CONVOLUTION AND S'-CONVOLUTION OF DISTRIBUTIONS

by

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Abstract. In the first section of this paper we prove a characterization of integrable distributions which shows that the two definitions for the convolution of distributions given by L. SCHWARTZ and W.S. WLADIMIROW are equivalent. In the second section we consider the corresponding situation for the S'-convolution which was introduced by Y. HIRATA and H. OGATA. We then give an example of two tempered measures whose convolution is a non-tempered measure. This answers a question of R. SHIRAISHI.

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0. *Introduction and Notations.*

For spaces of functions and distributions we use the notations of HORVATH [4]. In particular, $\mathcal{B}_0(\mathbf{R}^n)$ denotes the space of all C^∞-functions defined on $\mathbf{R}^n$ which together with all their derivatives vanish at infinity. The topology on $\mathcal{B}_0(\mathbf{R}^n)$ is defined by the sequence of norms $\varphi \longmapsto p_m(\varphi) := \max\{|\partial^\alpha \varphi(x)|;\, x \in \mathbf{R}^n,\, |\alpha| \leqslant m\}$ $(m \in \mathbf{N}_0)$. The topological dual $\mathcal{B}_0(\mathbf{R}^n)'$ of $\mathcal{B}_0(\mathbf{R}^n)$ is the space of all *integrable distributions* which is denoted $\mathcal{D}'_{L^1}$ by SCHWARTZ [9, Ch. VI, § 8, p. 200]. For $\varphi \in \mathcal{E}(\mathbf{R}^n)$ we denote by φ^Δ the function belonging to $\mathcal{E}(\mathbf{R}^{2n})$ defined by $(x, y) \longmapsto \varphi(x+y)$.

L. SCHWARTZ gave the following definition for the convolution of two distributions.

(0.1) *Definition* (SCHWARTZ [8, Exposé nº 22, p. 2]). Two distributions $S, T \in \mathcal{D}(\mathbf{R}^n)'$ are said to be *convolvable* if $\varphi^\Delta (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$. The convolution $S * T \in \mathcal{D}(\mathbf{R}^n)'$ is then defined by $\langle S * T, \varphi \rangle := \iint S(x)\ T(y)\ \varphi(x + y)\ dx\ dy := \langle \varphi^\Delta (S \otimes T), \mathbf{1} \rangle$ $(\varphi \in \mathcal{D}(\mathbf{R}^n))$.

It was shown by HORVATH [5, p. 185 et seq.] that the above definition extends the usual definition of convolution between $\mathcal{E}(\mathbf{R}^n)'$ and $\mathcal{D}(\mathbf{R}^n)'$, $\mathcal{O}_C(\mathbf{R}^n)'$ and $\mathcal{S}(\mathbf{R}^n)'$, $L^p(\mathbf{R}^n)$ and $L^q(\mathbf{R}^n)$ $(p^{-1} + q^{-1} \geqslant 1)$.

W.S. WLADIMIROW defined the convolution of two distributions by means of special approximate units.

(0.2) A sequence $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^n)$ will be called an *approximate unit* if $(\eta_k; k \in \mathbf{N})$ converges to $\mathbf{1}$ in $\mathcal{E}(\mathbf{R}^n)$ and $\{\eta_k; k \in \mathbf{N}\}$ is bounded in $\mathcal{B}_0(\mathbf{R}^n)$. $(\eta_k; k \in \mathbf{N})$ will be called a *special approximate unit* if in addition the following holds: For every compact set $K \subset \mathbf{R}^n$ there is $k(K) \in \mathbf{N}$ such that $\eta_k(x) = 1$ for all $x \in K$, $k \geqslant k(K)$.

It is easy to see that for every $\varphi \in \mathcal{B}_0(\mathbf{R}^n)$ and every (special) approximate unit $(\eta_k; k \in \mathbf{N})$ the sequence $(\eta_k \cdot \varphi; k \in \mathbf{N})$ converges to φ in $\mathcal{B}_0(\mathbf{R}^n)$.

(0.3) *Definition* (WLADIMIROW [14, Ch. II, § 7, p. 101-102]). Two distributions $S, T \in \mathcal{D}(\mathbf{R}^n)'$ are said to be *convolvable* if for every special approximate unit $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^{2n})$ and every $\varphi \in \mathcal{D}(\mathbf{R}^n)$ the scalar sequence $(\langle S \otimes T, \eta_k \cdot \varphi^\Delta \rangle; k \in \mathbf{N})$ converges. The convolution $S * T \in \mathcal{D}(\mathbf{R}^n)'$ is then defined by $\langle S * T, \varphi \rangle := \lim_{k\to\infty} < S \otimes T, \eta_k \cdot \varphi^\Delta > (\varphi \in \mathcal{D}(\mathbf{R}^n))$.

