$l_{\Phi,\varphi}$ OPERATORS AND (Φ, φ)-SPACES

by

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INTRODUCTION

A new class of linear and bounded operators between normed spaces is introduced. This class is more general than the classes of operators from [4], [5], [8].

Using this class $l_{\Phi,\varphi}$ is also introduced a class of locally convex spaces wich is more general than the classes of the nuclear spaces [2], [3] and φ-nuclear spaces [6].

For this class of operators are established similar properties to the properties of the well known classes l_p, l_φ, l_Φ [3], [5], [8] and also the stability of the tensor product is proved.

The stability of the tensor product is also proved for the (Φ, φ)-spaces.

1. PRELIMINARIES

Throughout the paper E, F denote normed spaces and $L(E, F)$ the space of bounded linear operators of E into F.

For $T \in L(E, F)$ the approximations numbers are defined in the following way [4], [9].

a) $\alpha_n(T) = \inf\limits_{K} \|T - K\| \quad , \quad \dim K \leqslant n$

b) $\delta_n(T) = \inf\limits_{F_n} \{\delta > 0 \mid T U_E \subset \delta U_F + F_n\}$, $U_{\bar{E}} = \{x \in E \mid \|x\| \leqslant 1\}$,
$U_F = \{y \in F \mid \|y\| \leqslant 1\}$, $\quad F_n \subset F \quad , \quad \dim F_n \leqslant n$

c) $d_n(T) = \inf\limits_{E_n} \{\|T|_{E_n}\|\} \quad , \quad \operatorname{codim} E_n \leqslant n \quad , \quad E \subset E$

($T|_{E_n}$ is the restriction of T to E_n) .

Let c_0 be the space of all scalar sequences converging to o and let $\hat{c}$ be the subspace of c_0 wich contains the finite rank sequences ($x \in \hat{c}$ if $x = \{x_1, x_2, ..., x_n, 0, 0, ...\}$, $n < \infty\}$.

The norm functions Φ (norming functions) [7] are defined in the following way

$\Phi : \hat{c} \to R_+$; $\Phi(x) = 0$ iff $x = \{0, 0, ...\}$; $\Phi(x + y) \leqslant \Phi(x) + \Phi(y)$, $x, y \in \hat{c}$;

$\Phi(\lambda x) = |\lambda| \Phi(x)$, $\lambda \in R$, $x \in \hat{c}$; $\Phi(1, 0, 0, ...) = 1$; $\Phi(x_1, ..., x_n, ...) = \Phi(|x_{i_1}|, ..., |x_{i_n}|, ...)$,

where $i_1, i_2, ... i_n$ is a permutation of $1, 2, ..., n$ (*)

The functions φ are introduced in [5] in the following way

$\varphi : R_+ \to R_+$; $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$; $\varphi(0) = 0$; $\varphi(x) < \varphi(y)$ iff $x < y$

and φ is continuous.

2. $l_{\Phi,\varphi}$ OPERATORS

Definition 2.1. *The operator* $T \in l_{\Phi,\varphi}(E, F)$ *if* $\Phi\{(\varphi(\alpha_n(T)))_n\} < \infty$

Replacing the elements $\alpha_n(T)$ by the elements $\delta_n(T)$ or $dn(T)$ result the classes $\tilde{l}_{\Phi,\varphi}(E, F)$ and respective $\bar{l}_{\Phi,\varphi}(E, F)$.

Remark 2.1. If $\varphi(x) = x$ and $\Phi(x) = \Phi_p(x) = (\sum |x_i|^p)^{\frac{1}{p}}$, $p \geqslant 1$ result the classes l_p, $\tilde{l}_p$, $\bar{l}_p$ [4] and if $\Phi(x) = \Phi_1(x) = \sum |x_i|$ results the class $l\varphi$ [5].

Proposition 2.1. $l_{\Phi,\varphi}(E, P)$ *is a linear quasinormed space with the quasinorm* $||T||_{\Phi\ \varphi} = \Phi\{\varphi(\alpha_n(T))\}$.

