ON THE EXISTENCE AND CLASSIFICATION OF Co-H-STRUCTURES

por

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ABSTRACT.

This work has been motivated by [B], where it is shown that $S^3 \bigcup_f e^{2p+1}$ does not admit any homotopy associative comultiplication, when $p \neq 2$ is a prime and f representes a generator of $Z_p \subset \subset \pi_{2p}(S^3)$.

If X is a 1-connected Co-H-space every comultiplication on X induces a loop structure in the set of homotopy classes $[X, Y]$, for all Y.

There is (see, e.g. [G]) an one-to-one correspondence between homotopy classes of comultiplications on X and homotopy classes of coretractions for the evaluation map.

The main result of this paper is the following theorem *Theorem* (4.1). If X is a 1-connected co-H-space then there is an one-to-one correspondence betwen the set of co-H-structures on X and $[X, X \# X]$.

Here $X \# X$ denotes the homotopy fiber of the inclusion map $i = X \vee X \subset X \times X$.

In the third section, an obstruction $\omega \in [X_1, X_2 \# X_2]$ to the primitivity of a map from X_1 into X_2 with respect to two fixed comultiplications is determined. In the same section, equivalent conditions for X_1 or X_2 to admit different comultiplications making a fixed map primitive, are given.

All spaces are assumed to have the homotopy type of connected CW-complexes with base-point.

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1. — Introduction.

A comultiplication on a space X is a map

$$\mu : X \to X \vee X$$

such that $i\mu \simeq \Delta$, where Δ is the diagonal map.

If $\mu \simeq \mu'$, it is clear that μ' is also a comultiplication. The homotopy class of a comultiplication is called a *co-H-structure*.

A homotopy theory on X is a system of binary operations defined in $[X, Y]$, for all Y, accomplishing the folowing conditions.

a) For each Y, the homotopy class of the constant map is a two-sided identity element.

b) For all Y_1 and Y_2 and for each map

$$f: Y_1 \to Y_2$$

the induced function

$$f_* : [X, Y_1] \to [X, Y_2]$$

is a homomorphism of binary systems.

If $\mathcal{B}$ is the category of binary systems and homomorphisms, it is known (see, e.g. [C]) that a space X admite a comultiplication if and only if the functor $[X, -]$ take values in $\mathcal{B}$. In particular if $[X, X \vee X]$ is a binary system, $[i_1] + [i_2]$ is a co-H-structure on X.

A loop M is a binary system such that the equations $a + x = b$, $y + a = b$, have unique solutions $x, y \in M$, for all a, b in M.

The functor $[X, -]$ take values in the category of loops if and only if there are maps

$$l, r: X \to X$$

such that $\nabla (l \vee 1) \mu \simeq 0 \simeq \nabla(1 \vee r) \mu$.

Let $\phi, \psi: X \vee X \to X \vee X$ be maps defined by $\phi i_1 = i_1, \psi i_1 = \mu$ and $\psi i_1 = \mu, \psi i_2 = i_2$. Then ϕ and ψ are homology equivalences.

A comultiplication μ induces loop structure in $[X, Y]$, for all Y, if and only if ϕ and ψ are homotopy equivalences. If X is 1-connected, then every comultiplication on X induces a loop structure in $[X, Y]$, for all Y (see, e.g. [G]).

In what follows we will not distinguish between maps and homotopy classes.

If X is a 1-connected co-H-space, the unique solution in $[X, Y]$ of the equation $x + g = f$ (resp. $g + y = f$) will be denoted by $D(f, g)$ (resp. $D'(f, g)$).

Let ϕ be a map from Y_1 into Y_2, then $\phi D(f, g) = D(\phi f, \phi g)$ and $\phi D'(f, g) = D'(\phi f, \phi g)$.

Furthemore, $D(f_1, f_2) = \nabla (f_1 \vee f_2) D(i_1, i_2)$ and, in this sense, we can think of $D(i_1, i_2)$ as an universal difference.

2. — An exact sequence of loops.