The convergence of $(\langle S \otimes T, \eta_k \cdot \varphi^\Delta \rangle; k \in \mathbf{N})$ for every special approximate unit $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^{2n})$ and every $\varphi \in \mathcal{D}(\mathbf{R}^n)$ implies that the limit does not depend on the choice of the sequence $(\eta_k; k \in \mathbf{N})$. It follows from the sequential completeness of $\sigma(\mathcal{D}(\mathbf{R}^n)', \mathcal{D}(\mathbf{R}^n))$ that $\varphi \longmapsto \lim_{k\to\infty} \langle S \otimes T, \eta_k \cdot \varphi^\Delta \rangle$ is again a distribution.

We didn't use different notations for the above defined convolutions since the proposition in the next section will show that the two definitions are equivalent.

1. *A characterization of integrable distributions.*

(1.1) *Proposition.* For a distribution $R \in \mathcal{D}(\mathbf{R}^n)'$ the following are equivalent:

(a) R is continuous for the topology induced by $\mathcal{B}_0(\mathbf{R}^n)$ on $\mathcal{D}(\mathbf{R}^n)$.
(b) There exists $m \in \mathbf{N}_0$ such that for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbf{R}^n$ with the property: $\varphi \in \mathcal{D}(\mathbf{R}^n)$, $\operatorname{supp}(\varphi) \cap K = \varnothing \Rightarrow |\langle R, \varphi \rangle| \leqslant \varepsilon \cdot p_m(\varphi)$.
(c) For every approximate unit $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^n)$ the sequence $(\langle R, \eta_k \rangle; k \in \mathbf{N})$ is convergent.
(d) For every special approximate unit $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^n)$ the sequence $(\langle R, \eta_k \rangle; k \in \mathbf{N})$ is convergent.
(e) There exist a compact set $K \subset \mathbf{R}^n$, $C > 0$ and $m \in \mathbf{N}_0$ such that $|\langle R, \varphi \rangle| \leqslant C \cdot p_m(\varphi)$ holds for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$ satisfying $\operatorname{supp}(\varphi) \cap K = \varnothing$.

Proof.

(a) $\Rightarrow$ (b): From (a) we obtain the existence of $m \in \mathbf{N}_0$ and $C > 0$ such that $|\langle R, \varphi \rangle| \leqslant C \cdot p_m(\varphi)$ holds for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$. Assume there exists $\varepsilon_0 > 0$ such that for every compact set $K \subset \mathbf{R}^n$ there is $\varphi_K \in \mathcal{D}(\mathbf{R}^n)$ satisfying $\operatorname{supp}(\varphi_K) \cap K = \varnothing$ and $\langle R, \varphi_K \rangle > \varepsilon_0 \cdot p_m(\varphi_K)$. Then we find inductively an increasing sequence $(K_l; l \in \mathbf{N})$ of compact subsets of $\mathbf{R}^n$ and a sequence $(\varphi_l; l \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^n)$ satifying $\operatorname{supp}(\varphi_l) \subset \mathring{K}_{l+1} \setminus K_l$, $p_m(\varphi_l) = 1$ and $\langle R, \varphi_l \rangle > \varepsilon_0$ for all $l \in \mathbf{N}$. We set $\psi_k := \sum_{l=1}^{k} \varphi_l$ $(k \in \mathbf{N})$. Since the functions φ_l $(l \in \mathbf{N})$ have disjoint supports we obtain $p_m(\psi_k) = 1$ and thus $|\langle R, \psi_k \rangle| \leqslant C$ for all $k \in \mathbf{N}$. On the other hand we have $\langle R, \psi_k \rangle > k \cdot \varepsilon_0 \to \infty$ $(k \to \infty)$ which contradicts the above estimate.

(b) $\Rightarrow$ (c): We choose $\theta \in \mathcal{D}(\mathbf{R}^n)$ satisfying $\theta(x) = 1$ in a neighbourhood of $0 \in \mathbf{R}^n$ and set $\theta_r(x) := \theta(x/r)$ $(x \in \mathbf{R}^n, r > 0)$. From the LEIBNIZ formula we obtain that for every $j \in \mathbf{N}_0$ there exists $C(\theta, j) > 0$ such that $p_j((1 - \theta_r) \cdot \varphi) \leqslant C(\theta, j) \cdot p_j(\varphi)$ holds for all $\varphi \in \mathcal{B}_0(\mathbf{R}^n)$ and all $r \geqslant 1$. Now let $(\eta_k; k \in \mathbf{N})$ be an approximate unit, take $m \in \mathbf{N}_0$ according to (b) and set $M := 4 \cdot \sup\{p_m(\eta_k); k \in \mathbf{N}\}$. Let $\varepsilon > 0$ be given. By hypothesis (b) there exists a compact set $K \subset \mathbf{R}^n$ such that $|\langle R, \varphi\rangle| \leqslant (M \cdot C(\theta, m))^{-1} \cdot \varepsilon \cdot p_m(\varphi)$ holds for all $\varphi \in D(\mathbf{R}^n)$ satisfying $\operatorname{supp}(\varphi) \cap K = \varnothing$. Now we choose $r \geqslant 1$ such that $\theta_r(x) = 1$ holds in a neighbourhood of K. We then have

