UNIQUENESS OF DIRICHLET, NEUMANN, AND MIXED BOUNDARY VALUE PROBLEMS FOR LAPLACE'S AND POISSON'S EQUATIONS FOR A RECTANGLE

by

J. B. Díaz †
Rensselaer Polytechnic Institute, Troy, N.Y. 12181

R. B. Ram
State University of New York, Oneonta, N.Y. 13820

ABSTRACT: See Title and Introduction. Consider, specifically, for brevity, the boundary value problems with «truly mixed» boundary conditions. A casual reader, misled into a false security by the particular nature of the domain in question, a rectangle (about this circumstance, see under (v) in § 4 of the Concluding Remarks), might, at first blush, get the impression that everything said in this paper has been known for a long time. However, nothing could be farther from the truth than this superficial view. As for uniqueness (that is, the proof that there is at most one "solution", a subject which is dealt with accurately in this paper), it is systematically ignored in the literature, and, where it is not ignored, then the crucial "growth condition at the corners" is generally overlooked. As for existence (that is, the proof that there is at least one true "solution", a subject which is *not* dealt with in this paper), all that one finds in the literature are, largely, *formal* trigonometric, or other, expansions, without any precise proof that a true «solution» has been obtained.

INDEX

1. Introduction.

It is the purpose of the present paper to consider sixteen different boundary value problems for a plane rectangle, and to prove a uniqueness theorem which is stronger than those generally known (to appreciate the nature of the additional conditions, which are usually required in uniqueness theorems for these boundary value problems, attention is called, for example, to the treatment of the Neumann problem, as given in Petrovskii's book [1, pp. 275-277]).

In the uniqueness proof, to be given here, use will be made of the geometry of the particular domain considered, whose boundary is piece-wise rectilinear. The three main tools employed are:

(a) a precise reflection principle for solutions of Laplace's equation, which satisfy Dirichlet or Neumann homogeneous boundary conditions, in a plane domain bounded partially by a straight line (this principle is generally associated with the name of Schwarz);

(b) a theorem on the nature of an allowable isolated singularity for a bounded solution of Laplace's equation (this theorem is usually associated with the name of Riemann);

(c) a theorem to the effect that a solution of Laplace's equation, for all x, y in the plane, which is bounded in absolute value, must be a constant function (this theorem is often referred to as Liouville's theorem for harmonic functions).

The guiding idea behind the uniqueness proof, using these three tools, may be described briefly as follows. First, item (a) is used to extend the definition of a certain harmonic function, from the initial interior of a rectangle, to the entire x, y plane, with the possible exception of a set of "lattice points" (this set of "excepted" lattice points arises from the four vertices of the original rectangle). Secondly, item (b) is used to "remove" the possible singularities of this extended harmonic function at the excepted lattice points. Thirdly, item (c) is used to deduce that the extended harmonic function, which is defined at all finite points of the plane, must be a constant.

The precise nature of these three results is crucial for the proof of the uniqueness theorem to be given here. Therefore, in order to make this paper self-contained, § 2 includes the detailed statements of items (a), (b) and (c) (see Theorems 1, 2, 3 and 4). The proofs of these statements, in the case of items (b) and (c), are readily available, see, for example, Petrovskii [1, pp. 258-259]. However, in the case of the homogeneous Neumann boundary condition in item (a), the proof of the precise result needed here does not seem to be readily accessible: see, for example, Courant and Hilbert [2], p. 272, where the desired result is stated, but not proved, for Laplace's equation in several variables; and Díaz and Ludford [3], where the desired result is stated and proved for linear elliptic second order partial differential equations with constant coefficients in any number of variables, but, under a general "linear" boundary condition, which renders their proof more complicated than the proof in the special "Neumann" boundary condition, two variable, case considered here. Consequently, § 2 also contains a detailed proof of the reflection principle for the Neumann homogeneous boundary condition (see Theorem 2).

Section 3 contains the general uniqueness Theorem 5 for Laplace's equation, and § 4 contains the uniqueness Theorem 6 for boundary value problems for Poisson's equation; remarks on the possible extensions to more general partial differential equations and several independent variables; and the construction of examples in which the uniqueness conclusion of Theorem 5 is not valid, because the function u is not bounded, in absolute value, in the open rectangle.

2. Background.

2(a_1) *Schwarz reflection principle* [4] *("odd reflection principle"), corresponding to the homogeneous Dirichlet boundary condition* $u = 0$.

