GAUSS-CODAZZI TENSOR FIELDS AND THE BONNET IMMERSION THEOREM

by

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INTRODUCTION. This paper deals with isometric immersions of Riemannian m-manifolds into Euclidean n-space. If $\varphi: M \to \mathbf{R}^n$ is such an immersion we introduce first an operator θ which in the case $n = m + 1$ reduces to the second fundamental form. Then we derive four intrinsic relations corresponding to the classical equations of Gauss-Codazzi. An operator which satisfies these relations will be called a *Gauss-Codazzi tensor field.*

Finally we show that if θ is a Gauss-Codazzi tensor field on a simply connected manifold M, then there exists a global isometric immersion of M into $\mathbf{R}^n$ such that θ is the second fundamental tensor.

The proof is based on the following idea: consider the vector bundle

$$\xi = \tau_M \oplus \varepsilon^r$$

where τ_M denotos the tangent bundle of M and ε^r is the trivial bundle of rank r. Define a certain linear connection in ξ and use the Gauss-Codazzi equations to show that this connection has zero curvature. Now apply the theorem, which states that every vector bundle over a simply connected manifold with a flat connection is trivial, to construct the immersion.

NOTATION. If M is a smooth manifold, $S(M)$ denotos the ring of smooth functions on M. The vector fields on M form a module over $S(M)$ wich is witten as $X(M)$. The derivative of a smooth map $f\colon M \to \mathbf{R}^n$ with respect to a vector field X is denoted by $d_X f$. It is again a smooth map from M to $\mathbf{R}^n$. Finally, the module of cross-sections in a vector bundle ξ is denoted by Sec ξ. Thus in particular, Sec $\tau_M = X(M)$, where τ_M denotes the tangent bundle of M.

1. IMMERSIONS INTO $\mathbf{R}^n$. Let M be a smooth manifold of dimension m and let $\varphi: M \to \mathbf{R}^n$ be an immersion of M into an n-dimensional vector space. It determines a vector bundle η over M whose fibre at x is the space $Im(d\varphi)_x$ and $d\varphi$ is a strong bundle isomorphism from the tangent bundle τ_M onto η. Under this map every vector field X on M defines a smooth map $\Phi_X : M \to \mathbf{R}^n$ given by

$$\Phi_X(x) = (d\varphi)_x X(x) \qquad x \in M.$$

Clearly,

$$\Phi_{X+Y} = \Phi_X + \Phi_Y$$

and

$$\Phi_{fX} = f \cdot \Phi_X \qquad f \in S(M).$$

Moreover, the map $X \to \Phi_X$ is injective.

Henceforth we shall identify every vector field X with the corresponding map Φ_X.

LEMMA I: Let X and Y be vector fields on M.
Then

(1) $$d_X Y - d_Y X = [X, Y],$$

where $[,]$ denotes the Lie product.

PROOF: Choose a basis $a_1, \dots a_n$ of $\mathbf{R}^n$ and write

$$\varphi(x) = \sum_i f^i(x) \cdot a_i$$

Then we have for every vector field X

$$(d\varphi)_x X(x) = \sum_i d_X f^i(x)\, a_i.$$

Under our identification this equation reads

$$X = \sum_i d_X f^i \cdot a_i.$$

It follows that

$$d_Y X = \sum_i d_Y d_X f^i . a_i$$

whence

$$d_X Y - d_Y X = \sum_i (d_X d_Y - d_Y d_X) f^i . a_i = \sum_i d_{[X,Y]} f^i . a_i = d_{[X,Y]} \mathscr{S}.$$

Suppose now that M is a *Riemannian* manifold with metric tensor g and that $\mathbf{R}^n$ is a *Euclidean* space with inner product $\langle , \rangle$.

Then φ is an isometric immersion if and only if for any two vector fields X, Y

(2) $$\langle X, Y \rangle = g(X, Y)$$

as follows from the definition.

THE OPERATORS θ_X AND θ_X^*. Let $\tau_M^\perp$ denote the normal bundle of M with respect to immersion φ. Its fibre at x is the orthogonal complement of the space $Im\,(d\varphi)_x$ in $\mathbf{R}^n$. Let $\pi^\perp : \tau_{\mathbf{R}^n} \to \tau_M^\perp$ be the strong bundle map obtained from the obvious projection.