$$|\langle R, (1-\theta_r)\cdot(\eta_k - \eta_l)\rangle| \leqslant (M \cdot C(\theta, m))^{-1} \cdot \varepsilon \cdot p_m((1-\theta_r)\cdot(\eta_k - \eta_l)) \leqslant M^{-1} \cdot \varepsilon \cdot (p_m(\eta_k) + p_m(\eta_l)) \leqslant \varepsilon/2$$

for all $k, l \in \mathbf{N}$.

Since $(\eta_k; k \in \mathbf{N})$ converges in $\mathcal{E}(\mathbf{R}^n)$ and $\theta_r \cdot R$ has compact support, there is $k_0 \in \mathbf{N}$ such that
$|\langle R, \theta_r \cdot (\eta_k - \eta_l)\rangle| = |\langle \theta_r \cdot R, \eta_k - \eta_l\rangle| \leqslant \varepsilon/2$ holds for all $k, l \geqslant k_0$. Putting these two estimates together we obtain the convergence of $(\langle R, \eta_k\rangle; k \in \mathbf{N})$.

(c) ⇒ (d) is obvious.

(d) ⇒ (e): Assume (e) is false and set $K_m := \{x \in \mathbf{R}^n; |x| \leqslant m\}$ $(m \in \mathbf{N})$. Then there exists a sequence $(\varphi_m; m \in \mathbf{N})$ in $D(\mathbf{R}^n)$ satisfying $\operatorname{supp}(\varphi_m) \cap K_m = \varnothing$ $(m \in \mathbf{N})$ such that $|\langle R, \varphi_m\rangle| > m^2 \cdot p_m(\varphi_m)$ holds for all $m \in \mathbf{N}$. We set $\psi_m := (m \cdot p_m(\varphi_m))^{-1} \cdot \varphi_m$ $(m \in \mathbf{N})$. This sequence satisfies $\operatorname{supp}(\psi_m) \cap K_m = \varnothing$ and $|\langle R, \psi_m\rangle| > m$ $(m \in \mathbf{N})$, and for every $k \in \mathbf{N}$ the estimate $p_k(\psi_m) \leqslant m^{-1}$ holds for all $m \geqslant k$.

Now let $(\eta_k; k \in \mathbf{N})$ be a special approximate unit in $D(\mathbf{R}^n)$. Then the sequence defined by $\tilde{\eta}_k := \eta_k + \psi_k$ $(k \in \mathbf{N})$ is again a special approximate unit in $D(\mathbf{R}^n)$, and we obtain
$|\langle R, \tilde{\eta}_k\rangle - \langle R, \eta_k\rangle| = |\langle R, \psi_k\rangle| > k$ $(k \in \mathbf{N})$. Thus at least one of the sequences $(\langle R, \tilde{\eta}_k\rangle; k \in \mathbf{N})$ and $(\langle R, \eta_k\rangle; k \in \mathbf{N})$ is not convergent.

(e) ⇒ (a): It follows from the Leibniz formula that for every $\theta \in B(\mathbf{R}^n)$ and every $m \in \mathbf{N}_0$ there exists $M(\theta, m) > 0$ such that $p_m(\theta \cdot \varphi) \leqslant M(\theta, m) \cdot p_m(\varphi)$ holds for all $\varphi \in B_0(\mathbf{R}^n)$. Now take K, C and m according to (e). We choose an open relatively compact neighbourhood U of K and a function $\eta \in D(\mathbf{R}^n)$ such that $\operatorname{supp}(\eta) \subset U$ and $\eta(x) = 1$ holds in a neighbourhood of K. Since $R|D(U)$ is of finite order there exist $k \in \mathbf{N}$, $k \geqslant m$, and $C_1 > 0$ such that $|\langle R, \psi\rangle| \leqslant C_1 \cdot p_k(\psi)$ for all $\psi \in D(U)$. Using (e) we obtain

$$\begin{aligned}|\langle R, \varphi\rangle| &\leqslant |\langle R, (1-\eta)\cdot\varphi\rangle| + |\langle R, \eta\cdot\varphi\rangle| \\ &\leqslant C \cdot p_m((1-\eta)\cdot\varphi) + C_1 \cdot p_k(\eta\cdot\varphi) \\ &\leqslant (C \cdot M(1-\eta, m) + C_1 \cdot M(\eta, k)) \cdot p_k(\varphi)\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$. Thus R is continuous for the topology induced by $\mathcal{B}_0(\mathbf{R}^n)$ on $\mathcal{D}(\mathbf{R}^n)$. (Since $\mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{B}_0(\mathbf{R}^n)$, R has a unique extension $R_0 \in \mathcal{B}_0(\mathbf{R}^n)'$.)