By this we mean:

THEOREM 1. Let D be a domain (i.e., an open, non-empty, connected set) in a real two dimensional x, y plane. Suppose that the domain D is symmetric about a straight line L, and that d denotes the (supposed non-empty) intersection of the line L and the domain D, while D^+ and D^- designate, respectively, the two open symmetric parts into which $D - d$ is divided by the line L, so that $D = D^+ + d + D^-$. Suppose that $u(x, y)$ is a real, singlevalued, twice continuously differentiable solution of Laplace's equation

$$u_{xx} + u_{yy} = 0, \tag{2.1}$$

in D^+, and that, further,

$$\lim_{(x,y) \to (\bar{x}, \bar{y})} u(x, y) = 0,$$

for (x, y) in D^+ and $(\bar{x}, \bar{y})$ in d. Then, the real valued function U, defined in D by "odd reflection":

$$U(x, y) = \begin{cases} u(x, y) & , \text{ for } (x, y) \text{ in } D^+, \\ 0 & , \text{ for } (x, y) \text{ in } d \;, \\ -u(x^*, y^*), & \text{ for } (x, y) \text{ in } D^-, \end{cases}$$

where the point (x^*, y^*) is the mirror image of the point (x, y) with respect to the line L, is an analytic solution of Laplace's equation (2.1) throughout the domain D.

Let us recall the "well known" proof. Although this proof seems to be "well known", it is worth recalling here, since it differs essentially from the argument given by Schwarz in [4] (because Schwarz goes back to 1869, while "Koebe's converse" dates only from 1906).

Since the partial differential equation (2.1) is invariant under translations and rotations, the line L may be taken to be the x-axis, $y = 0$, without loss of generality, with D^+ lying in the upper half plane $y > 0$, and D^- in the lower half plane $y < 0$. With this specific choice of the line L, the Theorem 1, above, now reads:

Let D be a domain (i.e., an open, non-empty, connected set) in a real two dimensional x, y plane. Suppose that the domain D is symmetric about the x-axis $y = 0$ (i.e., whenever the point (x, y)

belongs to the domain D, then the point $(x, -y)$ also belongs to D; so that, if d denotes the, supposed non-empty, intersection of the x-axis, $y = 0$, and the domain D, while D^+ and D^- designate, respectively, the two open symmetric parts into which $D - d$ is divided by the x-axis $y = 0$, one has $D = D^+ + d + D^-$). Suppose that $u(x, y)$ is a real, singlevalued, twice continuously differentiable solution of Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

in D^+, and that, further

$$\lim_{(x,y) \to (\bar{x}, 0)} u(x, y) = 0$$

for (x, y) in D^+ and $(\bar{x}, 0)$ in d. Then, the real valued function U, defined in D by "odd reflection":

$$U(x, y) = \begin{cases} u(x, y) & , \text{ for } (x, y) \text{ in } D^+, \\ 0 & , \text{ for } (x, 0) \text{ in } d \ , \\ -u(x, -y), & \text{for } (x, y) \text{ in } D^-, \end{cases}$$

where the point $(x, -y)$ is the mirror image of the point (x, y) with respect to the x-axis $y = 0$, is an analytic solution of Laplace's equation (2.1) throughout the domain D.

The function U is continuous on D, and, clearly, possesses the Gauss mean value property, locally, at every point of D. Hence, by the converse of the Gauss mean value theorem (see, for example, Kellogg [5, pp. 224-228]), it follows that $U_{xx} + U_{yy} = 0$ throughout D.

2(a_2) *Reflection principle corresponding to the homogeneous Neumann boundary condition* $\frac{\partial u}{\partial n} = 0$ *("even reflection principle").*

By this we mean:

THEOREM 2. Let D be a domain (i.e., an open, non-empty, connected set) in a real two dimensional x, y plane. Suppose that the domain D is symmetric about a straight line L, and that d denotes the (supposed non-empty) intersection of the line L and the domain D, while D^+ and D^- designate, respectively, the two open symmetric parts into which $D - d$ is divided by the line L, so that $D = D^+ + d + D^-$. Suppose that $u(x, y)$ is a real, singlevalued, twice continuously differentiable solution of Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

in D^+, and that, further,

$$\lim_{(x, y) \to (\bar{x}, \bar{y})} \frac{\partial u}{\partial n}(x, y) = 0,$$

for (x, y) in D^+ and $(\bar{x}, \bar{y})$ in d, where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the outward normal to the line L. Then, the real valued function U, defined in D by "even reflection":

$$U(x, y) = \begin{cases} u(x, y & , \text{ for } (x, y) \text{ in } D^+, \\ \lim\limits_{(x, y) \to (\bar{x}, \bar{y})} u(x, y), & \text{for } (x, y) \text{ in } D^+ \text{ and } (\bar{x}, \bar{y}) \text{ in } d \ , \\ u(x^*, y^*) & , \text{ for } (x, y) \text{ in } D^-, \end{cases}$$

where the point (x^*, y^*) is the mirror image of the point (x, y) with respect to the line L, is an analytic solution of Laplace's equation (2.1) throughout the domain D.

(It should be noticed that the existence of the limit, which occurs in the definition of the function U, is not a part of the hypotheses of the theorem, but must be shown in the course of the argument.)

Proof. Since the partial differential equation (2.1) is invariant under translations and rotations, the line L may be taken to be the x-axis, without loss of generality, with D^+ lying in the upper half plane $y > 0$, and D^- in the lower half plane $y < 0$. With this specific choice of the line L, Theorem 2, above, now reads:

Let D be a domain (i.e., an open, non-empty, connected set) in a real two dimensional x, y plane. Suppose that the domain D is symmetric about the x-axis (i.e., whenever the point (x, y) belongs to the domain D, then the point $(x, -y)$ also belongs to D; so that, if d denotes the, supposed non-empty, intersection of the x-axis and the domain D, while D^+ and D^- designate, respectively, the two open symmetric parts into which $D - d$ is divided by the x-axis, one has $D = D^+ + d + D^-$). Suppose that $u(x, y)$ is a real, single-valued, twice continuously differentiable solution of Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

in D^+, and that, further,

$$\lim_{(x, y) \to (\bar{x}, 0)} \frac{\partial u}{\partial y}(x, y) = 0,$$

for (x, y) in D^+ and $(\bar{x}, 0)$ in d. Then, the real valued function U, defined in D by "even reflection":

$$U(x, y) = \begin{cases} u(x, y) & , \text{ for } (x, y) \text{ in } D^+, \\ \lim\limits_{(x, y) \to (\bar{x}, 0)} u(x, y), & \text{for } (x, y) \text{ in } D^+ \text{ and } (\bar{x}, 0) \text{ in } d \ , \\ u(x, -y) & , \text{ for } (x, y) \text{ in } D_-, \end{cases}$$

where the point $(x, -y)$ is the mirror image of the point (x, y) with respect to the x-axis, is an analytic solution of Laplace's equation (2.1) throughout the domain D.

There are two points of difficulty to watch out for in the proof: (i) the existence of the limit

$\lim_{(x, y) \to (\bar{x}, 0)} u(x, y)$, for (x,y) in D^+, and for each point $(\bar{x}, 0)$ in d,

which appears in the definition of U; (ii) the proof that $U_{xx} + U_{yy} = 0$, whenever $(\bar{x}, 0)$ is in d (a priori, it is not even clear whether the partial derivatives in question exist for a point $(\bar{x}, 0)$ in d, but it is clear that $U_{xx}(x, y) + U_{yy}(x, y) = 0$, whenever the point (x, y) is in either D^+ or D^-).

These two difficulties will be taken care of simultaneously, by showing that, for each point $(x_0, 0)$ of d, there is a square interior, $D_{x_0}(r)$, consisting of all the points (x, y) in D such that

$$x_0 - r \leq x \leq x_0 + r \quad \text{and} \quad -r \leq y \leq r,$$

where r is a sufficiently small positive number; together with a real valued function $W(x, y) \equiv W(x_0, r; x, y)$, defined for (x, y) in $D_{x_0}(r)$, and satisfying Laplace's equation

$$W_{xx}(x, y) + W_{yy}(x, y) = 0, \qquad \text{for } (x, y) \text{ in } D_{x_0}(r),$$

and that, furthermore, the function W has the property that $U(x, y) = W(x, y)$, whenever the point (x, y) is in $D_{x_0}(r)$.

Let $(x_0, 0)$ be a point in d, and choose a positive number r so small that the square interior $D_{x_0}(r)$, with "center" at $(x_0, 0)$, is contained in D. The construction of the function W can be carried out as follows.

