THE GEOMETRIC MEANS OF AN ENTIRE FUNCTION OF ORDER ZERO (1)

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The geometric means $G(r)$, $g_k(r)$ and $g_k^*(r)$ for a non-constant entire function of order zero are considered and certain relations involving the comparative growth of the same relative to each other are obtained. It is shown that the logarithmic orders of the logarithms of these means are separately equal to the logarithmic order of the function. The difference in the results regarding the growth of the pairs $(G(r),\ g_k(r))$ and $(G(r), g_k^*(r))$ have also been observed.

1. INTRODUCTION

For a non-constant entire function $f(z)$ of order zero, the logarithmic order, ϱ^*, and the lower logarithmic order, λ^*, are given as [8]:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r,f)}{\log\log r} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix} \quad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty),$$

$$\text{where, } M(r,\ f) = \max_{|z| = r} |f(z)|.$$

Let us define the following mean values of $|f(z)|$:

(1.1) $$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta\right\},$$

(1.2) $$g_k(r) = \exp\left\{\frac{k+1}{r^{k+1}}\int_0^r x^k \log G(x)\, dx\right\},\ (0 < k < \infty),$$

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and

$$(1.3)\quad g_k^*(r) = \exp\left\{\frac{k+1}{(\log r)^{k+1}}\int_1^r (\log x)^k \log G(x)\frac{dx}{x}\right\}, \ (0 < k < \infty).$$

The mean value (1.2) was introduced by Kamthan [6] and a number of properties regarding its growth with respect to $G(r)$ and other auxiliary functions for an entire function of order ϱ were obtained in ([2] [3] [6] [7]). We introduce a new geometric mean $g_k^*(r)$ as defined in (1.3) and call it logarithmic geometric mean. Our aim in this paper is to investigate certain relations involving the comparative growth of $G(r)$, $g_k(r)$ and $g_k^*(r)$ relative to each other for entire functions of order zero. In section 2, we discuss certain preliminaries, whereas the remaining sections are devoted to our main results.

2. Preliminaries

Throughout this paper we assume $f(z)$ to be a nonconstant entire function of order zero. For these functions, we have

$$\varrho_1 = g.\ l.\ b.\ \{\alpha\colon \alpha > 0 \text{ and } \sum_{n=1}^{\infty} r_n^{-\alpha} < \infty\} = 0,$$

where, $\{r_n\}_{n=1}^{\infty}$ denotes the sequence of the moduli of the zeros of $f(z)$. To have a more precise description of the distribution of the zeros of such functions, let us consider

$$\varrho_1^* = g.\ l.\ b.\ \{\alpha : \alpha > 0 \text{ and } \sum_{n=1}^{\infty} (\log r_n)^{-\alpha} < \infty\}.$$

In analogy with the convergence exponent, ϱ_1, of the zeros of $f(z)$, ϱ_1^* will be called the logarithmic convergence exponent of the zeros of $f(z)$.

In a recent paper [4], we have proved that if $f(z)$ has atleast one zero, then

$$(2.1)\qquad \limsup_{r\to\infty} \frac{\log n(r)}{\log\log r} = \varrho_1^* \ (0 \leqslant \varrho_1^* \leqslant \infty),$$

where, $n(r)$ denotes the number of the zeros of $f(z)$ in $|z| \leqslant r$. Also, it been shown therein that $\varrho^* = \varrho_1^* + 1$.

Further, in analogy with the lower convergence exponent, λ_1, the limit inferior in (2.1) may be named as lower logarithmic convergence exponent of the zeros of $f(z)$ and denoted by λ_1^*, i.e.

$$\liminf_{r\to\infty} \frac{\log n(r)}{\log \log r} = \lambda_1^*, \ (0 \leqslant \lambda_1^* \leqslant \infty). \tag{2.2}$$

Also, let

$$N(r) = \int_0^r \frac{n(x)}{x}\, dx, \tag{2.3}$$

where, it is assumed without any loss of generality, that $n(r) = 0$ for $r \leqslant 1$.