(1.2) *Remarks.*

(a) As indicated by SCHWARTZ [8, Exposé n° 21, p. 2] and HORVATH [5, p. 184] an alternative proof of the implications (1.1a) $\Rightarrow$ (1.1b) and (1.1a) $\Rightarrow$ (1.1c) respectively may be given by using the representation theorem for integrable distributions (cf. HORVATH [4, Ch. 4, § 6, p. 347]). On the other hand (1.1b) may be used instead of the representation theorem to prove that the strong bidual $(\mathcal{B}_0(\mathbf{R}^n)'', \beta(\mathcal{B}_0'', \mathcal{B}_0'))$ of $\mathcal{B}_0(\mathbf{R}^n)$ is topologically isomorphic to $\mathcal{B}(\mathbf{R}^n)$ provided with the topology defined by the sequence of norms $(p_m;\ m \in \mathbf{N}_0)$.

Property (1.1b) is similar to condition (M) of BOURBAKI [1, § 5, n° 2, Prop. 5, p. 58].

(b) Let $\eta \in \mathcal{D}(\mathbf{R}^n)$ satisfy $\eta(x) = 1$ in a neighbourhood of $0 \in \mathbf{R}^n$. Then the sequence defined by $\eta_k(x) := \eta(x/k)$ $(x \in \mathbf{R}^n, k \in \mathbf{N})$ is a special approximate unit. Additionally this sequence has the following property:

$$(*) \begin{cases} \text{For every } \alpha \in \mathbf{N}_0^n \text{ there is } C_\alpha > 0 \text{ such that} \\ (1 + |x|^2)^{|\alpha|/2} \cdot |\partial^\alpha \eta_k(x)| \leqslant C_\alpha \text{ holds for all } x \in \mathbf{R}^n, k \in \mathbf{N}. \end{cases}$$

Let us note, however, that it is not possible to replace the special approximate units in (1.1d) by the smaller class of all special approximate units which in addition satisfy the uniform growth condition (*). For example the distribution generated by $f(x) := (\sin x)/x$ $(x \in \mathbf{R})$ does not belong to $\mathcal{B}_0(\mathbf{R})'$ although for every approximate unit $(\eta_k;\ k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R})$ which satisfies (*), the sequence $\langle f, \eta_k \rangle = \int f(x) \cdot \eta_k(x)\, dx$ $(k \in \mathbf{N})$ is convergent.

(c) Conditions (1.1a), (1.1b) and (1.1e) remain equivalent if $\mathbf{R}^n$ is replaced by an open subset $\Omega \subset \mathbf{R}^n$: The proofs of (1.1a) $\Rightarrow$ (1.1b) and (1.1e) $\Rightarrow$ (1.1a) given above apply also to this case, and the implication (1.1b) $\Rightarrow$ (1.1e) is obvious. (d) Using (1.1a) $\Leftrightarrow$ (1.1c) and B. LEVI's theorem we obtain $\mathcal{B}_0(\mathbf{R}^n)' \cap L^1_{\mathrm{loc}}(\mathbf{R}^n)^+ = L^1(\mathbf{R}^n)^+$. In general this equality is not true with $\mathbf{R}^n$ replaced by an open subset

$\Omega \subsetneq \mathbf{R}^n$ as the following example shows: $f(x) := \log x$ on $\Omega := (0,1) \subset \mathbf{R}$; then we obtain $\partial f \in \mathcal{B}_0(\Omega)' \cap L^1_{\mathrm{loc}}(\Omega)^+$ and $\partial f \notin L^1(\Omega)$.

(e) An application of the equivalence (1.1a) $\Leftrightarrow$ (1.1d) to the distributions $\varphi^\Delta \cdot (S \otimes T)$ $(\varphi \in \mathcal{D}(\mathbf{R}^n))$ shows that the Definitions (0.1) and (0.3) are equivalent.

In the following theorem we collect the equivalent conditions for the existence of the convolution of two distributions we know of.

(1.3) *Theorem.* For two distributions $S, T \in \mathcal{D}(\mathbf{R}^n)'$ the following are equivalent:

(a) $(S * \psi) \cdot (\check{T} * \varphi) \in L^1(\mathbf{R}^n)$ for all $\varphi, \psi \in \mathcal{D}(\mathbf{R}^n)$.

(b) $\varphi^\Delta \cdot (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$.