First, notice that the function $\frac{\partial u}{\partial y}$ satisfies the hypotheses of Theorem 1, and, hence, the function V, defined by odd reflection:

$$V(x, y) = \begin{cases} u_y(x, y) & , \text{ for } y > 0, \\ 0 & , \text{ for } y = 0, \\ -u_y(x, -y), & \text{for } y < 0, \end{cases}$$

is an analytic solution of Laplace's equation $V_{xx} + V_{yy} = 0$ throughout the domain D. Let $h > 0$ be a real positive number, with $r > h > 0$. The function W, which clearly depends upon the choices of x_0, r, and h, is then defined in $D_{x_0}(r)$ by the following equation

$$W(x, y) = u(x, h) + \int_h^y V(x, t)\, dt,$$

where the integration is the usual Riemann integration.

The function W is certainly analytic in $D_{x_0}(r)$, since the function V is analytic in D. The difficulty (i), mentioned above, will be taken care of once it is shown that

$$W(x, y) = \begin{cases} u(x, y) & , \text{ for } y > 0, \ (x, y) \text{ in } D_{x_0}(r), \\ u(x, -y), & \text{for } y < 0, \ (x, y) \text{ in } D_{x_0}(r). \end{cases}$$

This needed assertion may be verified as follows: clearly, by the fundamental theorem of the Riemann integral calculus,

$$W(x, y) = u(x, h) + \int_h^y u_y(x, t)\, dt = u(x, y),$$

for $y > 0$ and (x, y) in $D_{x_0}(r)$; while, when $y < 0$ and (x, y) belongs to $D_{x_0}(r)$, then (using carefully the definition of V)

$$
\begin{aligned}
W(x, y) &= u(x, h) + \int_h^y V(x, t)\, dt, \\
&= u(x, h) + \int_h^0 u_y(x, t)\, dt + \int_0^{-h} - u_y(x, -t)\, dt + \int_{-h}^{y} - u_y(x, -t)\, dt, \\
&= u(x, h) - \int_0^h u_y(x, t)\, dt + \int_{s=0}^{h} u_y(x, s)\, ds + \int_{s=h}^{-y} u_y(x, s)\, ds, \\
&= u(x, -y)\,.
\end{aligned}
$$

It still remains to take care of the difficulty (ii) mentioned above, and this will complete the proof. It has to be shown that $W_{xx}(x, y) + W_{yy}(x, y) = 0$ for every point (x, y) in $D_{x_0}(r)$, and this can be verified by a direct computation; since, from the definition of the function W, it follows that (for any point (x, y) in $D_{x_0}(r)$):

$$
\begin{aligned}
W_{xx}(x, y) &= u_{xx}(x, h) + \int_h^y V_{xx}(x, t)\, dt, \\
&= u_{xx}(x, h) - \int_h^y V_{yy}(x, t)\, dt\,, \\
&= u_{xx}(x, h) - V_y(x, y) + V_y(x, h), \\
&= u_{xx}(x, h) - V(x, y) + u_{yy}(x, h), \\
&= - V_y(x, y)\,;
\end{aligned}
$$

while, on the other hand, again from the definition of W, one has that

$$W_y(x, y) = V(x, y),$$

and, therefore,

$$W_{yy}(x, y) = V_y(x, y).$$

Finally, by addition of the equations for $W_{xx}(x, y)$ and $W_{yy}(x, y)$, the desired result

$$W_{xx}(x, y) + W_{yy}(x, y) = 0$$

follows.

2 (b) *The isolated singularity theorem.*

By this we mean:

THEOREM 3. Let D be a domain (i.e., an open, non-empty, connected set) in a real two dimensional x, y plane. Suppose that the function $u(x, y)$ is a real, single-valued, twice continuously differentiable solution of Laplace's equation $u_{xx} + u_{yy} = 0$ in $D - \{P_0\}$, where $P_0 = (x_0, y_0)$ is a point of the domain D; and that, further, the absolute value of the function u is bounded on $D - \{P_0\}$, that is to say, there exists a positive real number M such that

$$|u(x, y)| < M, \text{ for any point } (x, y) \text{ in } D - \{P_0\}.$$

Then, the function $u(x, y)$ has a "removable singularity" at the point P_0; that is to say, there is a real number m such that the function v defined by

$$v(x, y) = \begin{cases} u(x, y), & \text{for } (x, y) \text{ in } D - \{P_0\}, \\ m & \text{, for } (x, y) = (x_0, y_0), \end{cases}$$

is twice continuously differentiable, and satisfies Laplace's equation

$$v_{xx} + v_{yy} = 0 \qquad \text{throughout the domain } D.$$

For the proof of this theorem, see, for example, Petrovskii [1, p. 259].