Now fix a vector field X on M and consider the map

$$\theta_X : X(M) \to \text{Sec}\ \tau_M^\perp$$

given by

$$\theta_X(Y) = \pi^\perp (d_X Y).$$

If f is a function on M we have

$$\theta(fY) = \pi^\perp (d_X f . Y + f . d_X Y) = f . \pi^\perp (d_X Y) = f . \theta_X(Y)$$

and so θ_X is $S(M)$-linear in Y. Clearly,

$$\theta_{fX}(Y) = f . \theta_X(Y).$$

Thus the operators θ_X define an $S(M)$-bilinear map

$$\theta : X(M) \times X(M) \to Sec\ \tau_M.$$

We shall call θ the *second fundamental tensor field for the immersion* φ. Lemma I implies that

$$\theta_X(Y) - \theta_Y(X) = \pi^\perp (d_X Y - d_Y X) = \pi^\perp [X, Y] = 0,$$

whence

$$\theta(X, Y) = \theta(Y, X).$$

The adjoint operator

$$\theta_X^* : X(M) \leftarrow \text{Sec}\, \tau_M^\perp.$$

is determined by the equation

$$(3) \qquad g(\theta_X^* e, Y) = \langle e, \theta_X Y \rangle \qquad Y \in X(M) \qquad e \in \text{Sec}\, \tau_M^\perp.$$

Next, consider the bundle map $\pi : \tau_{\mathbf{R}^n} \to \tau_M$ obtained from the orthogonal projections π_x: $\mathbf{R}^n \to T_x(M)$ (recall that we identify $T_x(M)$ with its image under $(d\varphi)_X$.

LEMMA II: The operator θ_X^* satisfies the relation

$$\pi\, d_X(e) = -\theta_X^*(e) \qquad e \in \text{Sec}\, \tau_M^\perp.$$

PROOF: In fact, let $Y \in X(M)$. Then formula (3) yields

$$g(\theta_X^*(e), Y) = \langle e, \theta_X(Y) \rangle = \langle e, \pi^\perp d_X Y \rangle = \langle e, d_X Y \rangle.$$

On the other hand,

$$g(\pi\, d_X e, Y) = \langle \pi\, d_X e, Y \rangle = \langle d_X e, Y \rangle.$$

Thus

$$g(\theta_X^*(e) + \pi\, d_X e, Y) = \langle e, d_X Y \rangle + \langle d_X e, Y \rangle = d_X \langle e, Y \rangle = 0.$$

Since Y is arbitrary it follows that

$$\theta_X^*(e) + \pi\, d_X e = 0.$$

REMARK: If $n = m + 1$ and if e denotes the unit normal field, then θ and θ^* are given (respectively) by

$$\theta_X(Y) = H(X, Y).\, e$$

and

$$\theta_X^*(e) = -h(X),$$

where H is the second fundamental form and h is the bundle map $\tau_M \to \tau_M$ defined by

$$g(h(X), Y) = H(X, Y) \qquad X, Y \in X(M).$$

3. THE LINEAR CONNECTIONS ∇ AND $\nabla^\perp$. The immersion φ determines linear connections ∇ and $\nabla^\perp$ in τ_M and $\tau_M^\perp$ by

$$\nabla_Y Z = \pi(d_Y Z) \qquad Z \in X(M)$$

and

$$\nabla_Y^\perp e = \pi^\perp(d_Y e) \qquad e \in \operatorname{Sec} \tau_M^\perp.$$

PROPOSITION: Let $Z \in X(M)$ and $e \in \operatorname{Sec} \tau_M^\perp$. Then the following decompositions hold:

$$d_Y Z = \nabla_Y Z + \theta_Y Z \tag{4}$$

and

$$d_Y e = -\theta_Y^* e + \nabla_Y^\perp e. \tag{5}$$

PROOF: In fact,

$$d_Y Z = \pi(d_Y Z) + \pi^\perp(d_Y Z) = \nabla_Y Z + \theta_Y(Z)$$

and

$$d_Y e = \pi(d_Y e) + \pi^\perp(d_Y e) = -\theta_Y^\circ e + \nabla_Y^\perp e.$$

