FINITELY GENERATED PROJECTIVE MODULES AND FITTING IDEALS

by

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Through all this paper, A will be denote a commutative ring with identity and M a finitely generated A-module.

The main result gives a characterization of finitely generated projective modules in terms of its Fitting ideals. Namely we shall prove the following:

THEOREM. — A finitely generated A-module M is projective if and only if the Fitting ideals $\mathcal{F}_\nu(M)$, $\nu \geqslant 0$, are of the form Ae_ν where e_ν is an idempotent element of A.

In [4, § 4.4, Theorem 18, Corollary and Theorem 19], D. G. Northcott proves this result under the assumption of finite presentation for M and also proves the part «only if» in the theorem. We shall give here a complete and autonomous proof of it.

1. PRELIMINARY RESULTS

PROPOSITION 1.1. — Let A and M as above and $q \geqslant 0$ an integer. The following conditions are equivalent:

(1) M is free of rank q.

(2) q is the minimum number of generators of M and $\mathcal{F}_\nu(M) = (0)$, $0 \leqslant \nu < q$, $\mathcal{F}_\nu(M) = A$, $\nu \geqslant q$.

Proof. This follows from [4, § 3.1., Theorem 2 and exercice 1].

DEFINITION 1.2. — We shall say that M satisfies condition (FT) if and only if there is an integer $r \geqslant 0$ such that

$$\mathcal{F}_\nu(M) = \begin{cases} (0) & \text{for } 0 \leqslant \nu < r \\ A & \text{for } \nu \geqslant r. \end{cases}$$

(When $r = 0$ this means $\mathcal{F}_\nu(M) = A$, for all $\nu \geqslant 0$).

Remark 1.3. — If M is free of finite rank r, then M satisfies condition (FT). The converse is not true because, as we shall see below, weaker assumptions on M imply (FT).

PROPOSITION 1.4. — Let us assume that A is a local ring with maximal ideal m and M is a above. Then, the following conditions are equivalent:

(1) M is free.

(2) M satisfies condition (FT).

Proof. That (1) implies (2) is trivial. Conversely, assume that

$$\mathcal{F}_\nu(M) = \begin{cases} (0) & \text{for } 0 \leqslant \nu < r \\ A & \text{for } \nu \geqslant r, \end{cases}$$

and let h be the minimum number of generators for M. We have $h \geqslant r$ and therefore, by Proposition 1.1., it is enough to show that $h = r$. Since A is local, if $h > r$ and $\mathbf{m} = \{m_1, \ldots, m_h\}$ is a minimal set of generators for M, the condition $\mathcal{F}_{h-1}(M) = A$ implies that there is a relation

$$\sum_{1 \leq i \leq h} a_i \; m_i = 0,$$

such that some a_i does not belong to m, and so it is a unit in A. Then $\mathbf{m} - \{m_i\}$ is a set of generators for M, which contradicts the minimality of m.

2. Proof of the theorem and consequences

THEOREM 2.1. — A finitely generated A-module M is projective if and only if the Fitting ideals $\mathcal{F}_\nu(M)$, $\nu \geqslant 0$, are of the form $A\, e_\nu$ where e_ν is an idempotent of A.

Proof. We shall use the following criterion for a module to be projective: A finitely generated A-module M is projective if and only if, for every $p \in \mathrm{Spec}(A)$, the A_p-module M_p is free and, if r_p is its rank, the function

$$\varphi: X = \mathrm{Spec}(A) \longrightarrow \mathbf{Z}\ (p \longmapsto r_p)$$

is continuous (see [2, II, § 5.2., Theorem 1]).