Also, for a real and non-negative function $P(r)$, increasing for $r_0 < r < \infty$, where $r_0 > 1$, the logarithmic order, μ, and the lower logarithmic order, ν, are defined as

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log P(r)}{\log \log r} = \begin{matrix}\mu \\ \nu\end{matrix} \ (0 \leqslant \nu \leqslant \mu \leqslant \infty). \tag{2.4}$$

3. Main results

Theorem 1. *The logarithmic (lower logarithmic) orders of the functions* $P_1(r) = \log G(r)$, $P_2(r) = \log g_k(r)$ *and* $P_3(r) = \log g_k^*(r)$ *are the same.*

The proof makes use of the following lemmas:

Lemma 1. For $1 < r < R$,

$$\log g_k^*(r) \leqslant \log G(r) \leqslant \frac{(\log R)^{k+1}}{(\log R)^{k+1} - (\log r)^{k+1}} \log g_k^*(R). \tag{3.1}$$

Proof of the lemma. We have

$$\log g_k^*(r) = \frac{k+1}{(\log r)^{k+1}} \int_1^r (\log x)^k \log G(x) \frac{dx}{x}$$

$$\leqslant \log G(r). \tag{3.2}$$

Further

$$\log g_k^* (R) = \frac{k+1}{(\log R)^{k+1}} \int_1^R (\log x)^k \log G(x) \frac{dx}{x}$$

$$\geqslant \frac{(k+1)}{(\log R)^{k+1}} \int_r^R (\log x)^k \log G(x) \frac{dx}{x}$$

$$\geqslant \frac{(k+1) \log G(r)}{(\log R)^{k+1}} \int_r^R (\log x)^k \frac{dx}{x}$$

$$= \frac{(\log R)^{k+1} - (\log r)^{k+1}}{(\log R)^{k+1}} \log G(r). \tag{3.3}$$

Combining (3.2) and (3.3), the lemma follows.

LEMMA 2. [7]. *This is*

$$\log g_k(r) \leqslant \log G(r) \leqslant \frac{R^{k+1}}{R^{k+1} - r^{k+1}} \log g_k(R), \; R > r. \tag{3.4}$$

Proof of the theorem. By JENSEN's theorem (see BOAS [1], p. 2), we find that

$$\log G(r) = \int_0^r \frac{n(t)}{t} \mathrm{dt} + \log |f(0)|. \tag{3.5}$$

Therefore

$$\log G(r^2) \geqslant \int_r^{r^2} \frac{n(t)}{t} dt \geqslant n(r) \int_r^{r^2} \frac{dt}{t} = n(r). \log r,$$

which, in view of (2.1) and (2.2), implies

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log G(r)}{\log \log r} \geqslant \frac{\varrho_1^* + 1}{\lambda_1^* + 1},$$

Again, from (3.5), we get

$$\log G(r) = \int_{r_0}^{r} \frac{n(t)}{t}\, dt + 0(1),\ r_0 > 0$$

$$\leqslant n(r) \log r\,(1 + 0(1)).$$

The above inequality coupled with (2.1) and (2.2) gives

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log G(r)}{\log \log r} \leqslant \begin{matrix}\varrho_1^* + 1 \\ \lambda_1^* + 1\end{matrix}.$$

Hence

(3.6) $$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log G(r)}{\log \log r} = \begin{matrix}\varrho_1^* + 1 \\ \lambda_1^* + 1\end{matrix}.$$

Further, by putting $R = r^2$ in (3.1) and $R = 2r$ in (3.4), we obtain respectively

(3.7) $$\log g_k^*(r) \leqslant \log G(r) \leqslant \frac{2^{k+1}}{2^{k+1} - 1} \log g_k^*(r^2)$$

and

(3.8) $$\log g_k(r) \leqslant \log G(r) \leqslant \frac{2^{k+1}}{2^{k+1} - 1} \log g_k(2r).$$

Making use of (3.6) in (3.7) and (3.8), theorems 1 follows.

