{n} X_i\|^p \leqslant C \sum_{i=1}^{n} E \|X_i\|^p$, which we apply to obtain a Rademacher-Mensov law containing the Warren and Howell result (see [12]).

O. — INTRODUCTION

Let E be a separable real Banach space. We shall denote by $L^p(E)$ the space of P-equivalence classes of E-valued random variables X defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\|X\|^p$ is integrable.

Warren and Howell [12] have shown that if E is a G_α-space, $0 < \alpha \leqslant 1$, and $X_1, X_2, \ldots, X_n$ is a family of mutually orthogonal (in the James sense) random variables of $L^{1+\alpha}(E)$, then they satisfy an inequality of the following type:

A) $$E \|\sum_{i=1}^{n} X_i\|^{1+\alpha} \leqslant A \sum_{i=1}^{n} E \|X_i\|^{1+\alpha},$$

where $A > 0$ is a constant characteristic of the space.

From this inequality A) Warren and Howell obtain a generalization of the classical Rademacher-Mensov Strong Law:

B) *Let E be a G_α-space, $0 < \alpha \leqslant 1$, and let the sequence of E-valued random variables (X_n) be mutually orthogonal in $L^{1+\alpha}(E)$. If*

$$\sum_{n=1}^{\infty} n^{-(1+\alpha)} E\,||X_n||^{1+\alpha} \log^{1+\alpha} n < \infty ,$$

then

$$\lim_n n^{-1} \sum_{i=1}^{n} X_i = 0 \qquad \text{[a.s.]}$$

It is known that:

C) *E is a Banach space of type p, $1 \leqslant p \leqslant 2$, iff there exists a constant $C > 0$ so that for each n*

$$E\,||\sum_{i=1}^{n} X_i||^p \leqslant C \sum_{i=1}^{n} E\,||X_i||^p$$

for all independent and zero mean random elements $X_1, X_2, \ldots, X_n$ in $L^p(E)$.

In this paper we set out to show how well the unconditional basic sequences behave in Banach spaces of type p. It will be seen that an inequality similar to the one in C) is also valid for unconditional basic sequences in $L^p(E)$, which E is of type p, and from this we obtain a result analogous to B) for these sequences of random variables.

1. — Definitions and preliminary results

We recall that a *basic sequence* (x_n) in a Banach space E is a sequence which is a Schauder basis for the closure of its linear span. A basic sequence (x_n) in E is called *unconditional basic sequence* if for any pair $s, t \in N$ and for any scalars $\alpha_{i_1}, \ldots, \alpha_{i_s}, \alpha_{j_1}, \ldots, \alpha_{j_t}$, whose indexes satisfy $\{i_1, \ldots, i_s\} \cap \{j_i, \ldots, j_t\} = \varnothing$, there exists a constant A $(1 \leqslant A < \infty)$ such that:

$$||\sum_{k=1}^{s} \alpha_{i_k} x_{i_k}|| \leqslant A\,||\sum_{k=1}^{s} \alpha_{i_k} x_{i_k} + \sum_{k=1}^{t} \alpha_{j_k} x_{j_k}|| \qquad (1)$$

If $A = 1$, then (x_n) is called an *orthogonal basic sequence*. A basic sequence of either of the two types being considered which is basis of E is called respectively unconditional or orthogonal basis.

We recall that a $x \in E$ is said to be *orthogonal* to a $y \in E$ and we write $x \perp y$ if we have $||x + \alpha y|| \geqslant ||x||$ for all scalars α; x and y are called *mutually orthogonal* if $x \perp y$ and $y \perp x$. A subspace F of E is said to be *orthogonal* to a subspace G of E if we have $x \perp y$ for all $x \in F$ and $y \in G$. It is clear that (x_n) is an orthogonal basic sequence if and only if the closed linear subspaces generated by

$x_{i_1}, \ldots, x_{i_s}$ and $x_{j_1}, \ldots, x_{j_t}$, with $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \varnothing$, are orthogonal:

$$[x_{i_1}, \ldots, x_{i_s}] \perp [x_{j_1}, \ldots, x_{j_t}]$$

PROPOSITION 1

If (x_n) is an unconditional basic sequence of E, there exists a constant B $(1 \leqslant B < \infty)$ such that

$$B^{-1} \inf_{1 \leq k \leq n} (|\beta_k|) \, ||\sum_{k=1}^{n} \alpha_k x_k|| \leqslant ||\sum_{k=1}^{n} \beta_k \alpha_k x_k|| \leqslant$$
$$\leqslant B \sup_{1 \leq k \leq n} (|\beta_k|) \, ||\sum_{k=1}^{n} \alpha_k x_k|| \tag{2}$$

for each $n \in N$ and any $\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n \in \mathbf{R}$.

