THE MAXIMAL FUNCTION AND DERIVATION OF INTEGRALS IN A SPACE OF LOCALLY INTEGRABLE FUNCTIONS

BALDOMERO RUBIO

(University of Madrid)

Given an increasing continuous function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and $\varphi(t) \geq t$ for large t, we can define the space L_φ of measurable functions $f: R^n \to R$ which verify

$$\int_{R^n} \varphi(|f(y)|)\, dy < \infty.$$

It is clear that every function in L_φ is locally integrable. When $\varphi(t) = t^p$, $1 \leq p < \infty$, this space is $L^p(R^n)$.

If a homothecy invariant Busemann-Feller basis $\mathfrak{R}$ in R^n differentiates $\int f$ for every $f \in L_\varphi$, then the Hardy-Littlewood maximal function M associated wit $\mathfrak{R}$ has a property which generalizes the property of weak type (p, p). We proved in [3] this theorem in the case $\varphi(t) = t^p$ and more general bases.

We begin with some definitions. A Busemann-Feller basis in R^n is a collection $\mathfrak{R}$ of bounded open sets such that for every $x \in R^n$ there exists at least a sequence $\{R_k\} \subset R$ such that $R_k \to x$ (i. e. $x \in R_k$ and, given a neighborhood U of x is $R_k \subset U$ for large k). $\mathfrak{R}$ is homothecy invariant means $\mathfrak{R}$ contains the homothetic sets of its members. Being $f: R^n \to R$ locally integrable, the upper derivative of $\int f$, with respect to $\mathfrak{R}$, in x is defined by

$$\overline{D}(\int f, x) = \sup \limsup \frac{1}{|R_k|} \int_{R_k} f(y)\, dy$$

where the sup is taken over all the sequences $\{R_k\} \subset \mathfrak{R}$ such that $R_k \to x$. The lower derivative $\underline{D}(\int f, x)$ is defined setting inf lim inf above. When $\overline{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$ a. e. in R^n, we say that $\mathfrak{R}$

differentiatiates $\int f$. The Hardy-Littlewood maximal function M associated with R is defined by

$$Mf(x) = \sup_{x \in R \in \mathfrak{R}} \frac{1}{|R|} \int |f(y)| \, dy.$$

Since $\{x : Mf(x) > \lambda\}$ is an open set, Mf is measurable.

Then we will use the following covering lemma, which is a special case of the Vitali theorem and can be proved easily.

LEMMA. *Let P be a compact subset of R^n with positive measure, and $\delta > 0$. Given a bounded open set G in R^n, there exists a disjoint sequence $\{P_j\}$ of sets homothetic to P such that $|G - \cup P_j| = 0$ and, for every j, $P_j \subset G$ and diameter $P_j < \delta$.*

Now we can state our theorem.

THEOREM. *Let $\mathfrak{R}$ be an homothecy invariant Busemann-Feller basis in R^n. If $\mathfrak{R}$ differentiates $\int f$ for every $f \in L_\varphi$, then there exists $c > 0$ such that*

$$|\{x : Mf(x) > \lambda\}| \leq c \int_{R^n} \varphi\left[\frac{|f(y)|}{\lambda}\right] dy$$

holds for every $f \in L_\varphi$ and $\lambda > 0$.

Proof. If the theorem is not true, for every k we can choose a non negative function $f_k \in L_\varphi$, $\lambda_k > 0$ and $\alpha_k > 0$ such that

$$|\{x : M_k f_k(x) > \lambda_k\}| > c_k \int_{R^n} \varphi\left[\frac{f_k(y)}{\lambda_k}\right] dy$$

where M_k is the maximal function associated with the basis $\{R \in \mathfrak{R} :$ diameter $R < \alpha_k\}$, and the c_k's will be fixed later on. Taking $g_k = f_k/\lambda_k$, it is clear that $\{x : M_k f_k(x) > \lambda_k\} = \{x : M_k g_k(x) > 1\}$, and some compact P_k contained in this set verifics

$$|P_k| > c_k \int_{R^n} \varphi(g_k(y)) \, dy.$$

Now for every k we can use the lemma to cover the unit interval Q in R^n by a disjoint sequence $\{P_{kj}\}_{j=1,2,\ldots}$ of sets homothetic to P_k, such that the ratios ϱ_{kj} of these homothetic transformations verify $\varrho_{kj} \alpha_k \to 0$ as $k \to \infty$.

Define g_{kj} in the following manner: $g_{kj}(x) = 0$ if $x \notin P_{kj}$ and $g_{kj}(x) = g_k(y)$ being x the image of y in the homothetic transformation which carries P_k onto P_{kj}. So we have

$$\frac{\int_{P_{kj}} \varphi(g_{kj}(y))\, dy}{|P_{kj}|} = \frac{\int_{P_k} \varphi(g_k(y))\, dy}{|P_k|} < \frac{1}{c_k}$$

and

$$\int_{P_{kj}} \varphi(g_{kj}(y))\, dy < \frac{1}{c_k} |P_{kj}|.$$

Now, define $h_k = \Sigma_j g_{kj}$ for every k, and $h = \sup h_k$. We have

$$\int_Q \varphi(h_k(y))\, dy = \sum_j \int_Q \varphi(g_{kj}(y))\, dy \leq \frac{1}{c_k}$$

and

$$\int_Q \varphi(h(y))\, dy \leq \sum_k \int_Q \varphi(h_k(y))\, dy \leq \sum_k \frac{1}{c_k}.$$

It is possible to choose the c_k's to obtain $\int_Q \varphi(h(y))\, dy < \frac{1}{2} \varphi(1)$

On the other hand, the construction above proves that $\overline{D}(\int h, x) \geq 1$ a. c. in Q, and so we have also $h(x) \geq 1$ a. c. in Q, and

$$\int_Q \varphi(h(y))\, dy \geq \int_Q \varphi(1) = \varphi(1).$$

This is a contradiction.

* * *

As an aplication of this theorem we obtain the following: Let φ be the function $\varphi(t) = t(1 + \log^+ t)$ for $t \geq 0$. It is known (soe Jessen, Marcinkiewicz, Zygmund [2]) that the basis of intervals in R^2 differentiates $\int f$ for every $f \in L_\varphi$. Then, there exists $c > 0$ such that

$$|\{x : Mf(x) > \lambda\}| \leq c \int_{R^2} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \frac{|f(y)|}{\lambda}\right) dy$$

for every $f \in L_\varphi$ and $\lambda < 0$. In particular, if f is the characteristic function of a set E with finite measure, it holds

$$|\{x: M\chi_E(x) > \lambda\}| \leq \frac{c}{\lambda}\left(1 + \log\frac{1}{\lambda}\right)|E|$$

for $0 < \lambda < 1$.

With other methods, Burkill [1] obtained the inequality

$$|\{x: M\chi_E(x) > \lambda\}| \leq \frac{c_1}{\lambda}\log\frac{c_2}{\lambda}|E|$$

in the same conditions.

Acknowledgment. These results are contained in my doctoral thesis at the University of Madrid, being M. de Guzmán my thesis adviser, to whom I wish to thank for his help.

REFERENCES

[1] J. C. BURKILL, *On the Differentiability of Multiple Integrals*, J. London Math. Soc. (1951) 244-249.

[2] B. JESSEN, J. MARCINKIEWICZ and A. ZYGMUND, *Note on differentiability of multiple integrals*, Fund. Math. 25 (1935) 217-234.

[3] B. RUBIO, *The property of weak type (p, p) for the Hardy-Littlewood maximal operator and derivation of integrals*. Studia Mathematica, t. LVII (1976).