4. THE EQUATIONS OF GAUSS-CODAZZI. Applying d_X to (4) and using (5) we obtain

$$\begin{aligned} d_X d_Y Z &= d_X \nabla_Y Z + d_X(\theta_Y Z) = \\ &= \nabla_X \nabla_Y Z + \theta_X \nabla_Y Z - \theta_X^* \theta_Y Z + \nabla_X^\perp \theta_Y Z. \end{aligned}$$

Now interchange X and Y and subtract observing that

$$d_X\, d_Y - d_Y\, d_X = d_{[X,Y]}.$$

This yields the relation

$$d_{[X,Y]}\, Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\, Z + (\theta_X \nabla_Y - \theta_Y \nabla_X)\, Z - \\ - (\theta^*_X\, \theta_y - \theta^*_Y\, \theta_X)\, Z + (\nabla^\perp_X\, \theta_Y - \nabla^\perp_Y\, \theta_X)\, Z.$$

Finally, apply π and $\pi^\perp$ to this equation and use the formulae

$$\pi\, d_{[X,Y]}\, Z = \nabla_{[X,Y]}\, Z$$

and

$$\pi^\perp\, d_{[X,Y]}\, Z = \theta_{[X,Y]}\, Z$$

to obtain

(I) $$\mathcal{R}(X, Y) = \theta^*_X\, \theta_Y - \theta^*_Y\, \theta_X$$

and

(II) $$\theta_{[X,Y]} = (\theta_X \nabla_Y - \theta_Y \nabla_X) + (\nabla^\perp_X\, \theta_Y - \nabla^\perp_Y\, \theta_x)$$

where $\mathcal{R}$ denotes the curvature tensor of the linear connection ∇.

Similary, applying d_X to (5) and using (4) we find that

$$d_X\, d_Y\, e = - d_X\, \theta^*_Y\, e + d_X\, \nabla^\perp_X\, e = \\ = - \nabla_X\, \theta^*_X\, e - \theta_X\, \theta^*_Y\, e - \theta^*_X\, \nabla^\perp_Y\, e + \nabla^\perp_X\, \nabla^\perp_Y\, e.$$

It follows that

$$d_{[X,Y]}\, e = - \nabla_X\, \theta^*_Y\, e + \nabla_Y\, \theta^*_X\, e - \theta_X\, \theta^*_Y\, e + \\ + \theta_Y\, \theta^*_X\, e - \theta^*_X\, \nabla^\perp_Y\, e + \theta^*_Y\, \nabla^\perp_X\, e + \nabla^\perp_X\, \nabla^\perp_Y\, e - \nabla^\perp_Y\, \nabla^\perp_X\, e.$$

Now apply the projections π and $\pi^\perp$ observing that

$$\pi\, d_{[X,Y]}\, e = - \theta^*_{[X,Y]}\, e \text{ and } \pi^\perp\, d_{[X,Y]}\, e = \nabla^\perp_{[X,Y]}\, e$$

to obtain the equations

(III) $$\theta^*_{[X,Y]} = (\theta^*_X \nabla^\perp_Y - \theta^*_Y \nabla^\perp_X) + (\nabla_X \theta^*_Y - \nabla_Y \theta^*_X)$$

and

(IV) $$R^\perp (X, Y) = \theta_X \theta^*_Y - \theta_Y \theta^*_X$$

where $R^\perp$ denotes the curvature of the connection $\nabla^\perp$.

5. Gauss-Codazzi tensors. Let M be a Riemannian manifold of dimension m and let η be a vector bundle of rank r over M equipped with a Riemannian metric $\hat{g}$ and a Riemannian connection $\hat{\nabla}$ with curvature tensor $\hat{R}$. Consider a symetric bilinear map

$$\theta : X(M) \times X(M) \to Sec\, \eta$$

and let

$$\theta_X : X(M) \to Sec\ \eta$$

be the operator given by $\theta_X (Y) = \theta (X, Y)$. Let

$$\theta^*_X : X (M) \leftarrow Sec\, \eta$$

be the adjoint operator. We shall call θ a Gauss-Codazzi tensor field, if θ and θ^* satisfy equations I-IV (with $\nabla^\perp$ and $R^\perp$ replaced by $\hat{\nabla}$ and $\hat{R}$, respectively).