Assume that M is projective and let ν be a fixed integer, $\nu \geqslant 0$. Let $\mathcal{F} = \mathcal{F}_\nu(M)$ be the corresponding Fitting ideal of M. For every $p \in X$, one has that $\mathcal{F}_p$ is the $\nu - th$ Fitting ideal of M_p as a A_p-module (see [4, § 3.1, Theorem 3]). Since M_p is free of rank r_p, then $\mathcal{F}_p = (0)$ (resp. $\mathcal{F}_p = A_p$) if $r_p > \nu$ (resp. $r_p \leqslant \nu$). By the continuity of φ the (disjoint) sets

$$X^\nu{}_1 = \{p \in X \mid r_p > \nu\}\ ,$$
$$X^\nu{}_2 = \{p \in X \mid r_p \leqslant \nu\}\ ,$$

form an open covering of X. We must consider several cases:

Case 1. $X^\nu{}_1 = \varnothing$. Then $\mathcal{F}_p = A_p$, for all $p \in X$, and, therefore, $\mathcal{F} = A$, since, if otherwise, it would be contained in some maximal ideal m of A, and so we would have $\mathcal{F}_m \neq A_m$.

Case 2. $X^\nu{}_2 = \varnothing$. In this case, $\mathcal{F}_p = (0)$ for all $p \in X$, whence $\mathrm{Supp}\,(\mathcal{F}) = \varnothing$ and, therefore, $\mathcal{F} = (0)$.

It is obvious that in the above cases $\mathcal{F}_\nu(M)$ is of the form $A\,e_\nu$ with e_ν an idempotent.

Case 3. $X^\nu{}_1 \neq \varnothing$ and $X^\nu{}_2 \neq \varnothing$. In this case, there are two non-zero idempotents e and f, in A, such that $e + f = 1$ and $X^\nu{}_1 = V(A\,e)$, $X^\nu{}_2 = V(A\,f)$, (see [2, II, § 4.3, Proposition 15]). We have relations

$$X^\nu{}_2 = \{p \in X \mid r_p \leqslant \nu\} = \{p \in X \mid \mathcal{F}_p = A_p\} = \{p \in X \mid \mathcal{F} \subsetneq p\} = V(A\,f)\,,$$

$$X^\nu{}_1 = \{p \in X \mid r_p > \nu\} = \{p \in X \mid \mathcal{F}_p = (0)\} = X - X^\nu{}_2 = V(\mathcal{F}) = V(A\,e)\,.$$

From this last equality, we have that $\sqrt{\mathcal{F}} = \sqrt{A\,e.}$ But actually, we can show $\mathcal{F} = A\,e$.

In fact, for every $p \in X^\nu{}_1$, one has $\mathcal{F}_p = (0)$ and also $(A\,e)_p = (0)$ since $f \notin p$ and $e\,.\,f = 0$. Therefore, for every $\tilde{p} \in \mathrm{Spec}\,(A/A\,e)$ the $(A/A\,e)$-module $N = (\mathcal{F} + A\,e)/A\,e$ is such that $N_{\tilde{p}} = (0)$, whence $N = (0)$ and so $\mathcal{F} \subseteq A\,e$. Similary, for every $\tilde{q} \in \mathrm{Spec}\,(A/\mathcal{F})$ the

$(A/\mathcal{F})$-module $N' = (\mathcal{F} + A\,e)/\mathcal{F}$ is such that $N'_{\tilde{q}} = (0)$, so $N' = (0)$ and $A\,e \subseteq \mathcal{F}$. We then have $\mathcal{F} = A\,e$, so the condition in the theorem is necessary.

Conversely, let $\mathcal{F}_\nu(M) = A\,e_\nu$, $\nu \geqslant 0$, where e_ν is an idempotent in A. If h is the minimum number of generators of M, we may assume that $e_\nu = 1$, $\nu \geqslant h$.

For each integer $\nu \geqslant 0$, we have an open covering of X,

$$X = X^\nu{}_1 \cup X^\nu{}_2\,,$$

where

$$X^\nu{}_1 = \{p \in X \mid e_\nu \in p\}\,,$$
$$X^\nu{}_2 = \{p \in X \mid e_\nu \notin p\}\,.$$

On the other hand, if $p \in X$, we have $(A\,e_\nu)_p = (0)$ or $(A\,e_\nu) = A_p$ according as $p \in X^\nu{}_1$ or $p \in X^\nu{}_2$, respectively. Therefore, for each $p \in X$, we have $\mathcal{F}_\nu(M_p) = (0)$ or $\mathcal{F}_\nu(M_p) = A_p$. Since A_p is local, then, by Proposition 1.4, M_p is free and its rank r_p is given by

$$r_p = \min\{\nu \mid \mathcal{F}_\nu(M_p) = A_p\} = \min\{\nu \mid e_\nu \notin p\} \leqslant h\,.$$