Let ε be the evaluation map

$$\varepsilon : S\,\Omega\, X \to X$$

(i.e. $\varepsilon[t, \omega] = \omega(t)$). We can characterize co-H-spaces in terms of ε, as follows: A space X has a comultiplication if and only if there is a coretraction for ε.

Lemma (2.1). The induced homomorphism

$$f_* : [X, Y_1] \to [X, Y_2]$$

is surjective for all co-H-space X if and only if Ωf admites a coretraction.

Proof: If f_* is surjective for all co-H-space X, in particular for $X = S\,\Omega\, Y_2$ let $\gamma \in [S\,\Omega\, Y_2, Y_1]$ be such that $f\gamma = \varepsilon_2$. Then if $\xi = \operatorname{adj}(\gamma)$, it is clear that $(\Omega f)\,\xi = 1_{\Omega Y_2}$. Conversely, if $(\Omega f)\,\xi = 1_{\Omega Y_2}$ and X is a co-H-space there is γ such that $\varepsilon\gamma = 1$. Then, if $\phi \in [X, Y_2]$ and $\lambda = \operatorname{adj}(\phi\,\varepsilon)$, $\psi = \operatorname{adj}(\varepsilon\,\lambda)$, it is easy to see that $f_*(\psi\gamma) = f\psi\gamma = \phi$. Hence f_* is surjective.

Lemma (2.2). A space X admites comultiplication if and only if the induced map

$$i_* : [X, Y_1 \vee Y_2] \to [X, Y_1 \times Y_2]$$

is surjective, for all Y_1 and Y_2.

Proof: If $\phi \in [X, Y_1 \times Y_2]$ it is clear that

$$i_*[(p_1\phi \vee p_2\phi)\mu] = \phi$$

Conversely, if $Y_1 = Y_2 = X$ and i_* is surjective, then $i_*^{-1}(\Delta) \neq \varnothing$. Every element in $i_*^{-1}(\Delta)$ is a comultiplication.

Let $Y_1 \# Y_2$ be the homotopy fiber of $i: Y_1 \vee Y_2 \subset Y_1 \times Y_2$. We apply the functor $[X, -]$ to the Eckmann-Hilton fiber sequence,

$$\ldots \longrightarrow [X, \Omega(Y_1 \vee Y_2)] \xrightarrow{(\Omega i)_*} [X, \Omega(Y_1 \times Y_2)] \xrightarrow{\partial}$$

$$\xrightarrow{\partial} [X, Y_1 \# Y_2] \xrightarrow{k_*} [X, Y_1 \vee Y_2] \xrightarrow{i_*} [X, Y_1 \times Y_2]$$

If X is a co-H-space, i_* is sujective and Ωi admites a coretraction, then $(\Omega i)_*$ is surjective. Hence $k_*^{-1}(0) = 0$. If X is 1-connected, $[X, Y_1 \# Y_2]$ is a loop and then $k_*^{-1}(0) = 0$ is equivalent to the injectivity of k_*. Furthemore, if $ig = if$, then

$$0 = D(if, ig) = iD(f, g)$$

Thus $D(f, g) \in \operatorname{im} k_*$ and we have proved the following.

Proposition (2.3). If X is a 1-connected co-H-space

$$0 \longrightarrow [X, Y_1 \# Y_2] \xrightarrow{k_*} [X, Y_1 \vee Y_2] \xrightarrow{i_*} [X, Y_1 \times Y_2] \longrightarrow 0$$

is a exact sequence of loops.

In what follows all co-H-spaces will be considered 1-connected.

3. Primitive maps and obstruction.

Let (X_i, μ_i) be co-H-spaces $i = 1, 2$. A map

$$f: X_1 \to X_2$$

is called *primitive* (*co-H-map* or *homomorphism*) with respect to μ_1 and μ_2 if $\mu_2 f = (f \vee f)\mu_1$.

A map f is (μ_1, μ_2)-primitive if and only if the induced function

$$f^*: [X_2, Y] \to [X_1, Y]$$

is homomorphism for all Y.

If f is primitive we have

$$D(\phi f, \psi f) = D(\phi, \psi) f$$

for all $\phi, \psi \in [X_2, Y]$.