Proof. Let be $T_1, T_2 \in l_{\Phi,\varphi}$ ($\Phi\{\varphi((\alpha_n(T_k))\{ < \infty$, $k = 1, 2$).

$$\Phi(\varphi(\alpha_n(T_1 + T_2))) \leqslant 2\,\Phi(\varphi(\alpha_{2n}(T_1 + T_2))) \leqslant 2\,\Phi(\varphi(\alpha_n(T_1)) + \varphi(\alpha_n(T_2)) \leqslant 2\{\Phi(\varphi(\alpha_n(T_1))) + \Phi(\varphi(\alpha_n(T_2)))\} < \infty \ (**)$$

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*) Fo the others sequences $x \notin \hat{c}$, $\Phi(x) = \sup_n \Phi(x^1, x_2, ..., x_n, 0, 0, ...)$

**) For the properties of the elements α_n, δ_n, d_n see [4]. Since the properties of $\delta_n(T)$, $d_n(T)$ and $\alpha_n(T)$ are similar, the propositions 2.1, 2.2, 2.3 are true for $\tilde{l}_{\Phi,\varphi}$, $\bar{l}_{\Phi,\varphi}$.

Hence $T_1 + T_2 \in l_{\Phi,\varphi}$ and $||T_1 + T_2||_{\Phi,\varphi} \leqslant 2(||T_1||_{\Phi,\varphi} + ||T_2||_{\Phi,\varphi})$. If $\lambda \in R$ and $T \in l_{\Phi,\varphi}(E, P)$ results

$$||\lambda T||_{\Phi,\varphi} = \Phi(\varphi(\alpha_n(\lambda T))) = \Phi(\varphi(|\lambda|\alpha_n(T))) \leqslant [|\lambda|]\,||T||_{\Phi,\varphi},$$
$$([|\lambda|] = \inf\{n \in N \mid |\lambda| \leqslant n\}).$$

Hence $\lambda T \in l_{\Phi,\varphi}(E, P)$.

Proposition 2.2. *If* $T_1 \in L(E, F)$ *and* $T_2 \in l_{\Phi,\varphi}(F, G)$ *then* $T_2 T_1 \in l_{\Phi,\varphi}(E, G)$ and $||T_2 T_1||_{\Phi,\varphi} \leqslant [||T_2||] \cdot ||T_2||_{\Phi,\varphi}$.

Proof.

$$||T_2 T_1||_{\Phi,\varphi} = \Phi(\varphi(\alpha_n(T_2 T_1))) \leqslant \Phi(\varphi(\alpha_0(T_1) \cdot \alpha_n(T_2))) \leqslant [||T_1||] \cdot \Phi(\varphi(\alpha_n(T_2))).$$

Proposition 2.3. *If F is a Banach space, $l_{\Phi,\varphi}(E, F)$ is complete.*

Proof. Let be $\{T_n\}$ a Cauchy sequence in $l_{\Phi,\varphi}(E, F)$.

$(\forall\ \varepsilon > 0, \exists\ N(\varepsilon)$ such that $||T_n - T_m||_{\Phi,\varphi} < \varepsilon$ if $n, m > N(\varepsilon))$.

Since $\varphi(\alpha_0(T_n - T_m)) \leqslant \Phi(\varphi(\alpha_i(T_n - T_m))) < \varepsilon$, results $||T_n - T_m|| < \bar{\varepsilon}$ if $n, m > N(\varepsilon)$ $(\bar{\varepsilon} = \varphi^{-1}(\varepsilon))$.

Hence $\{T_n\}$ is a Cauchy sequence in $\mathsf{L}(E, F)$ and hence $\lim\limits_{n\to\infty} T_n = T \in \mathsf{L}(E, F)$.