Theorem: Let θ be a Gauss-Codazzi tensor field on a simply connected Riemannian manifold M. Then there exists an isometric immersion $\varphi : M \to \mathbf{R}^n$ $(n = m + r)$ and an isometric strong bundle map $\alpha : \eta \to \tau^\perp_M$ such that

(6) $$\pi(d_Y X) = \nabla_Y X$$

(7) $$\pi^\perp (d_Y X) = \alpha (\theta_Y X)$$

Proof: Consider the vector bundle

$$\xi = \tau_M \oplus \eta.$$

Define a Riemannian metric $\tilde{g}$ in ξ by

$$\tilde{g}(Z_1 \oplus \sigma_1,\ Z_2 \oplus \sigma_2) = g(Z_1, Z_2) + \hat{g}(\sigma_1, \sigma_2)$$
$$Z_i \in X(M),\ \sigma_i \in Sec\,\eta,\ i = 1,2$$

and a linear connection by

$$\tilde{\nabla}_Y (Z, \sigma) = (\nabla_Y Z - \theta_Y^* \sigma,\ \hat{\nabla}_Y \sigma + \theta_Y Z).$$

It is easy to check that $\tilde{\nabla}$ is a Riemannian connection with respect to $\tilde{g}$

LEMMA III: The connection $\tilde{\nabla}$ has curvature zero,

$$\tilde{R}(X, Y) = 0.$$

PROOF: We may assume that $[X, Y] = 0$. It follows from the definition of $\tilde{\nabla}$ that

$$\tilde{\nabla}_X \tilde{\nabla}_Y (Z, \sigma) = (\nabla_X W - \theta_X^* \tau,\ \hat{\nabla}_X \tau + \theta_X W)$$

where

$$W = \nabla_Y Z - \theta_Y^* \sigma \text{ and } \tau = \hat{\nabla}_Y \sigma + \theta_Y Z.$$

Set

$$\nabla_X W - \theta_X^* \tau = \Phi(X, Y)$$

and

$$\nabla_X \tau + \theta_X W = \Psi(X, Y).$$

Then

$$\Phi(X, Y) = \nabla_X \nabla_Y Z - \nabla_X \theta_Y^* \sigma - \theta_X^* \hat{\nabla}_Y \sigma - \theta_X^* \theta_Y Z$$

and so

$$\Phi(X, Y) - \Phi(Y, X) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z - \nabla_X \theta_Y^* \sigma + \nabla_Y \theta_X^* \sigma -$$
$$- \theta_X^* \hat{\nabla}_Y \sigma + \theta_Y^* \hat{\nabla}_X \sigma - \theta_X^* \theta_Y Z + \theta_Y^* \theta_X Z.$$

Hence equations (I) and (III) imply that

$$\Phi(X, Y) - \Phi(Y, X) = 0.$$

On the other hand,

$$\Psi(X, Y) = \hat{\nabla}_X (\hat{\nabla}_Y \sigma + \theta_Y Z) + \theta_X (\nabla_Y Z - \theta^*_Y \sigma)$$

and so

$$\Psi(X, Y) - \Psi(Y, X) = \hat{R}(X, Y) \sigma + \hat{\nabla}_X \theta_Y Z - \hat{\nabla}_Y \theta_X Z + $$
$$+ \theta_X \nabla_Y Z - \theta_Y \nabla_X Z - \theta_X \theta^*_Y \sigma + \theta_Y \theta^*_X \sigma.$$

Thus, by (II) and (IV),

$$\Psi(X, Y) - \Psi(Y, X) = 0.$$

It follows that

$$(\tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X) \ (Z, \sigma) = 0$$

whence

$$\tilde{R}(X, Y) = 0.$$

6. THE CROSS-SECTIONS $\tilde{\sigma}_i$. Since $\tilde{R} = 0$ and since M is simply connected there are n ($n = m + r$) parallel cross-sections $\tilde{\sigma}_i$ in ξ such that the vectors $\tilde{\sigma}_i(X)$ ($i = 1 \dots n$) are linearly independent for every $X \in M$ (cf. [3], p. 92 or [2] p. 361). More precisely, fix a base point $a \in M$ and choose an orthonormal basis $h_1, \dots h_m$ *of* $T_a(M)$ and an orthonormal basis $k_{m+1}, \dots, k_n$ in the fibre F_a of a in η. Set

$$z_i = \begin{cases} (h_i, 0) & (i = 1 \dots m) \\ (0_1 k_i) & (i = m + 1 \dots n). \end{cases}$$

