THE DIRICHLET BOUNDARY VALUE PROBLEM FOR THE SYSTEM $u_{x_i x_j} = 0,\ j \neq i$

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ABSTRACT

It is well known that the Dirichlet problem for hyperbolic equations is a classical «not well posed» problem. In this note we extend to n-dimension a 2-dimensional theorem that was done by Fritz John [1]. We treat uniqueness of solution of the Dirichlet boundary value problem for the system of hyperbolic equations $u_{x_i x_j} = 0$; $j \neq i$; $i, j = 1, \ldots, n$ in a closed domain which is a closed and bounded convex set in n-dim. such that any line parallel to the x_i-axis $\forall\, i = 1, \ldots, n$ will intersect the boundary in at most two points.

INTRODUCTION

Hadamard [2, 3] rejected the Dirichlet problem as unsuitable for hyperbolic equations. Bourgin and Duffin [4], and Fox and Pucci [5] treated the Dirichlet and Neumann problems for $u_{xx} - u_{yy} = 0$ for a rectangle in standard position. John [1] treated the Dirichlet problem for the equation $u_{xy} = 0$. The author and Diaz [6] treated the Dirichlet, Neumann and many mixed Dirichlet-Neumann boundary value problems including a «general mixed problem» where, at each point which is not a corner, the boundary condition is either of Dirichlet or of Neumann type. Dunninger and Zachmanoglou [7] treated uniqueness of solution of the Dirichlet problem for the equation $u_{x_1 x_1} + \ldots + u_{x_n x_n} - u_{tt} = 0$ in coordinate rectangles.

In this paper we consider the problem mentioned in the abstract above.

DEFINITIONS AND THEOREM

Our closed domain that we use will be denoted by G and it is a closed and bounded convex set in n-dim. such that any line parallel to the x_i-axis $\forall\ i = 1, \dots, n$, will intersect the boundary in at most two points.

We denote the boundary of G by bd.

We remark that in our domain the general solution of $u_{x_1 x_2} = 0$ is $u(x_1, x_2) = f_1(x_1) + f_2(x_2)$ and that the general solution for the system

$$u_{x_i x_j} = 0;\ j \neq i;\ i, j = 1, 2, \dots, n$$

$$\text{is } u(x_1, x_2, \dots, x_n) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n)$$

Now take $u(x_1, \dots, x_n) = f_1(x_1) + \dots + f_n(x_n)$ defined on G, where $f_i(x_i)$ is continuous on $G\ \forall\ i = 1, \dots, n$. We define a function v on the boundary as follows:

$$v(x_1, \dots, x_n) = u(x_1, \dots, x_n), \quad \forall\, p = (x_1, \dots, x_n) \in bd, \text{ so that}$$

$$v(x_1, \dots, x_n) = f_1(x_1) + \dots + f_n(x_n), \quad \forall\, p = (x_1, \dots, x_n) \in bd \text{ and}$$

$$v(p) = f_1(p) + \dots + f_n(p) \quad \forall\, p = (x_1, \dots, x_n) \in bd.$$

We define the following transformations that map the boundary into the boundary: Let $p = (x_1, x_2, \dots, x_i, \dots, x_n) \in bd$.

$$A_i: (x_1, x_2, \dots, x_i, \dots, x_n) \to (x_1, x_2 \dots, x_i', \dots, x_n) \in bd,\ i = 1, 2, \dots, n.$$

These transformations A_i are natural ones, since a line parallel to the x_i-axis will intersect the boundary in at most two points. So that if a line parallel to the x_i-axis passes through the point $p \in bd$, then $A_i(p) \in bd$ is also on this line. Of course if the line intersects the boundary at one point $p \in bd$ only, then $A_i(p)$. This is the same for all $i = 1, 2,, \dots, n$.

We define the mapping $T(p)$ on the boundary as:

$$T(p) = \prod_{i=1}^{n} A_i(p) = A_n A_{n-1} \dots A_2 A_1(p) = A_n(A_{n-1} \dots (A_2(A_1(p)))).$$

The sequence $p, T(p), T^2(p), \dots, T^n(p), \dots$ will be called $\lambda(p)$, where $p \in bd$.

We will say that a function $u(x_1, x_2, \ldots, x_n)$ is a solution of the Dirichlet problem for the boundary values v, if

(a) $u(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$

where $f_i(x_i)$ is continuous for $a_i \leqslant x_i \leqslant b_i \quad \forall\, i = 1. 2. \ldots, n.$

(b) $u(x_1, x_2, \ldots, x_n) = v \quad \forall\, (x_1, x_2, \ldots, x_n) \in bd.$

THEOREM:

The solution of the Dirichlet problem is uniquely determined if either one of the following two conditions is satisfied:

(i) If there exists $p \in bd$ such that $\lambda(p)$ is dense on the boundary;

(ii) If for every two points $p, q \in bd$, the intersection of the sets of limit points of $\lambda(p)$ and $\lambda(q)$ is non-empty.

PROOF:

Let $v(p) = 0 \quad \forall\, p \in bd,$

$$u(x_1, \ldots, x_n) = f_1(x_1) + \ldots + f_n(x_n),\ f_i(x_i) \text{ continuous}$$

$$v(x_1, \ldots, x_n) = f_1(x_1) + \ldots + f_n(x_n),\ p = (x_1, \ldots, x_n) \in bd$$

and $v(p) = f_1(p) + \ldots + f_n(p),\ p \in bd$

We have the following,

$$T(p) = \prod_{i=1}^{n} A_i(p)$$

$$f_i(p) = f_i\Big(\prod_{j \neq i} A_j(p)\Big) \text{ for any number of } j\text{'s}$$

and $f_i(A_i(p)) = f_i(T(p)) = f_i\big(A_i \prod_{j \neq i} A_j(p)\big)$ for any number of j's

Let $h_i = u - f_i$ everywhere, h_i is continuous, since u and f_i are continuous.