From the continuity of the functions Φ, φ and the continuity of the elements $\alpha_i(T)$ results

$$\lim_n \Phi(\varphi(\alpha_i(T_n - T_m))) = \Phi(\varphi(\alpha_i(T_n - T))) < \varepsilon \text{ if } n > N(\varepsilon).$$

Hence $\lim\limits_n T_n = T$ in the topology of the space $l_{\Phi,\varphi}$ and $T \in l_{\Phi,\varphi}(E, F)$

Using the methods from [5] and [3] results

Proposition 2.4. *The set of all finite rank operators is dense in* $l_{\Phi,\varphi}(E, F)$ if $\lim\limits_n \Phi(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+2}, \ldots) = 0$.

3. Stability of the tensor product

Let be $E_1 \otimes_\theta E_2$ the tensor product of the spaces E_1, E_2, endowed wirh the tensor norm $\theta(\varepsilon \leqslant \theta \leqslant \pi[3])$ and let $T_1 \otimes_\theta T_2$

be the tensor product operator $(T_1 \otimes_\theta T_2 : E_1 \otimes_\theta E_2 \to F_1 \otimes_\theta F_2;$ $T_i : E_i \to F_i \ (i = 1, 2))$.

Lemma 3.1. *The approximation numbers satisfy the following relations*

a) $\alpha_{m \cdot n}(T_1 \otimes_\theta T_2) \leqslant \alpha_m(T_1)\,||T_2|| + 2||T_1||\alpha_n(T_2)$

b) $\delta_{m \cdot n}(T_1 \otimes_\theta T_2) \leqslant \delta_m(T_1)\,\delta_n(T_2) + \delta_m(T_1)\,||T_2|| + ||T_1||\,\delta_n(T_2).$

c) $d_{m \cdot n}(T_1 \otimes_\theta T_2) \leqslant d_m(T_1)\,||T_2|| + d_n(T_2)\,||T_1||.$

Proof.

a) $\alpha_{m \cdot n}(T_1 \otimes T_2) \leqslant \inf\, ||T_1 \otimes T_2 - K_1 \otimes K_2||,$
$\dim K_1 \leqslant m, \dim K_2 \leqslant n.$

Hence

$$\begin{aligned}\alpha_{m \cdot n}(T_1 \otimes T_2) &\leqslant \inf_{K_1, K_2} ||(T_1 - K_1) \otimes T_2 + K_1 \otimes (T_2 - K_2)|| \\ &\leqslant \inf \{||T_1 - K_1|| \cdot ||T_2|| + ||K_1|| \cdot ||T_2 - K_2||\} \\ &\leqslant (\alpha_m(T_1) + \varepsilon_1)\,||T_2|| + (||T_1|| + \alpha_m(T_1) + \varepsilon_1) \\ &\quad (\alpha_n(T_2) + \varepsilon_2)\ ^{(*)}\end{aligned}$$

Or

$$\alpha_{m \cdot n}(T_1 \otimes T_2)) \leqslant \alpha_m(T_1) \cdot ||T_2|| + 2||T_1||\,\alpha_n(T_2).$$

b) For $\delta_1 > \delta_m(T_1)$ and $\delta_2 > \delta_n(T_2)$ are F_m, F_n such that $T_1 U_{E_1} \subset \delta_1 U_{F_1} + F_m$ and $T_2 U_{E_2} \subset \delta_2 U_{F_2} + F_n$, $\dim F_m \leqslant m$, $\dim F_n \leqslant n$.

Hence, for $x_1 \in U_{E_1}$, $x_2 \in U_{E_2}$, we can find $y_m \in F_m$, $y_n \in F_n$; $\bar{x}_1 \in U_{E_1}$, $\bar{x}_2 \in U_{E_2}$ with

$$T_1 x_1 = y_m + \delta_1 \bar{x}_1 \text{ and } T_2 x_2 = y_n + \delta_2 \bar{x}_2$$

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*) $||K_1|| = ||K_1 - T_1 + T_1|| \leqslant ||T_1|| + ||T_1 - K_1|| \leqslant ||T_1|| + \alpha_m(T_1) + \varepsilon_1$ $(\alpha_m(T_1) \leqslant ||T_1||)$.