Furthermore, the function $\varphi : X \longrightarrow \mathbf{Z}$, $\varphi(p) = r_p$ is continuous, because

$$\mathcal{F}^{-1}(\{\nu\}) = \{p \in X \mid e_\nu \notin p,\, e_{\nu-1} \in p\} = X^\nu{}_2 \cap X^{\nu-1}{}_1\,,\ \nu \geqslant 1\,,$$
$$\mathcal{F}^{-1}(\{0\}) = X^0{}_2\,.$$

Then by the criterion stated above, M is projective. This completes the proof.

Remark 2.2. — If M is projective (and always finitely generated), then the map φ is completely determined by the Fitting ideals $\mathcal{F}_\nu(M)$. In fact, if $\mathcal{F}_\nu(M) = A\,e_\nu$, $\nu \geqslant 0$, then φ is defined by

$$\varphi^{-1}(\{\nu\}) = \{p \in X \mid e_\nu \notin p,\, e_{\nu-1} \in p\}\,,$$

for $\nu \geqslant 1$, and

$$\varphi^{-1}(\{0\}) = \{p \in X \mid e_0 \notin p\}\,.$$

COROLLARY 2.3. — If M satisfies condition (FT), then M is projective.

COROLLARY 2.4. — If $X = \mathrm{Spec}(A)$ is connected, then, for every finitely generated A-module M, the following conditions are equivalent:

(1) M is projective.

(2) M satisfies condition (FT).

PROPOSITION 2.5. — Let M as above. Then M is projective of constant rank if and only if M satisfies condition (FT). In this case, if

$$r = \min \{\nu \mid \mathcal{F}_\nu(M) = A\} ,$$

then r is the arnk of M.

Proof. If M is projective of rank r, then, for every Fitting ideal $\mathcal{F} = \mathcal{F}_\nu(M)$ we have $\mathcal{F}_p = (0)$ (resp. $\mathcal{F}_p = A_p$) for all $p \in \mathrm{Spec}(A)$, if $0 \leqslant \nu < r$ (resp. $\nu \geqslant r$). In the first case, $X^\nu{}_2 = \varnothing$ and then $\mathcal{F}_\nu(M) = (0)$. In the other case, we have $X^\nu{}_1 = \varnothing$ and therefore, $\mathcal{F}_\nu(M) = A$.

Conversely, if M satisfies condition (FT), by the Theorem 2.1. M is projective and, moreover, for every $p \in X$, the A_p-module M_p satisfies condition (FT) for the same r.

COROLLARY 2.6. — $M = (0)$ if and only if $\mathcal{F}_0(M) = A$.

COROLLARY 2.7. — Let A be a semilocal ring and M a finitely generated A-module. Then, the following conditions are equivalent:

(1) M is free.

(2) M satisfies condition (FT).

Proof. We get our result from Proposition 2.5 above plus Proposition 5 of [2, II, § 5.3].

Remark 2.8. — In Proposition 1.1, we actually gave a characterization of the free A-modules of finite rank by means of the condition (FT) and the minimum number of generators h for M. If M satisfies condition (FT) and

$$\mathcal{F}_\nu(M) = \begin{cases} (0) & \text{for } 0 \leqslant \nu < q \\ A & \text{for } \nu \geqslant q, \end{cases}$$

then $h \geqslant q$. Note that, if $h > q$ then M is not free. We give an example to show that this situation can occur.

Example 2.9. — (See [4, Appendix A. Theorem 1]). Let $A = \mathbf{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = \mathbf{R}[x, y, z]$ and $M = A^3/(x, y, z)A$. By the exactness of the sequence

$$0 \longrightarrow (x, y, z)A \longrightarrow A^3 \longrightarrow M \longrightarrow 0,$$

the Fitting ideals of M are

$$\mathcal{F}_\nu(M) = \begin{cases} (0) & \text{for } \nu = 0, 1. \\ A & \text{for } \nu \geqslant 2. \end{cases}$$

Therefore M satisfies condition (FT), $h = 3, q = 2$, and M is not free, A being a noetherian domain.

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