REMARK ON PAPER OF M. VON RENTELN «FINITELY GENERATED IDEALS IN B-ALGEBRA H^∞»

by

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Abstract: The characterisation of all ring automorphisms of H^∞ is given.

1. *Introduction*

In the paper [1] is given characterisation of all normed (i.e. $\Phi(i) = i$) ring automorphisms Φ of H^∞, namely they are generated by the conformal mapping of D (Theorem 5.1). Important step in the proof of above result is the following fact (Theorem 4.5, [1]): If $M \in \mathcal{M}(H^\infty) - D$ then M is not finitely generated. We elementarise it's proof avoiding the Corona Theorem. We complete Theorem 5.1 of [1] by the following

Theorem: If Φ is ring automorphism of H^∞ then Φ has one of the following form: $\Phi(f)(z) = f(\varphi(z))$ or $\Phi(f)(z) = \overline{f(\overline{\varphi(z)})}$, where $f \in H^\infty$ and φ is conformal mapping of D.

Let us recall notation. D - denotes here open unit disc in the complex plane $\mathbf{C}$, i.e. $D = \{z \in \mathbf{C} : |z| < 1\}$; H^∞ is Banach algebra of all bounded analytic functions in D. $\mathcal{M}(H^\infty)$ denotes the maximal ideal space of H^∞ equipped with Gelfand topology, Z is the identity function in D, $\mathcal{M}_\alpha = \{\phi \in \mathcal{M}(H^\infty) : \phi(Z) = \alpha\}$, $|\alpha| = 1$.

Recall that $\mathcal{M}(H^\infty) = \bigcup_{|\alpha|=1} \mathcal{M}_\alpha \cup \bigcup_{\lambda \in D} M_\lambda$, where $M_\lambda = \{f \in H^\infty : f(\lambda) = 0\}$. There exists homeomorphic embedding $\tau: D \to \bigcup_{\lambda \in D} M_\lambda \subset \mathcal{M}(H^\infty)$, so we briefly write $\mathcal{M}(H^\infty) = \bigcup_{|\alpha|=1} M_\alpha \cup D$.

2. *Proof of the Theorem*

Firstly we give another proof of Theorem 4.5 of [1] which doesn't involve the Corona Theorem.

Proposition: If $M \in \mathcal{M}(H^\infty) - D$ then M is not finitely generated.

Proof: Assume that $M = (f_1, f_2, \dots, f_n)$ for some $f_1, \dots, f_n \in H^\infty$. Using Theorem 3.2 of [1] M need to be principal, i.e. there exists $h \in H^\infty$ such that $M = (h)$.

By [2] (p. 161) there exists $\alpha \in \partial D$ and sequence $(\lambda_n) \subset D$ such that $\lambda_n \to \alpha$ and $h(\lambda_n) \to 0$. Thus we choose ([2] p. 213) interpolation subsequence (λ_{n_k}) and function $f \in H^\infty$ such that $f(\lambda_{n_{2k}}) = 0$ and $f(\lambda_{n_{2k-1}}) = 1$. If $f \in (h)$ then there exists $g \in H^\infty$ such that $f = gh$, but $f(\lambda_{n_{2k-1}}) = 1$ and $h(\lambda_{n_{2k-1}}) \to 0$ so $f \notin (h)$.

As well $(f, h) \subsetneq H^\infty$. Indeed, *if* $(f, h) = H^\infty$ then $1 = fg_1 + hg_2$ for some $g_1, g_2 \in H^\infty$; $f(\lambda_{n_{2k}})$, $h(\lambda_{n_{2k}}) \to 0$ so we have contradiction. Thus M is not maximal.

Proof of the Theorem. Since Φ is ring automorphism we have $\Phi^2(i) = \Phi(-1) = -1$. Thus $\Phi(i)(z) = \pm i$, what means that analytic function $\Phi(i)(z)$ need to be constant.

If $\Phi(i) = i$, we use Theorem 5.1 of [1].

If $\Phi(i) = -i$, we slightly modify proof of above theorem. In what follows we show that $\Phi(\lambda) = \overline{\lambda}$ for all $\lambda \in \mathbf{C}$. For $g \in H^\infty$, $\lambda \in \mathbf{C}$ $g - \lambda$ is not invertible in H^∞ iff $\lambda \in R(g)^{cl}$. Since Φ is ring automorphism we have $\lambda \in R(g)^{cl}$ iff $\Phi(\lambda) \in R(\Phi(g))^{cl}$.

When $\lambda \in Q(i)$, $\lambda \in R(g)^{cl}$ iff $\overline{\lambda} \in R(\Phi(g))^{cl}$. $Q(i)$ is dense subset of $\mathbf{C}$ so $R(g)^{cl} = \overline{R(\Phi(g))^{cl}}$. If there exists $\lambda_0 \in \mathbf{C}$ such that $|\Phi(\lambda_0) - \overline{\lambda}_0| > 2\delta > 0$, then for $g(z) = \lambda_0 + \delta z$ we get: $\lambda_0 \in R(g)^{cl}$ so $\Phi(\lambda_0) \in (R(\Phi(g))^{cl} = \overline{R(g)^{cl}} = \{\overline{\lambda}_0 + z : |z| \leqslant \delta\}$. Thus $|\Phi(\lambda_0) - \overline{\lambda}_0| \leqslant \delta$, contradiction.

By the *Proposition* $\Phi(M_\lambda) = M_\mu$. If we denote $\tau : D \to \bigcup_{\lambda \in D} M_\lambda$, $\tau(\lambda) = M_\lambda$ and $\psi = \tau^{-1} \circ \Phi \circ \tau$ then $\Phi(M_\lambda) = M_{\psi(\lambda)}$. Since correspondence $\lambda \to \mu$ is bijective, so is ψ.

Next we show $\overline{\psi(z)}$ is analytic. If $\lambda \in D, f \in H^\infty$ then $f - f(\lambda) \in M_\lambda$ and $\Phi(f - f(\lambda)) \in M_{\psi(\lambda)}$, so $\Phi(f)(\psi(\lambda)) = \overline{f(\lambda)}$. Taking $f_{Id} = \Phi^{-1}(Z)$ we get $\psi(\lambda) = \overline{f_{Id}}(\lambda)$. Thus $\varphi(z) = \overline{\psi^{-1}}(z)$ is conformal and $\Phi(f)(z) = \overline{f(\overline{\varphi(z)})}$ for all $f \in H^\infty$. □

If $f \in H^{\infty}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, φ-conformal mapping of D, then

$$\Phi(f)(z) = \sum_{n=0}^{\infty} a_n(\varphi(z))^n \text{ or } \Phi(f)(z) = \sum_{n=0}^{\infty} a_n(\overline{\varphi(z)})^n.$$

Theorem is valid for the disc algebra $A(D)$ also.

REFERENCES

[1] M. von Renteln, *Finitely generated ideals in B-algebra* H^{∞}, Collect. Math. 26 /1975/, no 2, 115-126.

[2] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, N.Y; 1962.

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