Then

$T_1 \otimes T_2(\varkappa_1 \otimes \varkappa_2) - \mathfrak{I}_m \otimes \mathfrak{I}_n = (T_1 \varkappa_1 - \mathfrak{I}_m) \otimes T_2 \varkappa_2 + \mathfrak{I}_m \otimes (T_2 \varkappa_2 - \mathfrak{I}_n) = \delta_1 \bar{x}_1 \otimes T_2 \varkappa_2 + \mathfrak{I}_m \otimes \delta_2 \bar{x}_2$.

Since $||\mathfrak{I}_m|| \leqslant ||T_1|| + \delta_1$ results $||\delta_1 \bar{x}_1 \otimes T_2 x_2 + \mathfrak{I}_m \otimes \delta_2 \bar{x}_2||_\theta \leqslant \delta_1 ||T_2|| + (||T_1|| + \delta_1) \cdot \delta_2$

Hence

$$T_1 \otimes T_2(\varkappa_1 \otimes x_2) - \mathfrak{I}_m \otimes \mathfrak{I}_n \in (\delta_1 ||T_2|| + (||T_1|| + \delta_1) \delta_2) U_{E_1 \otimes_\theta E_2}.$$

Or

$$(T_1 \otimes T_2) U_{E_1 \otimes_\theta E_2} \subset F_m \otimes_\theta F_n + [\delta_1 ||T_2|| + (||T_1|| + \delta_1) \delta_2] U_{E_1 \otimes_\theta E_2}.$$

Thus results $\delta_{m \cdot n}(T_1 \otimes T_2) \leqslant \delta_m(T_1) \cdot \delta_n(T_2) + \delta_m(T_1) \cdot ||T_2|| + ||T_1|| \cdot \delta_n(T_2)$.

c) $\delta_{m \cdot n}(T_1 \otimes T_2) \leqslant \inf\limits_{M_1, M_2} \{||T_1 \otimes T_2|_{E_1 \otimes E_2 \ominus M_1 \otimes M_2}||\}$, $\dim M_1 \leq m$, $\dim M_2 \leq n$.

Hence

$$\begin{aligned} dm \cdot n(T_1 \otimes T_2) &\leqslant \inf_{M_1, M_2} \{||T_1 \otimes T_2|_{(E_1 \ominus M_1) \otimes E_2 \oplus M_1 \otimes (E_2 \ominus M_2)}||\} \\ &\leqslant \inf_{M_1, M_2} \{||T_1|_{E_1 \ominus M_1} \otimes T_2|| + ||T_1|_{M_1} \otimes T_2|_{E_2 \ominus M_2}||\} \\ &\leqslant dm(T_1) \cdot ||T_2|| + ||T_1|| \, dn(T_2). \end{aligned}$$

From these relations results

Proposition 3.1. *The classes* $l_{\Phi,\varphi}$, $\tilde{l}_{\Phi,\varphi}$, *and* $\bar{l}_{\Phi,\varphi}$, *are stable for the tensor product.* $(\Phi\{\varphi(\alpha_{n^2}(T_1 \otimes T_2)\} < \infty$ if $\Phi\{\varphi(\alpha_n(T_K)\} < \infty$, $K = 1,2)$.

This results from the fact that (for the $l_{\Phi,\varphi}$ class)

$$\Phi(\varphi(\alpha_{m \cdot n}(T_1 \otimes T_2)) \leqslant [||T_2||] \cdot ||T_1||_{\Phi,\varphi} + 2[||T_1||] \, ||T_2||_{\Phi,\varphi} < \infty$$

4. (Φ, φ)-SPACES

Let E be a locally convex separated space. We only consider absolutely convex neighbourhoods (of zero) in E. The gauge function p_U of a neighbourhood U in E induces a norm on $E/p^{-1}(0)$ [3].

Let E_U be this normed space and let $K_U : E \to E_U$ be the canonical map.