If in the exact sequence of Proposition (2.3) we make $Y_1 = Y_2 = X_2$ we obtain

$$0 \longrightarrow [X_1, X_2 \# X_2] \xrightarrow{k_*} [X_1, X_2 \vee X_2] \xrightarrow{i_*} [X_1, X_2 \times X_2] \longrightarrow 0.$$

Let $D(\mu_2 f, (f \vee f) \mu_1)$ be the unique solution in $[X_1, X_2 \vee X_2]$ of the equation $X + (f \vee f) \mu_1 = \mu_2 f$.

We have $i \mu_2 f = \Delta f = (f \times f) \Delta = (f \times f) i \mu_1 = i(f \vee f)\mu_1$ then $iD(\mu_2 f, (f \vee f) \mu_1) = 0$. Hence $D(\mu_2 f, (f \vee f) \mu_1) \in imk_*$.

Since k_* is injective there is an unique element

$$\omega = \omega(f; \mu_1, \mu_2) \in [X_1, X_2 \# X_2]$$

such that $k\omega = D(\mu_2 f, (f \vee f) \mu_1)$. If f is (μ_1, μ_2)-primitive, then $D(\mu_2 f, (f \vee f) \mu_1) = 0$. Hence $\omega = 0$. Conversely, if $\omega = 0$ then f is (μ_1, μ_2)-primitive.

Hence ω is an obstruction for f to be (μ_1, μ_2)-primitive and it is called *co-H-deviation* of f with respect μ_1 and μ_2.

If (X_i, μ_i), $i = 1, 2, 3$ are co-H-spaces and

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

a sequence of maps, then it is easy to prove the following:

Proposition (3.1). a) If f is (μ_1, μ_2)-primitive, then $\omega(gf; \mu_1, \mu_2) = \omega(g; \mu_2, \mu_1) f$

b) If g is (μ_2, μ_1)-primitive, then $\omega(gf; \mu_1, \mu_1) = (g \# g)\, \omega(f; \mu_1, \mu_2)$.

If f is a homotopy equivalence and g its homotopy inverse, we have

$$0 = \omega(1; \mu_1, \mu_1) = \omega(gf; \mu_1, \mu_1) = \omega(g; \mu_2, \mu_1) f$$

then $\omega(g; \mu_2, \mu_1) = 0$. Thus we have proved.

Corollary (3.2). The homotopy inverse of a (μ_1, μ_2)-primitive homotopy equivalence is also (μ_2, μ_1)-primitive.

Let (X_i, μ_i) be, $i = 1, 2$ two co-H-spaces and

$$f: X_1 \to X_2$$

a map not necessarily (μ_1, μ_2)-primitive, then

Proposition (3.3). There is a comultiplication $\hat{\mu}_1$ on X_1 such that f is $(\hat{\mu}_1, \mu_2)$-primitive if and only if $\omega(f; \mu_1, \mu_2) \in im\, (f \# f)_*$

Proof: Let $\hat{\mu}_1 \in i_*^{-1}(\Delta)$ be such that $\omega(f; \hat{\mu}_1, \mu_2) = 0$, then $D(\mu_2 f, (f \vee f)\, \mu_1) = D((f \vee f)\, \hat{\mu}_1, (f \vee f)\, \mu_1) = (f \vee f)\, D(\hat{\mu}_1, \mu_1)$. But $iD(\hat{\mu}_1, \mu_1) = 0$, then there is $\theta \in [X_1, X_1 \# X_1]$ such that $D(\hat{\mu}_1, \mu_1) = k\theta$. Hence $k\omega(f; \mu_1, \mu_2) = (f \vee f)\, k\theta = k(f \# f)\, \theta$. Since k_* is injective, we conclude that $\omega(f; \mu_1, \mu_2) = (f \# f)\, \theta$.

