# THE HOMOLOGY OF $\Omega (X \vee Y)$

By

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Abstract: We compute the homology with integral coefficients of the loop space of a wedge as well as that of the coproduct of two topological groups.

## 1. Introduction

If $X$ is a pointed space, let $\Omega X$ denote the loop space of $X$. The purpose of this paper is to compute the homology of the space $\Omega (X \vee Y)$ in terms of the homology of $\Omega X$ and $\Omega Y$. The problem is solved by formula (1) below. It is easy to see that $\Omega (X \vee Y)$ may be identified with the coproduct (or free product) $\Omega X * \Omega Y$ in the category of loop spaces. Since, following Milnor [5], the category of topological groups is enclosed in that of loop spaces, the computation of the homology of $\Omega (X \vee Y) = \Omega (X) * \Omega (Y)$ immediately provides us with a formula (2) for the homology of the coproduct of two topological groups.

Our proof is based on two results: the Adams cobar construction [1] and the Künneth formula for the homology of a tensor product of $n$ complexes [3].

After the preparation of this paper we became aware of [4], where (2) and (1) are proved under some more rectricted hypothesis. We believe our proof is shorter and conceptually simpler.

All spaces considered in this paper are pointed spaces of the homotopy type of a CW-complex.

## 2. Computation of $H(\Omega (X \vee Y))$

We briefly remember the definition of the coproduct of two connected differential graded $\mathbf{Z}$-algebras. Let $\overline{A}$ denote the reduced algebra associated to $A$. If $I = (i_1, \ldots, i_n)$, $i_j = 1{,}2$, we write $\overline{A}_I$ for the tensor product $\overline{A}_{i_1} \otimes \ldots \otimes \overline{A}_{i_n}$. With the above notation,

$$A_1 * A_2 = \mathbf{Z} \oplus (\bigoplus_I \overline{A}_I),$$

where the sum extends to all indices $I$ of the form $(1,2,1,2,\ldots)$ or $(2,1,2,1,\ldots)$. The multiplication and the differential in $A_1 * A_2$ are defined in the natural way.

Let $X$ be a 1-connected space and let $H(\Omega X)$ denote the Pontrjagin algebra of $\Omega X$ which can be computed using the Adams cobar construction [1]. In Adams paper two functors $C$ and $\mathcal{F}$ are defined, such that $\mathcal{F}\,C\,X$ is a free associative connected differential graded $\mathbf{Z}$-algebra whose homology is canonically isomorphic to that of $\Omega X$.

It is not difficult to show, ([2]), that the functors $C$ and $\mathcal{F}$ preserve coproducts by passing to homology. Hence

$$H(\Omega(X_1 \vee X_2)) \cong H(\mathcal{F}C(X_1 \vee X_2)) \cong H(\mathcal{F}C\,X_1 * \mathcal{F}C\,X_2).$$

In this way the problem is reduced to the computation of the homology of the coproduct of two algebras.

The main tool for this computation is the multiple Künneth formula. If $A^1, \ldots, A^n$ are abelian groups, let $\mathrm{Mult}_i^n(A^1, \ldots, A^n)$ denote the $i^{th}$ left derived functor of the tensor product $A^1 \otimes \ldots \otimes A^n$. Hungerford [3] has showed that there exists an isomorphism of graded groups

$$H(K^1 \otimes \ldots \otimes K^n) \cong \sum_{i=0}^{n-1} \mathrm{Mult}_i^n\ (HK^1, \ldots, HK^n)$$

for all chain complexes of abelian groups $K^1, \ldots, K^n$ such that $H(\mathrm{Mult}_i^n\ (K^1, \ldots, K^n)) = 0 \quad 0 < i \leq n-1$. Observe that the degree of $\mathrm{Mult}_i^n\ (H_{r_1} K^1, \ldots, H_{r_n} K^n)$ is $r_1 + \ldots + r_n + i$.

Since $A_1 = \mathcal{F}C\,X_1$ and $A_2 = \mathcal{F}C\,X_2$ are free algebras, the above formula applies to our case and so we have

$$H(\overline{A}_I) = H(\overline{A}_{i_1} \otimes \ldots \otimes \overline{A}_{i_n}) \cong \sum_{j=0}^{n-1} \mathrm{Mult}_j^n\ (H\overline{A}_{i_1}, \ldots, H\overline{A}_{i_n}).$$

But

$$\widetilde{H}(A_1 * A_2) = H(\bigoplus_I \overline{A}_I) = \bigoplus_I H(\overline{A}_I)$$

and we get

$$\widetilde{H}(\Omega(X_1 \vee X_2)) \cong \sum_{i_1, \ldots, i_n} \sum_{j=0}^{n-1} \mathrm{Mult}_j^n(\widetilde{H}(\Omega X_{i_1}), \ldots, \widetilde{H}(\Omega X_{i_n})) \quad (1)$$

where the first summation runs over all indices $(i_1, \ldots, i_n)$ of the form $(1,2,1,2, \ldots)$ or $(2,1,2,1, \ldots)$.