(c) $S(\hat{x} - \hat{y})\, T(\hat{y})$ is partially summable with respect to y (cf. SCHWARTZ [10, § 5, p. 130]).

(d) $S \cdot (\check{T} * \varphi) \in \mathcal{B}_0(\mathbf{R}^n)'$ for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$.

(e) For every $a > 0$ there exist $C > 0$ and $m \in \mathbf{N}_0$ such that $|\langle S \otimes T, \varphi\rangle| \leqslant C \cdot p_m(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbf{R}^{2n})$ satisfying supp $(\varphi) \subset \{(x, y) \in \mathbf{R}^{2n};\ |x + y| \leqslant a\}$.

(f) For every approximate unit $(\eta_k;\ k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^{2n})$ and every $\varphi \in \mathcal{D}(\mathbf{R}^n)$ the sequence $(\langle S \otimes T,\ \eta_k \cdot \varphi^\Delta\rangle;\ k \in \mathbf{N})$ is convergent.

(g) Same statement as (f) for special approximate units instead of approximate units.

(1.4) *Remarks.*

(a) Condition (1.3a) was first used by CHEVALLEY [2, p. 112] to define the convolution of two distributions $S, T \in \mathcal{D}(\mathbf{R}^n)'$.

(b) The conditions (1.3b) and (1.3c) are contained in SCHWARTZ [8, Exposé nº 22]; see also HORVATH [5] and ROIDER [7].

(c) SHIRAISHI [11, p. 24] proved the equivalence of (1.3a), ..., (1.3d); the equivalence of (1.3b) and (1.3e) is proved by HORVATH [6, Prop. 1]. The equivalence of (1.3b), (1.3f) and (1.3g) follows from Proposition (1.1).

(d) The notion of simultaneous convolution for more than two distributions is not considered in WLADIMIROW [14]. But a glance at the definition of the simultaneous convolution of a finite set of distributions in SHIRAISHI [11, p. 30] (cf. also HORVATH [5, p. 190]) immediately shows us how Definition (0.3) should be modified for this case. An application of the equivalence (1.1a) $\Leftrightarrow$ (1.1d) then shows that the definition obtained in this way is also equivalent to SHIRAISHI's definition of the simultaneous convolution of a finite set of distributions.

2. *The $\mathcal{S}'$-convolution for tempered distributions.*

In order to establish the validity of the exchange formula $\mathcal{F}(S * T) = \mathcal{F}(S) \cdot \mathcal{F}(T)$ for the Fourier transformation of tempered distributions, Y. HIRATA and H. OGATA introduced the notion of the $\mathcal{S}'$-convolution of two tempered distributions.

(2.1) *Definition* (HIRATA, OGATA [3, p. 148]). The $\mathcal{S}'$-convolution $S \circledast T$ of two tempered distributions $S, T \in \mathcal{S}(\mathbf{R}^n)'$ exists if $(S * \varphi) \cdot (\check{T} * \psi) \in L^1(\mathbf{R}^n)$ holds for all $\varphi, \psi \in \mathcal{S}(\mathbf{R}^n)$. In this case $S \circledast T$ is the unique tempered distribution satisfying $\langle (S \circledast T) * \varphi, \psi \rangle = \int (S * \varphi)(x) \cdot (\check{T} * \psi)(x)\, dx$ for all $\varphi, \psi \in \mathcal{S}(\mathbf{R}^n)$.

Y. HIRATA and H. OGATA (loc. cit., p. 151) showed that if the $\mathcal{S}'$-convolution of $S, T \in \mathcal{S}(\mathbf{R}^n)'$ is defined then $\mathcal{F}(S \circledast T) = \mathcal{F}(S) \cdot \mathcal{F}(T)$ holds in the following sense: For any two sequences of regularizations $(\varrho_k; k \in \mathbf{N})$ and $(\tilde{\varrho}_k; k \in \mathbf{N})$ the sequences $((\mathcal{F}(S) * \varrho_k) \cdot \mathcal{F}(T); k \in \mathbf{N})$ and $(\mathcal{F}(S) \cdot (\mathcal{F}(T) * \tilde{\varrho}_k); k \in \mathbf{N})$ converge in $\mathcal{D}(\mathbf{R}^n)'$ to the same distribution and this common limit is denoted by $\mathcal{F}(S) \cdot \mathcal{F}(T)$. (See SHIRAISHI, ITANO [12, Prop. 2. (3), p. 225] for a more convenient description of this multiplicative product.)