Conversely, if $\omega(f; \mu_1, \mu_2) = (f \# f)\, \theta$, then

$$k\omega = k(f \# f)\, \theta = (f \vee f)\, k\theta$$

Hence $(f \vee f)\, (k\theta + \mu_1) = (f \vee f) k\theta + (f \vee f)\, \mu_1 = \mu_2 f$. If we define $\hat{\mu}_1 = k\theta + \mu_1$, then $i\hat{\mu}_1 = ik\theta + i\mu_1 = \Delta$ and $\hat{\mu}_1$ is a comultiplication on X_1 such that $\omega(f; \hat{\mu}_1, \mu_2) = 0$.

Proposition (3.4). If there is $\hat{\mu}_2 \in i_*^{-1}(\Delta)$ on X_2 such that f is $(\mu_1, \hat{\mu}_2)$-primitive, then

$$\omega(f; \mu_1, \mu_2) \in im\, f^*$$

Proof: Let $\hat{\mu}_2$ be on X_2 such that $\omega(f, \mu_1, \hat{\mu}_2) = 0$, then $k\omega(f; \mu_1, \mu_2) = D(\mu_2 f, \hat{\mu}_2 f) = D(\mu_2, \hat{\mu}_2) f$. On the other hand $iD(\mu_2, \hat{\mu}_2) = 0$, then there is $\theta \in [X_2, X_2 \# X_2]$ such that $D(\mu_2, \hat{\mu}_2) = k\theta$. Hence $k\omega = k\theta f$. Since k_* is injective, $\omega = \theta f$.

In the next proposition, $\dim X$ will denote the infimum of the dimensions of all CW-complexes having the homotopy type of the space X.

Proposition (3.5). Let μ_1 be homotopy associative and $\dim X_2 < \infty$. If $-\omega(f; \mu_1, \mu_2) \in imf^*$, then there is a comultiplication $\hat{\mu}_2$ on X_2 such that $\omega(f; \mu_1, \hat{\mu}_2) = 0$.

Proof: If $-\omega(f; \mu_1, \mu_2) \in imf^*$, then there is $\theta \in [X_2, X_2 \# X_2]$ such that $-\omega = \theta f$. Let $\mu'_2 = k\theta + \mu_2$ be —with the binary operation induced by $\mu_2 -$. It is clear that μ'_2 is a comultiplication on X_2. Then

$$\mu'_2 f = \nabla(k\theta \vee \mu_2)\, \mu_2 f = \nabla(k\theta \vee \mu_2)\, k\omega + (f \vee f)\, \mu_1 .$$

If $\omega_1 = \omega(f; \mu_1, \mu'_2)$, we have $k\omega_1 = \nabla(k\theta \vee \mu_2)\, k\omega = k(k \# 1)(\theta \# \mu_2)\, \omega$. Since $i \nabla k(k \# 1) = 0$, there is a map

$$\alpha: (X_2 \# X_2) \# (X_2 \vee X_2) \to X_2 \# X_2$$

Such that $k\alpha = \nabla k(k \# 1)$. Then $k\omega_1 = k\alpha(\theta \# \mu_2)\, \omega$ and because k_* is injective we have $\omega_1 = \alpha(\theta \# \mu_2)\, \omega$. If $\theta_1 = \alpha(\theta \# \mu_2)\, \theta = \alpha(1 \# \mu_2)(\theta \# 1)\, \theta$ then $\theta_1 f = -\omega_1$ and $-\omega_1 \in imf^*$. Assume we have construded ω_n, θ_n and $\mu_2{}^{n+1} = k\theta_n + \mu_2{}^n$. Then

$$\mu_{n+1} = \alpha(\theta_n \# \mu_2{}^n)\, \omega_n = -\alpha(\theta_n \# \mu_2{}^n)\, \theta_n f$$

$$\theta_{n+1} = \alpha(\theta_n \# \mu_2{}^n)\, \theta_n = \alpha(1 \# \mu^n)(\theta_n \# 1)\, \theta_n = \hat{\alpha}_{n+1}\, \hat{\theta}_{n+1}$$

where $\hat{\alpha}_{n+1} = \alpha(1 \# \mu_2{}^n)(\hat{\alpha}_n \# 1)$ and

$$\hat{\theta}_{n+1} = (\hat{\theta}_n \# 1)\, \theta_n \quad , \quad \hat{\theta}_n : X_2 \to \mathop{\#}_{n+2} X_2$$