2(c) *Liouville's theorem for harmonic functions.*

By this we mean:

THEOREM 4. If the real valued function $u(x, y)$ is of class $C^{(2)}$ for all real x, y, satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

for all x, y, and is bounded in absolute value for all x, y, then the function u must be a constant.

For the proof of this theorem, see, for example, Petrovskii [1, p. 258].

3. *The Dirichlet problem, the Neumann problem, and the fourteen mixed Dirichlet-Neumann problems.*

In each of the boundary value problems to be considered in this paper, a certain real valued, twice continuously differentiable function, defined on an open plane rectangle, and satisfying Laplace's equation there, will be required to satisfy certain boundary conditions on the boundary of the rectangle: either the function itself, or its normal derivative, will be required to have limit zero. There are four open rectilinear boundary intervals, and two choices of two boundary conditions each; this will give $2^4 = 16$ distinct boundary value problems. The Dirichlet problem arises when the function itself is required to tend to zero on all four open boundary intervals; the Neumann problem arises when the normal derivative of the function is required to vanish on all four open boundary intervals; while the other fourteen "mixed Dirichlet-Neumann" boundary value problems arise whenever the function is required to tend to zero on some (but not all) open boundary intervals, while, at the same time, the normal derivative of the function is required to tend to zero on the remaining open boundary intervals.

All these boundary value problems are considered, simultaneously, in the following:

THEOREM 5. Let D be the interior of a finite plane rectangle, and suppose that the real valued, twice continuously differentiable function u is defined on D, and satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ throughout D. Further, suppose that the function u is bounded in absolute value on D. Let R denote the rectangular boundary of D (R is then the union of four open straight line intervals, call them I_1, I_2, I_3, I_4, plus four vertices). Suppose, further, that the function u satisfies, on each open interval I_i, where $i = 1, 2, 3, 4$, a boundary condition of the following form: either at all points of the open interval I_i one has $\lim u(x, y) = 0$, whenever the point $(x, y) \in D$ tends to a point of the open interval I_i; or else, at all points of the open interval I_i, one has $\lim \frac{\partial u}{\partial n}(x, y) = 0$, whenever the point $(x, y) \in D$

tends to a point of the open interval I_i, where $\frac{\partial}{\partial n}$ denotes the directional derivative in the direction normal to the open interval I_i. Then, unless, for every $i = 1, 2, 3, 4$, the boundary condition on the open interval I_i is $\lim \frac{\partial u}{\partial n}(x, y) = 0$, it can be concluded that the function u must be identically zero on D; while, in the Neumann case (when, for every $i = 1, 2, 3, 4$, the boundary condition on the open interval I_i is $\lim \frac{\partial u}{\partial n}(x, y) = 0$), it can only be concluded that the function u must have a constant value throughout D.

PROOF: The function u is defined, at the outset, only in the rectangular interior D. The idea behind the proof is to extend the function u to a function U, which is defined over the whole plane, and which satisfies the hypotheses of the Liouville Theorem 4; thus forcing U, and hence u, to be a constant. This extension of u to U will be carried out with the help of the reflection principles in Theorems 1 and 2, and the isolated singularity Theorem 3.

For convenience, the original rectangle D will be taken to be simply $0 < x < a, 0 < y < b$; the given function $u(x, y)$ is then defined for $0 < x < a, 0 < y < b$. By applying the reflection Theorem 1, or the reflection Theorem 2, whichever is needed, the function u may be extended to a function which is defined in the open rectangle $0 < x < a, \ -b < y < 2b$, which is three times the size of the original rectangle. This extension of u is bounded in absolute value, since its values on the (open) top and bottom intervals, of the original rectangle, $0 < x < a$, $y = 0$ and $0 < x < a$, $y = b$, are limits of values of the function u in the original rectangle, which are bounded, by hypothesis. Notice that this extended function satisfies the same homogeneous boundary condition (the Dirichlet or Neumann type) on the two open intervals $x = a, \ -b < y < 0$, and $x = a, b < y < 2b$, as the boundary condition satisfied by the original function u on the open interval $x = a, 0 < y < b$ (a similar remark applies to the boundary conditions relative to the straight line $x = 0$). Continuing in this way, "reflecting across the top and bottom sides of this bigger rectangle", and so forth..., one will obtain the harmonic extension of the function u to the whole open infinite strip $0 < x < a, \ -\infty < y < \infty$.