PROOF

The definition of an unconditional basic sequence given in (1) is equivalent to the existence of $B(1 \leqslant B < \infty)$, such that:

$$||\sum_{k=1}^{n} \gamma_k \alpha_k x_k|| \leqslant B ||\sum_{k=1}^{n} \alpha_k x_k||$$

for each $n \in N$, and any $\gamma_1, \ldots, \gamma_n; \alpha_1, \ldots, \alpha_n \in \mathbf{R}$ with $|\gamma_k| \leqslant 1$, $1 \leqslant k \leqslant n$ (see Singer [10], p. 500). So, if $\sup_{1 \leq k \leq n} |\beta_k| \neq 0$:

$$||\sum_{k=1}^{n} \beta_k \alpha_k x_k|| = ||\sum_{k=1}^{n} \frac{\beta_k}{\sup_{k \leq n} |\beta_k|} \sup_{k \leq n} |\beta_k| \alpha_k x_k|| \leqslant$$
$$\leqslant B \sup_{k \leq n} |\beta_k| \; ||\sum_{k=1}^{n} \alpha_k x_k||.$$

A similar reasoning allows us to obtain the left side inequality. #

PROPOSITION 2

Let (X_n) *be an unconditional basic sequence in* $L^p(E)$ $(p \geqslant 1)$. *If* (ξ_n) *are real-valued random variables so that* (X_n) *and* (ξ_n) *are independent, then there exists a constant* B $(1 \leqslant B < \infty)$ *such that:*

$$B^{-1} \, \| \inf_{k \leq n} |\xi_k| \, \|_p \cdot \| \sum_{k=1}^{n} \alpha_k X_k \|_p \leqslant \| \sum_{k=1}^{n} \alpha_k \xi_k X_k \|_p \leqslant$$

$$\leqslant B \, \| \sup_{k \leq n} |\xi_k| \, \|_p \cdot \| \sum_{k=1}^{n} \alpha_k X_k \|_p \tag{3}$$

for each $n \in N$ *and* $\alpha_1, \ldots, \alpha_n \in R$.

PROOF

This result follows from proposition 1, applied to $L^p(E)$, integrating with respect to the distribution law of $(\xi_1, \ldots, \xi_n)$. #

NOTE 1

If, in propositions 1 and 2, (x_n) and (X_n) are orthogonal basic sequences in E and $L^p(E)$ respectively, it can easily be shown that $B = 2$ and the inequalities (3) are a simple extension of corollary 4.2, in the Hoffmann-Jorgensen paper [6], which deals with the orthogonal basic sequence formed by independent and centered random elements in $L^p(E)$. The results derived from it (see [6], theorem 4.3 and corollary 4.4) can be generalizad in the same way.

2. — UNCONDITIONAL BASIC SEQUENCES IN TYPE p BANACH SPACES

If E is a space of type p, $1 \leqslant p \leqslant 2$, that is to say, if for a Bernoulli sequence (ε_n) and for each $n \in N$ and $x_1, \ldots, x_n \in E$ there exists a constant $C > 0$ such that

$$E \, \| \sum_{k=1}^{n} \varepsilon_k \, x_k \|^p \leqslant C \sum_{k=1}^{n} \|x_k\|^p \tag{4}$$

the previous proposition allow us to obtain the following results:

PROPOSITION 3

If (x_n) *is an unconditional basic sequence and* E *is of type* p*, we have*

$$\| \sum_{k=1}^{n} \alpha_k x_k \|^p \leqslant D \sum_{k=1}^{n} \| \alpha_k x_k \|^p, \; D = C B^p \tag{5}$$

for each $n \in N$ *and every* $\alpha_1, \ldots, \alpha_n \in R$.

PROOF

As

$$\| \sum_{k=1}^{n} \alpha_k x_k \|^p \leqslant B^p E \| \sum_{k=1}^{n} \varepsilon_k \alpha_k x_k \|^p$$

where (ε_n) is a Bernoulli sequence, the result can easily be obtained using (4). #

THEOREM 4

E is a type p space, $1 \leqslant p \leqslant 2$*, iff for every unconditional basic sequence* (X_n) *in* $L^p(E)$ *there exists a constant* $D > 0$ *such that:*

$$\| \sum_{k=1}^{n} \alpha_k X_k \|_p^p \leqslant D \sum_{k=1}^{n} \| \alpha_k X_k \|_p^p \tag{6}$$

for each $n \in N$ *and* $\alpha_1, \ldots, \alpha_n \in R$.

PROOF

A sequence (X_n) of independent and centered random variables of $L^p(E)$ is an orthogonal basic sequence in $L^p(E)$. If (X_n) satisfies inequality (6) then E will be of type p in accordance with proposition C) in paragraph 0.

Conversely, if E is of type p then so is $L^p(E)$ and (6) is just (5) for this space. #

From theorem 4 it can be deduced that any unconditional basic sequence in $L^p(E)$ such that

$$\sum_{n=1}^{\infty} n^{-p} E \,||X_n||^p < \infty \text{ satisfies } n^{-1} (X_1 + \dots + X_n) \underset{n\to\infty}{\to} 0$$

in probability and in $L^p(E)$ but it does not seem possible to obtain the a.s. convergence for every unconditional basic sequence. However, the authors have proved a strong law for conditionally independent sequences of random variables in $L^p(E)$ (see [1]), taking into account that each sequence forms an orthogonal basic sequence in $L^p(E)$.