Then there are n parallel cross-sections $\tilde{\sigma}_i$ in ξ such that

$$\tilde{\sigma}_i(a) = z_i \qquad (i = 1 \dots n)$$

Since the connection $\tilde{\nabla}$ is Riemannian,

$$\tilde{g}(\tilde{\sigma}_i, \tilde{\sigma}_j) = \delta_{ij} \qquad (i, j = 1 \dots n).$$

Now wite $\tilde{\sigma}_i = (Z_i, \sigma_i)$ $Z_i \in X(M)$, $\sigma_i \in Sec\, \eta$.

Then the relations $\widetilde{\nabla}_X \widetilde{\sigma}_i = 0$ imply that

(8) $$\nabla_X Z_i = \theta^*_X = (\sigma_i)$$

and $(i = 1 \dots n)$

(9) $$\hat{\nabla}_X \sigma_i = -\theta_X(Z_i).$$

Let ω^i denote the 1-form on M corresponding to the vector field Z_i,

$$\omega^i(X) = g(X, Z_i) \qquad (i = 1 \dots n).$$

LEMMA IV:

(1) $\sum_i \omega^i(X) Z_i = X$

$X \in X(M)$

(2) $\sum_i \omega^i(X) \sigma_i = 0$

(3) $\sum_i \omega^i(X) \omega^i(Y) = g(X, Y) \qquad X, Y \in X(M)$

(4) $\sum_i \hat{g}^i(\sigma, \sigma_i) = \sigma \qquad \sigma \in \operatorname{Sec} \eta.$

(5) The 1-forms ω^i are closed,

$$\delta \omega^i = 0.$$

PROOF: (1) and (2): In fact,

$$\sum_i \omega^i(X) \widetilde{\sigma}_i = \sum_i g(X, Z_i) \widetilde{\sigma}_i = \sum_i \widetilde{g}(X \oplus 0, \widetilde{\sigma}_i) \widetilde{\sigma}_i = X \oplus 0$$

(since the $\widetilde{\sigma}_i$ are orthonormal). It follows that

$$\sum_i \omega^i(X) Z_i = X$$

and

$$\sum_i \omega^i(X) \sigma_i = 0.$$

(3): Since

$$\omega^i(Y) = g(Y, Z_i)$$

it follows from (1) that

$$\sum_i \omega^i(X)\,\omega^i(Y) = \sum_i \omega^i(X)\,g(Y, Z_i) = g(Y, \sum_i \omega^i(X)\,Z_i) = \\ = g(Y, X) = g(X, Y).$$

(4) Observe that

$$\sum_i \hat{g}(\sigma, \sigma_i)\,(0 \oplus \sigma_i) = \sum_i \tilde{g}(0 \oplus \sigma, Z_i \oplus \sigma_i)(0 \oplus \sigma_i) = 0 \oplus \sigma$$

to obtain

$$\sum_i \hat{g}(\sigma, \sigma_i)\,\sigma_i = \sigma.$$

(5) Let X and Y be vector fields such that $[X, Y] = 0$. Then

$$\delta\,\omega^i(X, Y) = d_X\,\omega^i(Y) - d_Y\,\omega^i(X).$$

But, in view of (8),

$$d_X\,\omega^i(Y) = g(\nabla_X Y, Z_i) + g(Y, \nabla_X Z_i) = \\ = g(\nabla_X Y, Z_i) + g(Y, \overset{*}{\theta}_X \sigma_i) = g(\nabla_X Y, Z_i) + g(\theta_Y X, \sigma_i)$$