Now, the function which is defined on the infinite strip $0 < x < a$, $-\infty < y < \infty$ can be reflected, across the straight line $x = a$, to a function which is harmonic on the infinite strip $0 < x < 2a$, $-\infty < < y < \infty$ (which is twice as wide as the initial strip just started with), with the *sole* possible exception of the countable set of points (a, nb), where n is an integer, positive, negative or zero. But, from the isolated singularity Theorem 3, it then follows that there are indeed no exceptional points, and that the harmonic extension of the function u has been obtained, in this way, to the entire infinite strip $0 < x < 2a$, $-\infty < y < \infty$. In particular, this means that the harmonic extension of the function u to the whole open infinite strip $0 < x < a$, $-\infty < y < \infty$ satisfies, *at every point* of the straight line $x = a$, the same homogeneous boundary condition as the function u satisfies on the side $x = a$, $0 < y < b$ of the original rectangle (similarly, this harmonic extension of the function u satisfies, *at every point* of the straight line $x = 0$, the same homogeneous boundary condition that was satisfied by the original function u on the side $x = 0$, $0 < y < b$ of the original rectangle).

Proceeding in this way, by reflection across the straight lines $x = ma$, where m is an integer, positive, negative, or zero, one obtains the desired extension U of the function u to the whole x, y plane. Since the function u is bounded in absolute value in the original open rectangle, the extension U, as follows from its construction, will also be bounded in absolute value in the whole x, y plane. By Liouville's Theorem 4, the extension U will be a constant function, hence the original function u will be a constant.

If at least one of the boundary conditions satisfied by the function u is the Dirichlet boundary condition, that $\lim u(x, y) = 0$ on an open side of the original rectangle, then it follows that the constant value of the function u must be zero (this will be the case in the Dirichlet and the fourteen "mixed Dirichlet-Neumann" boundary value problems for the rectangle). If the four boundary conditions satisfied by the function u on the four open sides of the original rectangle are all of Neumann type (that is, the normal derivatives of the function u tend to zero), then it can not, of course, be asserted that the constant value of the function u must necessarily be zero.

4. *Concluding remarks.*

(I) It is clear that the uniqueness Theorem 5 in § 3, which is concerned with Laplace's equation $u_{xx} + u_{yy} = 0$, readily implies a uniqueness theorem for a rectangle for Poisson's equation $u_{xx} + u_{yy} = F(x, y)$, with appropriate "limiting boundary" conditions, of the form $\lim u = f$ and $\lim \frac{\partial u}{\partial n} = g$, where F, f and g are given functions (no continuity of any sort is required of these real valued functions). All that one has to do is to apply the uniqueness Theorem 5 of § 3 to the difference $u = u_1 - u_2$, where the functions u_1 and u_2 are assumed to be solutions of one and the same boundary value problem for Poisson's equation, to obtain a uniqueness theorem for Poisson's equation. In this way, one obtains the following theorem.

THEOREM 6. *Hypothesis.* Let D be the interior of a finite plane rectangle, and let R denote the rectangular boundary of D (R is then the union of four open straight line intervals, call them I_1, I_2, I_3, I_4, plus four vertices). Let F be a real valued function defined on the interior D, and, for each $i = 1, 2, 3, 4$, let f_i be a real valued function defined on the open interval I_i. Suppose that the real valued, twice continuously differentiable function u, is defined on D, and satisfies Poisson's equation $u_{xx} + u_{yy} = F(x, y)$ throughout D. Further, suppose that the function u is bounded in absolute value on D. Suppose, still further, that the function u satisfies, on each open interval I_i, where $i = 1, 2, 3, 4$, a boundary condition of the following form: Either at all points of the open interval I_i, one has $\lim u(x, y) = f_i(P)$, whenever the point (x, y) in D tends to a point P of the open interval I_i; or else, at all points of the open interval I_i, one has $\lim \frac{\partial u}{\partial n}(x, y) = f_i(P)$, whenever the point (x, y) in D tends to a point P of the open interval I_i, where $\frac{\partial}{\partial n}$ denotes the directional derivative in the direction normal to the open interval I_i.