Now $\forall\, p \in bd$, we have

$$h_i(p) - h_i(T(p)) = \sum_{\substack{j \neq i \\ j = 1}}^{n} [f_j(p) - f_j(T(p))] = \sum_{j \neq i}^{n} [f_j(p) - f_j(A_j(p))]$$

and

$$v(p) - v\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j(p)\Big) = \Bigg(\sum_{\substack{j \neq i \\ j=1}}^{n} f_j(p) + f_i(p)\Bigg) -$$

$$-\Bigg(\sum_{\substack{j \neq i \\ j=1}}^{n} f_j\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j(p)\Big) + f_i\Big(\prod_{\substack{j \neq j \\ j=1}}^{n} A_j(p)\Big)\Bigg)$$

$$= \sum_{\substack{j \neq i \\ j=1}}^{n} f_j(p) - \sum_{\substack{j \neq i \\ j=1}}^{n} f_j\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j(p)\Big), \text{ since } f_i(p) = f_i\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j(p)\Big)$$

$$= \sum_{\substack{j \neq i \\ j=1}}^{n} f_j(p) - \sum_{\substack{j \neq i \\ j=1}}^{n} f_j(A_j(p))$$

$$= \sum_{\substack{j \neq i \\ j=1}}^{n} [f_j(p) - f_j(A_j(p))] = h_i(p) - h_i(T_i(p)).$$

With the help of iteration we get,

$$h_i(p) - h_i(T^2(p)) = [h_i(p) - h_i(T(p))] + [h_i(T(p)) - h_i(T^2(p))]$$

$$= v(p) - v\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j(p)\Big) + v(T(p)) - v\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j T(p)\Big)$$

$$= \sum_{k=0}^{1} \Big[v(T^k(p)) - v\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j T^k(p)\Big)\Big]$$

By induction get for all integers $n > 0$,

$$h_i(p) - h_i(T^n(p)) = \sum_{k=0}^{n-1} \Big[v(T^k(p)) - v\Big(\prod_{\substack{j \neq i \\ j=1}}^{n} A_j T^k(p)\Big)\Big]$$

i.e., $h_i(p) - h_i(T^n(p)$ may be written in terms of v for the points of the boundary. But $v = 0$ on the boundary, hence $h_i(p) = h_i(T^n(p))$ $\forall$ integers $n > 0$, $p \in bd$. Hence h_i has the same value for all the members of the sequence $\lambda(p)$.

Now if $\lambda(p)$ is dense on the boundary then h_i is constant on the boundary, since h_i is continuous.

If for every two doundary points p, q, the set of limit points of the two sequences $\lambda(p)$, $\lambda(q)$ have a non-empty intersection, then clearly again h_i is constant on the whole boundary.

Now $f_i = u - h_i$, and on the boundary we have

$f_i = v - h_i = 0 - h_i = - h_i$, since $v = 0$ on the boundary.

But h_i is constant on the boundary, hence f_i is constant on the boundary.

Now to show that f_i is constant everywhere, we let $f_i(p) = c$, $p \in bd$. Then let $q = (x_1, x_2, \ldots, x_i, \ldots, x_n)$ be an interior point and take a line through q parallel to the x_j-axis (any $j \neq i$), then this line will intersect the boundary at some point $p = (x_1, x_2, \ldots, x_j', \ldots, x_i, \ldots, x_n)$ on the boundary whose ith coordinate is the same. Hence $f_i(p) = f_i(q)$, but since $p \in bd$ then $f_i(p) = c$, and hence $f_i(q) = c$ Hence f_i is constant everywhere.

The same can be done for all $i = 1, 2, \ldots, n$ to show that f_i is constant for all $i = 1, 2, \ldots, n$. Hence $u(x_1, x_2, \ldots, x_n)$ is constant everywhere. But $u = 0$ on the boundary. Hence $u(x_1, x_2, \ldots, x_n) \equiv 0$.

REFERENCES

[1] FRITZ JOHN, *The Dirichlet problem for the hyperbolic equation*, J. Amer. Math. Assoc. 63, No 1. Jan, 1941, 141-154.

[2] J. HADAMARD, *On some topics connected with linear partial differential equations*, Proc. Benares Math. Soc. 3 (1921), 33-48.

[3] J. HADAMARD, *Equations aux derivés partielles*, L'Enseignement Mathématique 35 (1936), 5-42.

[4] D. G. BOURGIN and R. DUFFIN, *The Dirichlet Problem for a Vibrating String Equation*, Bull. Amer. Math. Soc. 45 (1939), 851-858.

[5] D. W. FOX and G. PUCCI, *The Dirichlet Problem for the Wave Equation*, Annali Di Matematica pura ed applicata (IV), XLVI, 155-182.

[6] ALI I. ABDUL-LATIF and J. B. DIAZ, *Dirichlet, Neumann and Mixed Boundary value Problems for the Wave Equation* $u_{xx} - u_{yy} = 0$ *for a Rectangle*, Applicable Analysis, Vol. 1, (1971) 1-12.

[7] D. R. DUNNINGER and E. C. ZACHMANOGLOU, *The Condition for Uniqueness of solutions of the Dirichlet Problem for the Wave Equation in Coordinate Rectangles*, Journal of Math. Analysis and App., Vol. 20, No. 1, (1967), 17-21.