Next we give the Rademacher-Mensov theorem for unconditional basic sequences in $L^p(E)$:

THEOREM 5

If E is a type p space $(1 \leqslant p \leqslant 2)$, for every unconditional basic sequence (X_n) in $L^p(E)$ such that:

$$\sum_{n=1}^{\infty} n^{-p} \,||X_n||_p^p \log^p n < \infty$$

we have

$$\lim n^{-1} \sum_{k=1}^{n} X_k = 0 \qquad \text{[a.s.]}$$

PROOF

From the inequality obtained in Theorem 4 it is sufficient to use a reasoning similar to the one used by Doob [3], pages 156-157, in the scalar case. #

In a general Banach space, given a sequence of mutually orthogonal elements $(x_n) \subset E$, it does not always follow that

$$[x_{i_1}, \dots, x_{i_s}] \perp [x_{j_1}, \dots, x_{j_t}] \text{ if } \{i_1, \dots, i_s\} \cap \{j_i, \dots, j_t\} = \varnothing$$

(see Veic's remark [11] on Solomjak's paper).

However, as the class of type p space is strictly larger than the p-uniformly smooth (or G_α-spaces), as James has already shown (see

[8]) with the construction of a non-reflexive B-convex space, Theorem 5 is in fact a generalization of the Warren-Howell result mentioned in the introduction, as the following lemma shows:

LEMMA 6

If E is a p-uniformly space, a sequence (X_n) of mutually orthogonal random variables of $L^p(E)$ is a orthogonal basic sequence in $L^p(E)$ and consequently an unconditional basic sequence.

PROOF

.) According to Giles [4], if E is a smooth Banach space there exists a unique semi-inner product generating the norm (*), and such that $x \perp y$ iff $[y,x] = 0$

..) Warren and Howell have shown that if E is smooth and $x_1, \ldots, x_n$ are non zero mutually orthogonal elements of E, given $\alpha_1, \ldots, \alpha_n \in R$ there exists real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$f_{\alpha_1 x_1 + \ldots + \alpha_n x_n \mid T} = \lambda_1 f_{x_1 \mid T} + \cdots + \lambda_n f_{x_n \mid T} ,$$

f_x being the semi-inner product functional $[., x]$ and $\mid T$ denoting the restriction to the subspace $T = [x_1, \ldots, x_n]$.

...) According to R. C. James, see [7] Theorem 5.7, if E is reflexive, E must also be smooth if there is to be a unique norm-preserving extension of any functional defined in a subspace of E, and reciprocally.

Taking all this into account, if $(x_n) \subset E$ is a sequence of mutually orthogonal elements, E a G_α-space and $i_1, \ldots, i_s, j_1, \ldots, j_t$, positive integers such that

$$\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \varnothing :$$

(*) A semi-inner product in E is a function
$[. , .] : E \times E \to R$ satisfying:
i) $[x + y, z] = [x, z] + [y, z]$; $[\lambda x, y] = \lambda [x, y]$ $\forall \lambda \in R$
ii) $[x, x] > 0$ if $x \neq 0$
iii) $[x, y]^2 \leq [x, x] [y, y]$
iv) $[x, x] = ||x||^2$

$$\|\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}\|^4 = ([\sum_{k=1}^{s}\alpha_{i_k}x_{i_k},\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}] + [\sum_{k=1}^{t}\alpha_{j_k}x_{j_k},\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}])^2 =$$

$$= [\sum_{k=1}^{s}\alpha_{i_k}x_{i_k} + \sum_{k=1}^{t}\alpha_{j_k}x_{j_k},\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}]^2 \leqslant$$

$$\leqslant \|\sum_{k=1}^{s}\alpha_{i_k}x_{i_k} + \sum_{k=1}^{t}\alpha_{j_k}x_{j_k}\|^2 \cdot \|\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}\|^2 ;$$

then

$$\|\sum_{k=1}^{s}\alpha_{i_k}x_{i_k}\| \leqslant \|\sum_{k=1}^{s}\alpha_{i_k}x_{i_k} + \sum_{k=1}^{t}\alpha_{j_k}x_{j_k}\|$$

for any $\alpha_{i_1}, \ldots, \alpha_{i_s}, \alpha_{j_1}, \ldots, \alpha_{j_t} \in R$, where we use the previous remarks applied to $f_{x_{i_1}}, \ldots, f_{x_{i_s}}$ in the first identity.

Therefore (x_n) is an orthogonal basic sequence in E.

To prove the lemma, it is sufficient to apply what we have already explained to the sequence (x_n) of mutually orthogonal random variables of $L^p(E)$, which is a reflexive and smooth space. #

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