Conclusion. Then, unless, for every $i = 1, 2, 3, 4$, the boundary condition on the open interval I_i is $\lim \frac{\partial u}{\partial n}(x, y) = f_i(P)$, it can be concluded that there is at most one such function u; while, in the Neumann case, when, for every $i = 1, 2, 3, 4$, the boundary condition on the open interval I_i is $\lim \frac{\partial u}{\partial n}(x, y) = f_i(P)$, it can only be concluded that, if u_1 and u_2 satisfy all the conditions imposed on u, then their difference, $u_1 - u_2$, must have a constant value throughout D.

(II) It seems clear that extensions of the uniqueness Theorem 5 are possible, in various directions: to several independent variables, and to more general partial differential equations. To mention just one possible result, Laplace's equation $u_{xx} + u_{yy} = 0$ could be replaced by the Helmholtz equation $u_{xx} + u_{yy} - \lambda^2 u = 0$, using the results of Díaz and Ludford [3]. It is planned to consider these possible generalizations, in more detail, elsewhere.

(III) In a paper concerned with Dirichlet, Neumann, and mixed boundary value problems for a rectangle, but for the (hyperbolic) wave equation $u_{xx} - u_{yy} = 0$, Abdul-Latif and Díaz [6] have given a number of uniqueness theorems. It was in connection with some correspondence involving this paper of Abdul-Latif and Díaz [6] that Professor J. Barkley Rosser, in a letter, called our attention to the related uniqueness questions for the same boundary value problems, but for the (elliptic) Laplace's equation $u_{xx} + u_{yy} = 0$, and communicated to us that he was in the process of writing up his uniqueness results [7].

(IV) It is of interest to give examples in which the uniqueness conclusion of Theorem 5 is not valid, because the function u is not bounded, in absolute value, in the open rectangle D. It is easy to construct such an example for the Dirichlet boundary conditions, making use of the previously known fact that the Dirichlet problem for given continuous boundary values can be solved. Intuitively, using the terminology of plane hydrodynamics, all that has to be done is to consider the stream function, in the rectangle, due to four suitable dipoles situated at the vertices of the rectangle, and to adjust this stream function, by subtracting from it a certain solution of the Dirichlet problem (in order that the resulting difference function have zero Dirichlet data at every point of the boundary of the rectangle which is not a vertex). Consider the complex number $z = x + iy \neq 0$, where x and y are real. Basically, all that is being used is the fact that, when $z \neq 0$ is either real or purely imaginary, then the imaginary part of $\frac{1}{z^2}$ is zero. Notice that the imaginary part of $\frac{1}{z^2}$ equals $\frac{-2xy}{(x^2+y^2)^2}$.

Explicitly, consider, for (x, y) in the rectangle, the function

$$v_1(x,y) = -2\frac{xy}{(x^2+y^2)^2} + \frac{(x-a)y}{[(x-a)^2+y^2]^2} + \\ + \frac{x(y-b)}{[x^2+(y-b)^2]^2} + \frac{(x-a)(y-b)}{[(x-a)^2+(y-b)^2]^2},$$

which is just the stream function of four dipoles situated at the vertices of the rectangle. Notice that the function v_1 approaches continuous boundary values at all points of the boundary R of the rectangle, save at the vertices, and that it is unbounded in the rectangle, "near the vertices". Furthermore, the continuous boundary values of the function v_1 can easily be extended to a function which is continuous on the entire boundary of the rectangle, *including* the vertices; let the real, single valued, function v_2 be the solution of the Dirichlet problem, for the plane rectangle D, corresponding to these continuous boundary values. Then, the difference function $u = v_1 - v_2$ satisfies all the hypotheses of Theorem 5, save for the fact that it is not bounded.

In this example, use was made of the previous knowledge of the existence of the solution of the Dirichlet problem, for the plane rectangular domain, for given continuous boundary data. It seems clear that, proceeding in a similar way, for the other fifteen boundary value problems, one would require a knowledge of the existence theorem for each of the boundary value problems in question. To avoid having to rely on a previously proved existence theorem, one could perhaps proceed, in a manner reminiscent of similar constructions in the theory of elliptic functions, by placing suitable dipoles (or other suitable singularities) at the lattice points (na, mb), where $m, n = 0, \pm 1, \pm 2, \ldots$, which are "generated" by the four vertices of the rectangle. This is just an indication of one possible way of approaching the problem of the explicit construction of examples for the other fifteen boundary value problems.

