THE DISTRIBUTIONAL ${}_1F_1$-TRANSFORM

By

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1. Introduction

The conventional Laplace transform

$$F(x) = \int_0^\infty e^{-xy} f(y)\, dy \tag{1.1}$$

was studied in detail by Widder [8]. Some generalizations of (1.1) have been given from time to time by several mathematicians. Erdélyi [2] gave an important generalization of the Laplace transform (1.1) as

$$F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^\infty (xy)^\beta\, {}_1F_1(\beta+\eta+1;\; \alpha+\beta+\eta+1;\; -xy)\, f(y)\, dy \tag{1.2}$$

where ${}_1F_1$ function is a confluent hypergeometric function and α, β and η are complex parameters. For $\alpha = \beta = 0$, (1.2) reduces to (1.1). Joshi [3] had studied several preperties of (1.2) including convergence criteria and inversion formulas [4] with the restrictions Re $\beta \geqslant 0$, $Re\, \eta > 0$ and $Re\ \alpha + Re\, \beta + Re\, \eta + 1 \neq 0, -1, -2, \ldots$ [6, p. 2] and published recently several papers. Zemanian [9] has discussed some classical properties of the Laplace transform (1.1) for ordinary functions and then developed in detail the same for distributions. We propose to discuss in this paper the generalized Laplace transform (1.2) for ordinary functions and then extend the same for a certain class of distributions.

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2. Definitions and Notations

A function is said to be smooth if all its derivatives of all orders are continuous at all points of its domain.

The space of testing functions denoted by D, consists of all complex-valued functions $\varphi(t)$ that are smooth and zero outside some finite interval [5] and [9].

Distributions whose supports are bounded on the left are called right-sided distributions and are denoted by D'_R [9].

A distribution is said to be left-sided if its support is bounded on the right. D'_L represents the space of such distributions.

S denotes the space of testing functions of rapid descent, and S' its dual space, is the space of distributions of slow-growth. S_t is the space S of testing functions that are defined over R_t, where R_t is the one-dimensional Euclidean space containing all real values for t.

In what follows let us set

$$\begin{aligned} a &= \beta + \eta + 1 \\ b &= \alpha + \beta + \eta + 1 \\ P &= \Gamma(a)/\Gamma(b) \\ K(s, t) &= P(st)^{\beta}\ {}_1F_1\,(a; b; -st) \\ (a)_n &= a(a+1)\,(a+2)\,...\,(a+n-1). \end{aligned}$$

The symbol $\triangleq$ will be used to emphasize that a certain equality is a definition. Throughout this paper it will be assumed that Re $\beta \geqslant 0$, $Re\,\eta > 0$ and $Re\ \ \alpha + Re\,\beta + Re\,\eta + 1 \neq 0, \ -1, -2, ...$ unless otherwise stated.

3. The ${}_1F_1$-Transform of ordinary functions

Special conditions: Let $f(t)$ be a locally integrable function satisfying the following conditions.

(i) $f(t) = 0$ for $-\infty < t < T$.

(ii) There exists a negative real number c such that $f(t)\,e^{-ct}$ is absolutely integrable over $-\infty < t < \infty$.

The generalized Laplace transformation is defined as an operation L that assigns a function $F(s)$ of the complex variable s to each locally integrable function $f(t)$ that satisfies conditions above.

The operation L is defined by

$$L\,f(t) \triangleq F(s) \triangleq \int_{-\infty}^{\infty} K(s, t)\,f(t)\,dt.$$

$F(s)$ is called the ${}_1F_1$-transform of $f(t)$ and the operation L is defined to be the ${}_1F_1$-transformation. The right-sided ${}_1F_1$-transform is defined by

$$L\,f(t) \triangleq F(s) \triangleq \int_{T}^{\infty} K(s, t)\,f(t)\,dt. \tag{3.1}$$

LEMMA 3.1 *The integral in* (3.1) *is absolutely convergent for all* s *in the half-plane* $Re\ s > c$ *provided* $Re\ \beta > 0$ *and* $c < 0$.