If $V \subset U$ is a second neighbourhood in E then we have a well-defined continuous map $K_{U,V} : E_V \to E_U$ such that $K_{UV} \circ K_V = K_U$.

Definition 4.1. *Let* $\mathcal{U}$ *be a o-basis in* E *(basis of the filter of the neighbourhoods of zero). If we can find, for each* $U \in \mathcal{U}$ a $V \in \mathcal{U}$ *such that* $V > U$ *and* $K_{UV} \in l_{\Phi,\varphi}(E_U, E_V)$, *then we will call* E a (Φ, φ)*-space.*

Remark. If $\Phi(x) = \sum |x_i|$ results the φ-nuclear spaces [6] and if $\varphi(t) = t$ and $\Phi(x) = \Phi_p(x)$ results the nuclear spaces [3].

The (Φ, φ) spaces and φ-nuclear spaces [6] possess similar proporties.

In this paper we insist on the stability of the tensor product.

Let E, F be locally convex spaces and U, V neighbarhods of zero in E and F resp. (U^0, V^0 are their polars [3]).

On $E \otimes F$ we define the families of seminorms $W_{p,U,V}(Z)$ in the following way

$$W_{p,U,V}(Z) = \inf \{\sup_{x' \in U_0} (\sum |\langle x_i, x' \rangle|^p)^{\frac{1}{p}} \cdot \sup_{y' \in V^0} (\sum |\langle y_i, y' \rangle|^q)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where the infimum is taken over all representations $Z = \sum x_i \otimes y_i \in E \otimes F$. For the normed spaces E, F ire consider $E \otimes_{W_p} F$, where

$$W_p(Z) = \inf \{\sup_{\|x'\| \leq 1} (\sum |\langle x_i, x' \rangle|^p)^{\frac{1}{p}} \sup_{\|y'\| \leq 1} (\sum |\langle y_i, y' \rangle|^q)^{\frac{1}{q}} [1], Z = \sum x_i \otimes y_i \in E \otimes F.$$

We can prove the following proposition

Proposition 4.1. *If* E, F *are two* (Φ, φ)*-spaces, then* $E \otimes_{W_{p,U,V}} F$ *is a* (Φ, φ)*-space.*

This results from the proposition 3.1 and the following Lemma.

Lemma 4.1. *Let be* E, F *locally convex separated spaces. Then* $(E \otimes_{W_{p,U,V}} F)_{\tilde{U}}$ *is isometric with* $E_U \otimes_{W_p} F_V$, *where* $\tilde{U} = \{Z \in E \otimes F \mid W_{p,U,V}(z) \leqslant 1\}$.

Proof.

Let be $K_U : E \to E_U$, $K_V : F \to F_V$. We show that.

$$p_{\tilde{U}}(Z) = ||K_U \otimes K_V(Z)||_{W_p}, \quad Z \in E \otimes F.$$

The dual $(E_U)'$ can be identified with E'_{U° [3] by means of the cores-pondence $u' \to x'$ defined by

$$\langle K_U x, u' \rangle = \langle x, x' \rangle \quad , \quad x \in E \quad , \quad x' \in E'_{U^\circ} .$$

In the same way we identify $(F_V)'$ and F'_{V°.

Thus

$$p\tilde{v}(Z) = \inf \{\sup_{x' \in U^\circ} (\Sigma |\langle x_i, x' \rangle|^p)^{\frac{1}{p}} \cdot \sup_{y' \in V^\circ} (\Sigma |\langle y_i, y' \rangle|^q)^{\frac{1}{q}} =$$

$$= \inf \{\sup_{\|u'\| \leq 1} (\Sigma |\langle K_U x_i, u' | \rangle|^p)^{\frac{1}{p}} \cdot \sup_{\|V'\| \leq 1} (\Sigma |\langle K_V y_i, v' | \rangle|^q \}^{\frac{1}{q}} =$$

$$= \|\sum_i K_U x_i \otimes K_V y_i\|_{W_p} = \|K_U \otimes K_V (Z)\|_{W_p} .$$

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