(v) It may appear to the reader, upon first thinking about the problem, that the uniqueness theorem of the type given here, for a very particular domain, like a rectangle, must surely be contained, as a very special case, in the literature concerning what are called "discontinuous boundary value problems" (where, for example, the unknown function is assigned on the part of the boundary of the domain and the normal derivative of the function is assigned on the other part of the boundary). The standard reference for such problems for Laplace's equation is the book of G. C. Evans [8]. However, a study of this work of Evans, and of several of his later papers (see [9]), having to do with what he called "discontinuous Dirichlet and Neumann problems in the plane", will reveal that the theorems given here are not special cases of his results.

Still another remark is to be made about the fact that the rectangle appears, at first glance, to be a very special domain, and that one has the uneasy feeling that one would rather like to have a result about "a more general open set". But, this can be taken care of, in a nearly obvious way. All one has to do is to consider all the open sets which one obtains by "mapping conformally" the closed rectangle by means of a single valued function which is analytic (in the sense of "an analytic function of a single complex variable") in a domain containing the closed rectangle in its interior. It is not worthwhile to state here in detail separately the resulting theorem for these "open sets".

(VI) It is clear that the condition, in Theorem 5, that the function u be bounded in absolute value over the whole open rectangle, can be replaced by a much weaker condition. Because, this boundedness condition is only used in the proof for points which are in the immediate neighborhood of the vertices of the rectangle (when the apparent singularities, which are the vertices, are "removed", using what was called "Riemann's removable singularity theorem"). All that one has to do is to use a stronger form of the removable singularity theorem, which, instead of boundedness of the absolute value of the function in the neighborhood of the "singular point", merely requires that the absolute value of the function, at points near the "singular point", is bounded by a positive constant times the absolute value of the logarithm of the Euclidean distance, measured from the "singular point" (see Petrovskii [1], pp. 260-261). The point being made here is that, instead of assuming that the function u is bounded in absolute value in the open rectangle, one need only assume that, speaking informally, "near each vertex of the rectangle, the absolute value of the function u grows at most like a positive constant, depending on the vertex, times the absolute value of the logarithm of the Euclidean distance, measured from the vertex in question". Only a slight modification of the proof of the Theorem 5, as given above, is required. All that one has to do is to proceed as follows. The first step of the proof is as before: the original function, defined on the given rectangle, is extended, by a reflection process, to a function defined on a rectangle which is nine times as large, but this extended function has possible "singularities" at the vertices of the original rectangle; these apparent singularities must then be "removed" at once, before proceeding further with the reflection process which gives a function defined over the whole plane, and to which Liouville's theorem is applicable.

REFERENCES

[1] I. G. PETROVSKII, *Partial Differential Equations*, English Edition 1967, W. B. Saunders Company-Philadelphia.

[2] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. II, Intercience Publishers, New York 1962.

[3] J. B. DÍAZ and G.S.S. LUDFORD, *Reflection Principles for Linear Elliptic Second Order Partial Differential Equations with Constant Coefficients*. Ann. Mat. Pura Appl., Serie IV. Tomo XXXIX (1955), 87-95.

[4] H. A. SCHWARZ, *Ueber einige Abbildungsaufgaben*. J. Reine Angew. Math. *70*: 105-120, 1869. (See also "Gesammelte Mathematische Abhandlungen", Berlin, 1890, Vol. II, pp. 65-83, specially pages 66-67.)

[5] O. D. KELLOGG, *Foundations of Potencial Theory*. Dover Publications Inc., New York, 1953.

[6] A. I. ABDUL-LATIF and J. B. DÍAZ, *Dirichlet, Neumann, and Mixed Boundary Value Problems for the Wave Equation* $u_{xx} - u_{yy} = 0$ *for a Rectangle*. Applicable Anal. *1* (1971), 1-12.

[7] J. BARKLEY ROSSER, *unpublished manuscript*.

[8] G. C. EVANS, *The Logarithmic Potential, **Discontinuous Dirichlet and Neumann Problems*** Amer. Math Soc. Colloquium Publications, Vol. VI, American Mathematical Society, New York, 1927.

[9] G. C. EVANS, *Modern Methods of Analysis in Potential Theory*, Bull. Amer. Math. Soc. *43* (1937), 418-502.