PROOF. (3.1) may be written as

$$F(s) = \int_{T}^{0} K(s, t)\,f(t)\,dt + \int_{0}^{\infty} K(s, t)\,f(t)\,dt = I_1 + I_2$$

where

$$I_1 = \int_{T}^{0} K\,(s, t)\,f(t)\,dt$$

and

$$I_2 = \int_{0}^{\infty} K\,(s, t)\,f(t)\,dt.$$

Let us first consider I_1. The absolute convergence of $\int_T^0 K\,(s, t)\,f(t)\,dt$ is related to that of $\int_T^0 (st)^\beta\ f(t)\ dt$ since ${}_1F_1\,(a; b; -st) = O\,(1)$ as $t \to 0$. Since $f(t)$ is a locally integrable function and $Re\ \beta > 0$, I_1 is absolutely convergent for all (finite) values of s.

Again

$$I_2 = \int_{0}^{\infty} K\,(s, t)\,e^{ct}\,\{f(t)\,e^{-ct}\}\,dt.$$

Here $\int_0^\infty e^{-ct} f(t)\,dt$ is absolutely convergent according to the second of the 'Special conditions' in Section 3. Let us consider the function

$e^{ct} K(s, t)$. Let $Re\ s < 0$. Then from the asymptotic formula of the confluent hypergeometric function [6, p. 60]

$$ {}_1F_1(a; b; x) = \frac{\Gamma(b)}{\Gamma(a)} \times e^x x^{a-b}\{1 + O(|x|^{-1})\} $$
$$ (Re\ x > 0 \text{ and } x \to \infty) $$

we find that $e^{ct} K(s, t) \to 0$ as $t \to \infty$ when $c < Re\ s < 0$. Again let $Re\ s > 0$. Then from [6, p. 60]

$$ {}_1F_1(a; b; x) = \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \{1 + \underline{0}(|x|^{-1})\} $$
$$ (Re\ x < 0 \text{ and } x \to \infty) $$

we see that $e^{ct} K(s,t)$ again tends to zero as $t \to \infty$ when $Re\ s > 0$ and $c < 0$. The above function takes the value 1 when $t = 0$ because $Re\ \beta > 0$. Also we note that $K(s, t)\, e^{ct}$ is bounded when $t \geqslant T \geqslant 0$ and integrable in any closed interval $[T, T']$ since $Re\ \beta > 0$ and $c < 0$. Also $\int_0^\infty e^{-ct} f(t)\, dt$ is absolutely convergent. Hence by [1, p. 120] I_2 is absolutely convergent for all s such that $c < Re\ s < 0$ or $c < 0 < Re\ s$. It therefore follows that (3.1) converges absolutely for all $s \in \Gamma s$ where Γs is the set of all s satisfying $c < Re\ s < 0$ or $c < 0 < Re\ s$.

The greatest lower bound σ_a on all values of c for which the second of the 'Special conditions' holds, is called the abscissa of absolute convergence and the open half-plane $Re\ s > \sigma_a$ is called the region of absolute convergence for the ${}_1F_1$-transform (3.1). It is easy to show that ${}_1F_1$-transformation is linear.

4. The analyticity

THEOREM 4.1. *Let $f(t)$ be a function satisfying the 'Special conditions' and σ_a be the abscissa of absolute convergence for the ${}_1F_1$-transform (3.1). Let $Re\ \beta > 0$. Then, $F(s)$ is an analytic function for $Re\ s > \sigma_a$ and*

$$ F^{(k)}(s) = \int_T^\infty \frac{d^k}{ds^k} \{K(s, t)\}\, f(t)\, dt. $$

PROOF. (3.1) can be written as

$$(4.1)\ L\,f(t) = F(s) = \int_T^\infty K(s, t)\, e^{ct}\, \{f(t)\, e^{-ct}\}\, dt.$$

First let us suppose that $T \geqslant 0$. It has been proved already in Lemma 3.1 that the right-hand side of (4.1) is absolutely convergent for $Re\ s > \sigma_a$. With $Re\ \beta > 0$ and $c < 0$, $K(s, t)\, e^{ct}$ is bounded in $t \geqslant T \geqslant 0$ for all $s \in \Gamma s$ where Γs is as specified in Section 3. Also ${}_1F_1$ $(a; b; -st)$ is continuous in $[T, T']$ for all $s \in \Gamma s$. $(st)^\beta$ is also continuous in $[T, T']$ for all $s \in \Gamma s$ since $Re\ \beta > 0$. e^{ct} is also continuous in $[T, T']$. Hence $K(s, t)\, e^{ct}$ is continuous in $[T, T']$ for all $s \in \Gamma s$. Consequently it is integrable in $[T, T']$ for every s in Γs. Also $\int_T^\infty f(t)\, e^{-ct}\, dt$ is absolutely convergent. Hence by [1, p. 196] the right-hand side integral of (4.1) converges uniformly for all $s \in \Gamma s$. Also the integrand of (4.1) is a continuous function of (s, t) for all s and all t and an analytic function of s for every t. So by a result in [7, p. 99] $F(s)$ is analytic for $Re\ s > c > \sigma_a$. Hence we may differentiate the integral in (3.1) under the integral sign. Then we have