Since, by hypothesis, $\theta_X Y = \theta_Y X$

it follows that

$$\delta\,\omega^i(X,Y) = 0.$$

7. The immersion functions. Since the 1-forms ω^i are closed and since M is simply connected, there are functions f^i on M such that

$$\delta f^i = \omega^i \qquad (i = 1 \ldots n).$$

Now choose an orthonormal basis $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in $\mathbf{R}^n$ and define φ by

$$\varphi(x) = \sum_i f^i(x)\,\mathfrak{a}_i \qquad x \in M.$$

Then we have

(10) $$(d\varphi)_x h = \sum_i \omega^i (x; h)\, a_i \qquad \begin{array}{l} x \in M \\ h \in T_x(M) \end{array}$$

and so by Lemma IV, (3)

$$\langle (d\varphi)_x h, (d\varphi)_x k \rangle = g(x; h, k) \qquad \begin{array}{l} x \in M \\ h, k \in T_x(M). \end{array}$$

This shows that φ is an isometric immersion.

8. THE BUNDLE MAP α. Let E_η denote the total space of η and define functions λ^i on E_η by setting

$$\lambda^i(z) = \hat{g}(z, \sigma_i, (\pi_\eta z)) \quad z \in E_\eta ,$$

where $\pi_\eta : E \to M$ is the hundle projection. Let $\alpha : E \to M \times \mathbf{R}^n$ be the strong bundle map given by

$$\alpha(z) = \sum_i \lambda^i(z)\, a_i \qquad z \in E_\eta .$$

Then we have by Lemma IV, (2),

$$\langle (d\varphi)_x h, \alpha(z) \rangle = \sum_i \omega^i(x, h) \lambda^i(z) = \sum_i \omega^i(x, h)\, \hat{g}(z, \sigma_i(x)) =$$
$$= \hat{g}(z, \sum_i \omega^i(x, h)\, \sigma_i(x)) = 0 \qquad z \in E_\eta, \ x = \pi_\eta z.$$

This shows that $\alpha(z) \in T_x(M)^\perp$.

Moreover, if $u \in F_x$ and $v \in F_x$, we have

$$\langle \alpha u, \alpha v \rangle = \sum_i \lambda^i(u) \lambda^i(v) = \sum_i \hat{g}(u, \sigma_i(x))\, \hat{g}(v, \sigma_i(x)) =$$
$$= \hat{g}(u, \sum_i \hat{g}(v, \sigma_i(x))\, \sigma_i(x)) = \hat{g}(u, v)$$

(cf. Lemma IV, (4)).

Thus α is an isometric bundle map from η to the normal bundle of M,

$$\alpha : \eta \xrightarrow{\cong} \tau_M^\perp .$$

It remains to be checked that the immersion φ induces the given linear connection and that the second fundamental tensor of φ corresponds to θ under the bundle map α.

Write equation (10) in the form

$$X = \sum_i \omega^i(X)\,\mathfrak{a}_i = \sum_i g(Z_i, X)\,\mathfrak{a}_i \tag{11}$$

and apply d_Y. It follaws that

$$d_Y X = \sum_i g(\nabla_Y Z_i, X)\mathfrak{a}_i + \sum_i g(Z_i, \nabla_Y X)\mathfrak{a}_i = \sum_i g(\theta^*_Y \sigma_i, X)\mathfrak{a}_i +$$
$$+ \sum_i \omega^i(\nabla_Y X)\mathfrak{a}_i = \sum_i \hat{g}(\sigma_i, \nabla_Y X)\mathfrak{a}_i + \sum_i \omega^i(\nabla_Y X)\mathfrak{a}_i .$$

Now observe that, by (11),

$$\sum_i \omega^i(\nabla_Y X)\mathfrak{a}_i = \nabla_Y X.$$

On the other hand,

$$\alpha(\theta_Y X) = \sum_i \lambda^i(\theta_Y X)\mathfrak{a}_i = \sum_i \hat{g}(\theta_Y X, \sigma_i)\mathfrak{a}_i .$$

Thus we have

$$d_Y X = \nabla_Y X + \alpha(\theta_Y X)$$

whence

$$\pi(d_Y X) = \nabla_Y X$$

and

$$\pi^{\perp}(d_Y X) = \alpha(\theta_Y X).$$

This completes the proof of the theorem.

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