Note that if $H(\Omega X_1)$ and $H(\Omega X_2)$ are torsion free, (1) reduces to

$$H(\Omega(X_1 \vee X_2)) = H(\Omega X_1) * H(\Omega X_2).$$

## 3. The homology of a coproduct of topological groups

It is showed in [5] that every topological group is weak homotopy equivalent to a loop space. Therefore if $G_1$ and $G_2$ are connected topological groups, we have for their coproduct the following formula

$$\widetilde{H}(G_1 * G_2) \cong \sum_{i_1, \ldots, i_n} \sum_{j=0}^{n-1} \mathrm{Mult}_j^n (\widetilde{H}(G_{i_1}), \ldots, \widetilde{H}(G_{i_n})) \qquad (2)$$

where the first summation runs over all indices $(i_1, \ldots, i_n)$ of the form $(1,2,1,2, \ldots)$ or $(2,1,2,1, \ldots)$.

In particular if $H(G_1)$ and $H(G_2)$ are torsion free, formula (2) yields

$$H(G_1 * G_2) \cong H(G_1) * H(G_2).$$

This is the case, for instance, if $G_1$ and $G_2$ are unitary groups $U_n$ or symplectic groups $\mathrm{Sp}_n$.

## 4. Computation of the functors $\mathrm{Mult}_i^n$

In this section we give a method which allows us to compute $\mathrm{Mult}_i^n(A_1, \ldots, A_n)$, where $A_1, \ldots, A_n$ are finitely generated abelian groups. We need the following

*Proposition:* The following holds:

(i) $\mathrm{Mult}_i^n$ is an additive functor,
(ii) $\mathrm{Mult}_i^n$ is symmetric,
(iii) $\mathrm{Mult}_i^n(A_1, \ldots, A_{n-1}, \mathbf{Z}) \cong \mathrm{Mult}_i^{n-1}(A_1, \ldots, A_{n-1})$
(iv) $\mathrm{Mult}_i^n(\mathbf{Z}/r_1\mathbf{Z}, \ldots, \mathbf{Z}/r_n\mathbf{Z}) \cong \bigoplus_{\binom{n-1}{i}} \mathbf{Z}/(r_1, \ldots, r_n)\mathbf{Z}$

where $(r_1, \ldots, r_n)$ stands for the greatest common divisor of $r_1, \ldots, r_n$.

Proof: Since (i), (ii) and (iii) are quite trivial, we only give the proof of (iv). We proceed by induction on $n$, the case $n = 1$ being obvious. By the usual Künneth formula we have

$$\mathrm{Mult}_i^{n+1}(\mathbf{Z}/r_1\mathbf{Z}, \ldots, \mathbf{Z}/r_{n+1}\mathbf{Z}) \cong (\mathrm{Mult}_i^n(\mathbf{Z}/r_1\mathbf{Z}, \ldots, \mathbf{Z}/r_n\mathbf{Z}) \otimes \mathbf{Z}/r_{n+1}\mathbf{Z}) \oplus \oplus \mathrm{Tor}(\mathrm{Mult}_{i-1}^n(\mathbf{Z}/r_1\mathbf{Z}, \ldots, \mathbf{Z}/r_n\mathbf{Z}), \ \mathbf{Z}/r_{n+1}\mathbf{Z})$$

and by induction the proposition follows.

For a space $X$ let $P(X)$ denote the set of prime integers $p$ for which $HX$ has $p$-torsion.

*Corollary:* If $G_1$ and $G_2$ are topological groups (or loop spaces) with finitely generated homology groups in each dimension, then

$$P(G_1 * G_2) = P(G_1) \cup P(G_2).$$

Proof: It follows from formula (2) and the proposition above.

Notice that formula (2) and the corollary show that even if $G_1$ and $G_2$ have a simple homology, $G_1 * G_2$ may have a complicated one. For instance, $H_8(SO(3) * SO(3))$ is the direct sum of 104 copies of $\mathbf{Z}/2\mathbf{Z}$, while $H_i(SO(3)) = 0$ for $i > 3$.

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*References*

1. ADAMS, J. F. «*On the cobar construction*» Colloque de topologie algébrique. Louvain 1956.

2. CLARK, A. - SMITH, L. «*The rational homotopy of a wedge*» Pacific J. Math. 24 (1968), 241-246.

3. HUNGERFORD, T. W. «*Multiple Künneth formulas for abelian groups*» Trans. A. M. S. 118 (1965), 257-275.

4. KATZ, E. «*A Künneth formula for coproducts of simplical groups*» Proc. A. M. S. 61 (1976), 117-121.

5. MILNOR, J. «*Construction of universal bundles, II*», Ann. of Math. 63 (1953), 430-436.

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