SHIRAISHI [11, p. 26] showed that the $\mathcal{S}'$-convolution of two tempered distributions $S, T \in \mathcal{S}(\mathbf{R}^n)'$ exists if and only if $\varphi^\Delta \cdot (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ holds for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

In WLADIMIROW [14] the $\mathcal{S}'$-convolution is not considered, but the modification of Definition (0.3) to this case is straightforward. An appeal to the equivalence (a) $\Leftrightarrow$ (c) in Proposition (1.1) yields that $\varphi^\Delta \cdot (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ holds for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$ if and only if for all (special) approximate units $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^{2n})$ and all $\varphi \in \mathcal{S}(\mathbf{R}^n)$ the sequence $(\langle S \otimes T, \eta_k \cdot \varphi^\Delta \rangle; k \in \mathbf{N})$ is convergent.

The properties considered in the next proposition are analogous to (1.3b) and (1.3e) and provide another criterion for the existence of the $\mathcal{S}'$-convolution.

(2.2) *Proposition.* For $R \in \mathcal{D}(\mathbf{R}^{2n})'$ the following are equivalent:
(a) $\varphi^{\Delta} \cdot R \in \mathcal{B}_0(\mathbf{R}^{2n})'$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$.
(b) There exist $C > 0$, $m \in \mathbf{N}_0$ such that
$|\langle R, \psi \rangle| \leqslant C \cdot \sup \{(1 + |x + y|^2)^m \cdot |\partial^\gamma \psi(x, y)|;\ x, y \in \mathbf{R}^n,\ |\gamma| \leqslant m\}$
holds for all $\psi \in \mathcal{D}(\mathbf{R}^{2n})$.

Proof.

(a) $\Rightarrow$ (b): We set $\omega_m(x) := (1 + |x|^2)^m$ $(x \in \mathbf{R}^n,\ m \in \mathbf{Z})$. The topology of $\mathcal{S}(\mathbf{R}^n)$ may be defined by the sequence of norms $\varphi \longmapsto q_m(\varphi) := \sup \{\omega_m(x) \cdot |\partial^\alpha \varphi(x)|;\ x \in \mathbf{R}^n,\ |\alpha| \leqslant m\}$ $(m \in \mathbf{N}_0)$. Now let (a) be satisfied and consider the bilinear form $B : \mathcal{S}(\mathbf{R}^n) \times \mathcal{B}_0(\mathbf{R}^{2n}) \to \mathbf{C}$, $B(\varphi, \psi) := \langle \varphi^\Delta \cdot R, \psi \rangle$. By assumption $B(\varphi, \cdot) : \mathcal{B}_0(\mathbf{R}^{2n}) \to \mathbf{C}$ is continuous for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Let $(\eta_k;\ k \in \mathbf{N})$ be an approximate unit in $\mathcal{D}(\mathbf{R}^{2n})$. Then the sequence $\langle \eta_k \cdot \varphi^\Delta \cdot R, \psi \rangle = \langle \varphi^\Delta \cdot R, \eta_k \cdot \psi \rangle$ $(k \in \mathbf{N})$ converges to $\langle \varphi^\Delta \cdot R, \psi \rangle$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $\psi \in \mathcal{B}_0(\mathbf{R}^{2n})$; and for every $\psi \in \mathcal{B}_0(\mathbf{R}^{2n})$ and every $k \in \mathbf{N}$ the linear form $\varphi \longmapsto \langle \varphi^\Delta \cdot R, \eta_k \cdot \psi \rangle = \langle \eta_k \cdot \psi \cdot R, \varphi^\Delta \rangle$ is continuous on $\mathcal{S}(\mathbf{R}^n)$ (cf. Horvath [4, Ch. 4, § 9, p. 387], proof of Prop. 5). Thus $B(\cdot, \psi) : \mathcal{S}(\mathbf{R}^n) \to \mathbf{C}$, being the pointwise limit of a sequence of continuous linear forms, is continuous for all $\psi \in \mathcal{B}_0(\mathbf{R}^{2n})$ since $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space. Theorem 1 of Horvath [4, Ch. 4, § 7, p. 357] implies the continuity of $B : \mathcal{S}(\mathbf{R}^n) \times \mathcal{B}_0(\mathbf{R}^{2n}) \to \mathbf{C}$. Therefore there exist $C' > 0$ and $l \in \mathbf{N}_0$ such that $|\langle \varphi^\Delta \cdot R, \psi \rangle| \leqslant C' \cdot q_l(\varphi) \cdot p_l(\psi)$ holds for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$ and all $\psi \in \mathcal{D}(\mathbf{R}^{2n})$. The space $\mathcal{S}_l^l(\mathbf{R}^n)$ (Horvath [4, Ch. 2, § 4, Example 11, p. 90]) is canonically isomorphic to the completion of $(\mathcal{S}(\mathbf{R}^n), q_l)$. From Horvath [4, Ch. 2, § 5, Example 8, p. 101] we obtain $\omega_{-(l+1)} \in \mathcal{S}_l^l(\mathbf{R}^n)$. Since the above inequality remains true for all $\varphi \in \mathcal{S}_l^l(\mathbf{R}^n)$ we obtain:

$$|\langle R, \psi \rangle| = |\langle \omega_{-(l+1)}{}^\Delta \cdot R, \omega_{l+1}{}^\Delta \cdot \psi \rangle| \leqslant C' \cdot q_l(\omega_{-(l+1)}) \cdot p_l(\omega_{l+1}{}^\Delta \cdot \psi)$$
$$\leqslant C \cdot \sup \{(1 + |x + y|^2)^{l+1} |\partial^\gamma \psi(x, y)|;\ x, y \in \mathbf{R}^n,\ |\gamma| \leqslant l + 1\}$$

for all $\psi \in \mathcal{D}(\mathbf{R}^{2n})$, where the last estimate follows by a routine calculation (analogous to Horvath [4, Ch. 2, § 5, p. 102, (4) et seq.]). The above proof was inspired by Schwartz [9, Ch. VII, § 4, p. 240 b)].

(b) $\Rightarrow$ (a): Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be given. By the Leibniz formula we obtain from (b) the estimate

$|\langle \varphi^\Delta \cdot R, \psi\rangle| = |\langle R, \varphi^\Delta \cdot \psi\rangle| \leqslant C \cdot \sup \{(1 + |x+y|^2)^m |\partial^\gamma(\varphi^\Delta \cdot \psi)(x,y)|;$ $x, y \in \mathbf{R}^n, |\gamma| \leqslant m\} \leqslant C' \cdot p_m(\psi)$ for all $\psi \in \mathcal{D}(\mathbf{R}^{2n})$. This shows that $\varphi^\Delta \cdot R$ is continuous for the topology induced by $\mathcal{B}_0(\mathbf{R}^{2n})$ on $\mathcal{D}(\mathbf{R}^{2n})$.

As in the preceding section we collect in the following theorem the equivalent conditions for the existence of the $\mathcal{S}'$-convolution of two tempered distributions we know of.

(2.3) *Theorem.* For two tempered distributions $S, T \in \mathcal{S}(\mathbf{R}^n)'$ the following are equivalent:

(a) $(S * \psi) \cdot (\check{T} * \varphi) \in L^1(\mathbf{R}^n)$ for all $\varphi, \psi \in \mathcal{S}(\mathbf{R}^n)$.

(b) $(S * \psi) \cdot (\check{T} * \varphi) \in L^1(\mathbf{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n), \psi \in \mathcal{D}(\mathbf{R}^n)$.

(c) $S \cdot (\check{T} * \varphi) \in \mathcal{B}_0(\mathbf{R}^n)'$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

(d) $\varphi^\Delta \cdot (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

(e) There exist $C > 0, m \in \mathbf{N}_0$ such that
$|\langle S \otimes T, \psi\rangle| \leqslant C \cdot \sup \{(1 + |x+y|^2)^m |\partial^\gamma \psi(x,y)|;\ x, y \in \mathbf{R}^n, |\gamma| \leqslant m\}$ holds for all $\psi \in \mathcal{D}(\mathbf{R}^{2n})$.

(f) For every approximate unit $(\eta_k; k \in \mathbf{N})$ in $\mathcal{D}(\mathbf{R}^{2n})$ and every $\varphi \in \mathcal{S}(\mathbf{R}^n)$ the sequence $(\langle S \otimes T, \eta_k \cdot \varphi^\Delta\rangle; k \in \mathbf{N})$ is convergent.

(g) Same statement as (f) for special approximate units instead of approximate units.

(2.4) *Remarks.*

(a) The equivalence of (2.3a), ..., (2.3d) was proved by Shiraishi [11, p. 26].

(b) The equivalence of (2.3d) and (2.3e) follows from Proposition (2.2), and the equivalence of (2.3d), (2.3f) and (2.3g) follows from Proposition (1.1).

(c) The considerations of (1.4d) apply also to the simultaneous $\mathcal{S}'$-convolution of a finite set of tempered distributions (cf. Shiraishi [11, p. 30]).

3. *An example.*

Shiraishi [11, p. 27, Remark 1.] observed that if two distributions $S, T \in \mathcal{D}(\mathbf{R}^n)'$, $S \neq 0$, $T \neq 0$, satisfy $\varphi^\Delta \cdot (S \otimes T) \in \mathcal{B}_0(\mathbf{R}^{2n})'$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$ then it follows that S and T are tempered. He also observed (loc. cit., p. 28, Remark 2.) that if the convolution (in the sense of Definition (0.1)) of two tempered distributions were tempered whenever it is defined, the concept of S'-convolution would be superfluous. Therefore he asked (loc. cit., p. 20) whether the convolution $S * T$ of two tempered distributions is tempered whenever it is defined.