$$F^{(1)}(s) = \int_T^\infty \frac{d}{ds}\, \{K(s, t)\}\, f(t)\, dt.$$

By continuing the same argument, we get, after differentiating k times under the integral sign

$$F^{(k)}(s) = \int_T^\infty \frac{d^k}{ds^k}\, \{K(s, t)\}\, f(t)\, dt.$$

If $T < 0$, (4.1) may be broken up into a sum of two integrals I_1 and I_2 where

$$I_1 = \int_T^0 K(s, t)\, f(t)\, dt,$$

and

$$I_2 = \int_0^\infty K(s, t)\, f(t)\, dt.$$

I_2 can be seen to be analytic by the same argument which proved the analyticity of $F(s)$ in (4.1).

Taking up I_1 next, we observe that

$$(st)^\beta \, {}_1F_1 \, (\alpha; b; st)$$

is finite at $t = T$ and $t = 0$ (since $Re \, \beta > 0$) and is continuous in t and hence bounded in $[T, 0]$. Let M be its upper bound. Then, over every bounded domain Ω of the s plane, since

$$|(st)^\beta \, {}_1F_1 \, (a; b; -st)| = |(st)^\beta \, {}_1F_1 \, (\alpha; b; st) \, e^{-st}|$$

by a result [6, p. 6], we have

$$|K(s, t)| \leqslant Me^{-c't} \text{ for } T \leqslant t \leqslant 0,$$

where c' is any upper bound on $Re \, s$ for all s in Ω. All this shows that I_1 converges uniformly over Ω. We then proceed as above to establish the analyticity of $F(s)$ and the subsequent result.

5. The ${}_1F_1$-transform of right-sided distributions

We have already defined the ordinary ${}_1F_1$-transform $G(s)$ of a locally integrable function $g(t)$ satisfying the 'Special consitions' in the form

$$G(s) \triangleq Lg(t) \triangleq \int_T^\infty g(t) \, K(s,t) \, dt.$$

We can write this relation in the distributional sense as

$$G(s) \triangleq Lg(t) \triangleq \langle \, g(t), K(s, t) \, \rangle. \tag{5.1}$$

We shall now define the ${}_1F_1$-transform of a right-sided pistribution $f(t)$ by extending (5.1) to get

$$F(s) \triangleq Lf(t) \triangleq \langle \, f(t), K(s, t) \, \rangle. \tag{5.2}$$

We shall assume that there exists a negative real number c for which

$$e^{-ct} f(t) \in S'.$$

(5.2) can be written as

$$F(s) \triangleq Lf(t) \triangleq \langle \, e^{-ct} f(t), \lambda(t) \, K(s, t) \, e^{ct} \, \rangle \tag{5.3}$$

where $\lambda(t)$ is a smooth function with support bounded on the left and equals 1 over a neighbourhood of the support of $f(t)$.

We will now prove that

$$\lambda(t)\ K(s, t)\ e^{ct}$$

is a testing function in S_t. This can be done by showing that

$$|t^p\ \varphi_t^{(k)}(s, t)| \leqslant C_{p,k} \qquad -\infty < t < \infty \tag{5.4}$$

where $C_{p\,k}$ are constants with respect to t, depending upon p and k; p, k being any nonnegative integers [9, p. 100] and

$$\varphi(s, t) = \lambda(t)\ \ K(s, t)\ e^{ct}.$$

Now

$$\varphi_t^{(k)}(s, t)$$

$$= \sum_{m=0}^{k} \{\lambda(t)\ (st)^\beta\}_{(m)}\ \{{}_1F_1(a; b; -st)\ e^{ct}\}_{(k-m)}.$$

A typical term containing t in the above summation is

$$\lambda(t)\ t^{\beta-m}\ e^{ct}\ {}_1F_1\ (a; b; -st). \tag{5.5}$$

Let $Re\ s < 0$ and $t\ (> 0) \to \infty$. Then the above expression (5.5) is asymptotic to

$$\lambda(t)\ t^{\beta-m}\ e^{ct}\ e^{-st}\ (-\ st)^{-a}$$

which tends to 0 provided $Re\ s > c$ (*i. e.* $c < Re\ s < 0$).