Because $\mathop{\#}_{n+2} X_2$ is $(2n+2)$-connected, if $\dim X_2 < 2n+2$ we have $[X_2, \mathop{\#}_{n+2} X_2] = 0$. Hence $\theta_{n+1} = 0$ and $-\omega(f; \mu_1, \mu_2{}^{n+1}) = -\omega_{n+1} = \theta_{n+1} f = 0$. Therefore, if $\hat{\mu}_2 = \mu_2{}^{n+1}$, f is $(\mu_1, \hat{\mu}_2)$-primitive.

4. A theorem of classification.

We have the exact sequence of loops

$$0 \longrightarrow [X, X \# X] \xrightarrow{k_*} [X, X \vee X] \xrightarrow{i_*} [X, X \times X] \longrightarrow 0$$

and let μ be a fixed comultiplication on X. If $i_*{}^{-1}(\Delta)$ denotes the set of co-H-structures then we define two maps

$$\alpha = \alpha_\mu : i^{-1}(\Delta) \to [X, X \# X]$$

such that $\alpha(\hat{\mu},) = \omega(1; \hat{\mu}, \mu)$, and

$$\beta = \beta_\mu : [X, X \# X] \to i_*{}^{-1}(\Delta)$$

such that $\beta(\theta) = D'(\mu, k\theta)$. Note that β is well defined. In fact, $iD'(\mu, k\theta) = D'(i\mu, ik\theta) = D'(\Delta, 0) = \Delta$. Hence $D'(\mu, k\theta) \in i_*^{-1}(\Delta)$.

It is easy to see taht α and β are mutually inverses. Thus we have proved:

Theorem (4.1). If X is a 1-connected co-H-space then there is an one-to-one correspondence between the set of co-H-structures on X and $[X, X \# X]$.

Note that if μ and $\hat{\mu}$ are two comultiplications on X, then there is an unique $\theta \in [X, X \# X]$ such that $\hat{\mu} = \mu + k\theta$. In fact, it is sufficient to fix μ.

Examples. It is well known that if X is a $(n - 1)$-connected space, $n \geqslant 1$, and $\dim X \leqslant 2n - 1$ then X has a (uniques if $\dim X \leqslant 2n - 2$) co-H-structure. The Moore spaces $M(G, n)$, where G is an abelian group and $n \geqslant 2$, are $(n - 1)$-connected. Moreover $\dim M \leqslant n + 1$ and $\dim M = n$ is G is free, because then $M \simeq \bigvee_\alpha S^n$. Hence if $n \geqslant 3$, or $n \geqslant 2$ and G free, then $M(G, n)$ has an unique co-H-structure. In particular, if $G = \mathbf{Z}$ then $M(Z, n) = S^n$ has an unique co-H-structure for all $n \geqslant 2$. For $n = 1$, $[S', S' \# S']$ being a subgroup of $\mathbf{Z} * \mathbf{Z}$ is not finite.

Let X_1 and X_2 be two co-H-spaces such that X_1 is 1-connected and X_2 is $(n - 1)$-connected. If $\dim X_1 \leqslant 2n - 2$ then every map from X_1 into X_2 is primitive with respect to any comultiplication. In fact, the inclusion $i : X \vee X \subset X \times X$ is an $(2n - 1)$-equivalence. Since $\dim X_1 \leqslant 2n - 2$ the induced map i_* is a bijection. Hence $[X_1, X_2 \# X_2] = 0$.

If $X_1 = M(G_1, m)$, $m \geqslant 2$, and $X_2 = M(G_2, n)$, $2 \leqslant m \leqslant 2n - 3$, then every map

$$f : M(G_1, m) \to M(G_2, n)$$

is a primitive map. In particular if $G_1 = G_2 = \mathbf{Z}$ every map

$$f : S^m \to S^n$$

is primitive for $2 \leqslant m \leqslant 2n - 3$.

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