In order to answer this question we give an example of two tempered measures whose convolution is a non-tempered measure.

For $j, k \in \mathbf{N}, k \leqslant 2^j$ we define

$$(3.1) \quad \begin{cases} x(j, k) := \sum_{i=1}^{j-1} (2i) \cdot 2^i + (2j) \cdot k, \\ y(j, k) := x(j, k) - j. \end{cases}$$

For fixed $j \in \mathbf{N}$, the block $(x(j, k); k \leqslant 2^j)$ consists of 2^j even integers spaced at distance $2j$. The distance from the $j-th$ block to the $(j+1)-th$ block is $2(j+1)$. With the exception of $y(1,1)$ the points $y(j, k)$ are just the arithmetic means of the consecutive points $x(l, m)$ $(l, m \in \mathbf{N}, m \leqslant 2^l)$.

We note the following inequalities:

$$(3.2) \quad \begin{cases} x(j, k) \geqslant x(j, 1) > x(l, 2^l) \geqslant x(l, m), \\ y(j, k) \geqslant y(j, 1) > x(l, 2^l) \geqslant x(l, m) > y(l, m) \\ \text{for all } j > l, k \in \{1, \ldots, 2^j\}, m \in \{1, \ldots, 2^l\}. \end{cases}$$

In particular we have $x(j, k) \neq x(l, m)$ and $y(j, k) \neq y(l, m)$ for $(j, k) \neq (l, m)$.

For $z \in \mathbf{R}$ let $\delta(\hat{x} - z)$ denote the Dirac measure concentrated at z. Then the measures

$$(3.3) \quad \begin{cases} \mu(\hat{x}) := \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \delta(\hat{x} - x(j, k)), \\ \nu(\hat{x}) := \sum_{l=1}^{\infty} \sum_{m=1}^{2^l} \delta(\hat{x} + y(l, m)) \end{cases}$$

are tempered (cf. Treves [13, Part II, Ch. 25, p. 275, Exercise 25.7]). For every $\varphi \in \mathcal{K}(\mathbf{R}^2)$ we have

$$\langle \mu \otimes \nu, \varphi \rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \sum_{l=1}^{\infty} \sum_{m=1}^{2^l} \varphi(x(j, k), -y(l, m)).$$

Considering the three cases $j > l, j = l, j < l$ one easily proves the following inequality

$$(3.4) \left\{ \begin{array}{l} |x(j, k) - y(l, m)| \geqslant \max \{j, l\} \text{ for all } j, l \in \mathbf{N}, \\ \qquad k \in \{1, \ldots, 2^j\}, m \in \{1, \ldots, 2^l\}. \end{array} \right.$$

Thus for every compact set $K \subset \mathbf{R}$ the set $\operatorname{supp}(\mu \otimes \nu) \cap K^\Delta$ is finite and therefore the convolution of μ and ν exists even in the stronger sense of HORVATH [4, Ch. 4, § 9, p. 384]; we obtain

$$\langle \mu * \nu, \psi \rangle = \langle \mu \otimes \nu, \psi^\Delta \rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \sum_{l=1}^{\infty} \sum_{m=1}^{2^l} \psi(x(j, k) - y(l, m))$$

for all $\psi \in \mathcal{K}(\mathbf{R})$. For $\psi \in \mathcal{K}(\mathbf{R})$, $\psi \geqslant 0$ we have the estimate

$$\langle \mu * \nu, \psi \rangle \geqslant \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \psi(x(j, k) - y(j, k)) = \sum_{j=1}^{\infty} 2^j \psi(j)$$

which shows that $\mu * \nu$ is not tempered (TRÈVES [13, Part II, Ch. 25, p. 275, Exercise 25.7]).

(3.5) *Remarks.*

(a) Let $\varrho \in \mathcal{D}(\mathbf{R})$ satisfy $\operatorname{supp}(\varrho) \subset (-1/4, 1/4)$, $\varrho \geqslant 0$, $\int \varrho(x)\, dx = 1$. Then $\mu * \varrho$ and $\nu * \varrho$ are two bounded non-negative C^∞-functions whose convolution $(\mu * \varrho) * (\nu * \varrho) = (\mu * \nu) * (\varrho * \varrho)$ (cf. ROIDER [7, Prop. 1, p. 195]) is a (non-negative) C^∞-function which generates a non-tempered distribution (cf. SCHWARTZ [9, Ch. VII, § 5, p. 242]).
(b) In the above example the measure $\mu * \nu$ was estimated below by the measure $\langle \omega, \psi \rangle := \sum_{j=1}^{\infty} 2^j \psi(j)$ $(\psi \in \mathcal{K}(\mathbf{R}))$. The construction shows that instead of the weights 2^j $(j \in \mathbf{N})$ also faster growing sequences may be obtained.

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