Again if $Re\ s > 0$ and $t\ (> 0) \to \infty$ the expression (5.5) is asymptotic to

$$\lambda(t)\ t^{\beta-m}\ e^{ct}\ (st)^{-\beta-\eta-1}$$

which again tends to 0 provided $c < 0$ (i.e. $c < 0 < Re\ s$). Similarly the cases when $Re\ s > 0$ and $t\ (< 0) \to \infty$ and $Re\ s < 0$ and $t\ (< 0) \to \to \infty$ can be discussed when $c < 0 < Re\ s$ and $c < Re\ s < 0$ respectively. This proves (5.4).

The right-hand side of (5.3) possesses a sense, as the aplication of a distribution in S_t' to a testing function in S_t. Hence if $f(t) \in D'_R$ and if $f(t)$ is such that $e^{-ct} f(t) \in S'$ for some negative real number c, then the right hand side of (5.3) provides the definition of ${}_1F_1$-transform of $f(t)$ and the distribution $f(t)$ is said to be ${}_1F_1$-transformable. It may be noted that $F(s)$ is independent of λ since λ is a fixed smooth function with support bounded on the left and equals 1 over a neighbourhood of the support of $f(t)$.

REMARK. The generalization of Laplace transform of distributions reduces to the Laplace transform of distributions [9, p. 223] for $\alpha = \beta = 0$.

The greatest lower bound σ_1 of all real constants c for which $e^{-ct} f(t) \in S'$ is called the abscissa of convergence. The region of convergence for $L\, f(t)$ is the half-plane $Re\ s > \sigma_1$.

We observe that the ${}_1F_1$-transform $F(s)$ is independent of all choices of c in the definition (5.3) provided $\sigma_1 < c$ and this fact can be proved easily.

6. The linearity of distributional right-sided ${}_1F_1$-transformation

The distributional right-sided ${}_1F_1$-transformation is linear as the ordinary right-sided ${}_1F_1$-transformation is.

THEOREM 6.1 *Let f and g be ${}_1F_1$-transformable distributions in D'_R, having as their abscissas of convergence σ_f and σ_g respectively.*

Let

$$Lf = F(s) \text{ when } Re\ s > \sigma_f$$

and

$$Lg = G(s) \text{ when } Re\ s > \sigma_g.$$

If δ_1 and δ_2 be two constants, then

$$\begin{aligned} L(\delta_1 f + \delta_2 g) &= \delta_1\, Lf + \delta_2\, Lg \\ &= \delta_1\, F(s) + \delta_2 G(s) \end{aligned}$$

where $Re\ s > sup\ (\sigma_f, \sigma_g)$.

Proof is quite similar to [9, p. 224] and hence omitted.

7. The analyticity

THEOREM 7.1 *Let $f(t)$ be a ${}_1F_1$-transformable distribution belonging to D'_R. Let σ_1 be the abscissa of convergence for $Lf(t)$.*

Then,

$$F(s) \triangleq Lf(t) \triangleq \langle e^{-ct} f(t),\ \lambda(t)\, e^{ct}\, K(s,t) \rangle$$

is an analytic function in its region of convergence Re $s > \sigma_1$ *and*

$$\frac{dF}{ds} = \langle e^{-ct} f(t), \lambda(t) e^{ct} K(s, t) \left\{ \frac{\beta}{s} - \frac{ta}{b} \frac{{}_1F_1(a+1; b+1; -st)}{{}_1F_1(a; b; -st)} \right\} \rangle$$
$$= \rangle f(t), \frac{\partial}{\partial s} \{K(s, t)\} \rangle$$

where $Re\ s > c > \sigma_1$.

To prove this theorem we require the following Lemma.

LEMMA 7.1 *Let* $\lambda(t)$ *be the function specified in definition of* ${}_1F_1$-*transform of* $f(t)$. *Let* s *be fixed with* $Re\ s < 0$, *and let* $|\Delta s| \leqslant Re\ s - p > 0, p$ *being a real negative number. Also let* $p - c > 0$ *and*

$$\varphi_{\Delta s}(t) \triangleq \lambda(t)\, Pe^{ct} \{(s + \Delta s)^\beta t^\beta\, {}_1F_1(a; b; -t(s + \Delta s)) -$$
$$(st)^\beta\, {}_1F_1(a; b; -st)\} \div \Delta s$$

where $\Delta s \neq 0$.

Then, as $\Delta s \to 0$, $\varphi_{\Delta s}(t)$ *converges in* S *to*

$$\lambda(t)\, e^{ct} \left\{ -\frac{ta}{b} \frac{{}_1F_1(a+1; b+1; -st)}{{}_1F_1(a; b; ; -st)} + \frac{\beta}{s} \right\} K(s, t)$$

which is equal to

$\lambda(t)\ e^{ct} \frac{\partial}{\partial s} \{ K(s, t)\}$. *If* $Re\ s > 0$, *then* $|\Delta s| \leqslant Re\ s - p > 0$, p *being any real number and* $p - c - Re\ s > 0$.

PROOF. Let us put

$${}_1F_1(n) = {}_1F_1(a + n; b + n; -st) \quad (n = 0, 1, 2, \dots)$$

and

$$(A)_n = \frac{(a)_n}{(b)_n\, n!} \quad (n = 1, 2, \dots).$$

By using a result in [6, p. 22]

$${}_1F_1(a; b; x + y) = \sum_{n=0}^{\infty} (A)_n y^n\, {}_1F_1(a + n; b + n; x)$$

we have

$$\frac{\{(s + \Delta s)t\}^\beta\, {}_1F_1(a; b; -t(s + \Delta s)) - (st)^\beta\, {}_1F_1(a; b; -st)}{\Delta s}$$

$$= (st)^\beta [-t\{(A)_1\,{}_1F_1(1) - (A)_2\,{}_1F_1(2)\,(\Delta st) + \ldots\} + \frac{\beta}{s}\{| +$$

$$\frac{\beta - 1}{2!}\frac{\Delta s}{s} + \ldots\}\,\{{}_1F_1(0) - (A)_1\,(\Delta st)\,{}_1F_1\,(1) + \ldots\}].$$

We now use an asymptotic property of ${}_1F_1$ function [6, p. 60] and see that the right-hand side of the above equation is asymptotic to

$$(7.1)\qquad (st)^\beta P^{-1} e^{-st}(-st)^{-\alpha}\,[|t|\,\{1 + \frac{|\Delta st|}{2!} + \frac{|\Delta st|^2}{3!}$$

$$+ \ldots\} + \frac{\beta}{s}(1 + \frac{\beta - 1}{2!}\frac{|\Delta s|}{s} + \ldots)\,\{1 + |\Delta st| + \frac{|\Delta st|^2}{2!} + \ldots\}].$$

Now

$$|\text{ expression } (7.1)| < (st)^{\beta-\alpha} P^{-1} e^{-st}\,[|t| + \beta s^{-1} \times e^{\beta|\Delta s|s^{-1}}]\,e^{|\Delta st|}$$

Hence, for any nonnegative integer m,

$$|t^m \varphi_{\Delta s}(t)| < |t^m \lambda(t) e^{ct}\,(st)^{\beta-\alpha} \times$$
$$\times\, e^{-st+|\Delta st|}\,[|t| + \beta s^{-1} \times$$
$$\times\, e^{\beta|\Delta s|s^{-1}}]|$$

which is less than

$$|t^m\lambda(t)\,(st)^{\beta-\alpha}\,e^{-t(p-c)}\,\{|t| + \beta s^{-1}\,e^{\beta|\Delta s|s^{-1}}\}\,|.$$

The above expression is bounded uniformly for all t and for all $\Delta\, s$ under the given conditions. By using a similar argument as above and using the differential and asymptotic properties of ${}_1F_1\,(a;\,b;\,-st)$ [6, p. 15, 66] and proceeding as in Section 5, we can prove that, for each nonnegative integer k, $|t^m\,\varphi^{(k)}_{\Delta s}\,(t)|$ is also bounded uniformly foı all t and for all $\Delta\, s$ under the given conditions. Hence for each k, $\varphi^{(k)}_{\Delta s}\,(t)$ tends uniformly to a limit.

Further, we shall prove that as $\Delta\, s \to 0$, $\varphi_{\Delta s}(t)$ converges in S to

$$\lambda(t)\,e^{ct}\left\{-\,(A)_1 t\,\frac{{}_1F_1(1)}{{}_1F_1(0)} + \frac{\beta}{s}\right\} K(s,\,t).$$

For

$$\varphi_{\Delta s}(t) = P\lambda(t)\,e^{ct}\,(st)^\beta\,(\Delta s)^{-1}\,[(1 + \frac{\Delta s}{s}^{\beta}\,\{{}_1F_1(0) - (A)_1\,\Delta s.\,t\,{}_1F_1(1) +$$

$$(A)_2\,(\Delta s)^2\,t^2\,{}_1F_1(2) + \ldots\} - {}_1F_1(0)].$$

$$= P\,\lambda(t)\,e^{ct}\,(st)^{\beta}\,[-\,(A)_1\,t\cdot{}_1F_1(1) + (A)_2\Delta s\cdot t^2\,{}_1F_1\,(2) + \ldots$$

$$+\frac{\beta}{s}\Big(1 + \frac{\beta-1}{2!}\frac{\Delta s}{s} + \ldots\Big)\,\{{}_1F_1\,(0) - (A)_1\,\Delta s\cdot t\,{}_1F_1\,(1) +$$

$$+\,(A)_2\,(\Delta\, s)^2\,t^2\,{}_1F_1\,(2) + \ldots\}].$$

$$= \lambda(t)\,e^{ct}\,K(s,t)\,[-\,(A)_1\,t\cdot{}_1F_1\,(1)\,\{{}_1F_1(0)\}^{-1} + \beta s^{-1}]$$

$$as\ \Delta\ s \to 0.$$

Similarly the case when $Re\ s > 0$, can be proved.

PROOF OF THEOREM 7.1 Since $f(t)$ is a ${}_1F_1$-transformable distribution in D'_R,

$$F(s) = \langle e^{-ct}f(t),\ \lambda(t)\,e^{ct}\,K(s,t)\rangle.$$

Therefore

$$\frac{F(s+\Delta s) - F(s)}{\Delta s} = \langle e^{-ct}f(t),\ \lambda(t)\,Pe^{ct}\ \times$$

$$\{(s+\Delta s)^{\beta}\,t^{\beta}\,{}_1F_1a;\,b;\,-\,t(s+\Delta s))$$

$$-\,(st)^{\beta}\,{}_1F_1(a;\,b;\,-\,st)\}\ \div\ \Delta s\,\rangle$$

$$= \langle\, e^{-ct}f(t),\ \varphi_{\Delta s}\,(t)\,\rangle$$

for $\qquad |\Delta s| \neq 0.$

As $\Delta\ s \to 0$, we have from Lemma 7.1

$$\frac{dF}{ds} = \langle\, e^{ct}f(t),\ \lambda(t)\,e^{ct}\,K(s,t)\,[\beta s^{-1}\ -$$

$$(A)_1\,t\cdot{}_1F_1(1)\ \{{}_1F_1(0)\}^{-1}]\rangle$$

$$= \langle f(t), \frac{\partial}{\partial s}\,\{K(s,t)\}\,\rangle.$$

REMARK. For $\alpha = \beta = 0$, this theorem is reduced to Theorem 8.3 - 2 [9]

The linearity of the distributional ${}_1F_1$-transformation was already established and we now prove its continuity in the following theorem.

8. The continuity

THEOREM 8.1. Let $\{f_j\}_{j=1}^{\infty}$ *be a sequence of distributions with the properties:*

(i) *The supports of all* f_j *are contained in a fixed interval* $T \leqslant t < \infty$.

(ii) *There exists a negative number* c *such that the sequence* $\{e^{-ct} f_j(t)\}_{g=1}^{\infty}$ *converges in* S' *to* $e^{-ct}f(t)$.

Then, the sequence $\{F_j(s)\}_{j=1}^{\infty} = \{Lf_j\}_{j=1}^{\infty}$ *converges to* $F(s) = L\,f(t)$ *for* $Re\ s > c$.

The proof is similar to [9].

REMARK: For $\alpha = \beta = 0$, this theorem is reduced to Theorem 8.3-3 in [9].

The theory of ${}_1F_1$-transform of left - sided distributions can be similarly developed.

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