REMARK. It is interesting to note that inspite of the fact that the logarithmic orders of $\log g_k(r)$ and $\log g_k^*(r)$ are separately equal to the logarithmic order of the entire function $f(z)$, still it seems that $\log g_k(r)$ is of larger growth than $\log g_k^*(r)$. Consider, for example, the function

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{e^n}\right).$$

It is clear that $f(z)$ is an entire function of order zero and logarithmic order 2, since $n(r) \sim \log r$. Also, $\log G(r) \sim 1/2\, (\log r)^2$; $\log g_k^*(r) \sim \sim \frac{k+1}{2(k+3)} (\log r)^2$ and $\log g_k(r) \sim 1/2\, (\log r)^2$.

4. *Derivatives of* $G(r)$, $g_k^*(r)$ and $g_k(r)$.

THEOREM 2. *Let* $G'(r)$, $g_k^{*\prime}(r)$ and $g_k'(r)$ *be the first derivatives of* $G(r)$, $g_k^*(r)$ *and* $g_k(r)$ *respectively. Then*

$$\lim_{\substack{r\to\infty\\ r\notin E}} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{r\,G'(r)/G(r)\}}{\log\log r} = \frac{\varrho_1^*}{\lambda_1^*}, \tag{4.1}$$

$$\lim_{\substack{r\to\infty\\ r\notin E}} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{r g_k^{*\prime}(r)/g_k^*(r)\}}{\log\log r} = \frac{\varrho_1^*}{\lambda_1^*}, \tag{4.2}$$

and

$$\lim_{\substack{r\to\infty\\ r\notin E}} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{r\,g_k'(r)/g_k(r)\}}{\log\log r} = \frac{\varrho_1^*}{\lambda_1^*}, \tag{4.3}$$

where E is a set of the values of r of measure zero.

Proof. Differentiating (3.5) with respect to r, we get

$$\frac{r\,G'(r)}{G(r)} = n(r), \tag{4.4}$$

for all values of r excluding a set E of measure zero. Therefore, in view of (2.1) and (2.2), (4.1) follows.

Further, we find that

$$\frac{g_k^{*\prime}(r)}{g_k^*(r)} = \lim_{\epsilon\to 0} \frac{\log g_k^*(r) - \log g_k^*(r^{1-\epsilon})}{r - r^{1-\epsilon}}$$

$$= \frac{k+1}{(\log r)^{k+1}} \lim_{\epsilon\to 0} \frac{1}{(r-r^{1-\epsilon})}\left\{\int_1^r (\log x)^k \log G(x)\,\frac{dx}{x} - \frac{1}{(1-\epsilon)^{k+1}}\int_1^{r^{1-\epsilon}} (\log x)^k \log G(x)\,\frac{dx}{x}\right\}$$

$$= \frac{k+1}{(\log r)^{k+1}} \lim_{\epsilon\to 0} \frac{1}{(r-r^{1-\epsilon})}\left\{\int_1^r (\log x)^k \log G(x)\,\frac{dx}{x} - \int_1^r (\log x)^k \log G(x^{1-\epsilon})\,\frac{dx}{x}\right\}$$

(4.5) $$= \frac{k+1}{(\log r)^{k+1}} \lim_{\epsilon \to 0} \frac{1}{(r - r^{1-\epsilon})} \left[\int_1^r \left\{ \frac{\log G(x) - \log G(x^{1-\epsilon})}{(x - x^{1-\epsilon})} \right\} (1 - x^{-\epsilon}) (\log x)^k \, dx \right]$$

$$< \frac{k+1}{r (\log r)^{k+1}} \int_1^r \left\{ \lim_{\epsilon \to 0} \frac{\log G(x) - \log G(x^{1-\epsilon})}{(x - x^{1-\epsilon})} \right\} (\log x)^k \, dx$$

$$= \frac{K+1}{r (\log r)^{k+1}} \int_1^r \frac{G'(x)}{G(x)} (\log x)^k \, dx$$

$$= \frac{k+1}{r (\log r)^{k+1}} \int_{r_0}^r \frac{G'(x)}{G(x)} (\log x)^k \, dx + 0 (r^{-1} (\log r)^{-k-1}).$$

Since $\{xG'(x)/G(x)\}$ is an increasing funstion of x by (4.4), the above inequality implies

$$\frac{g_k^{*\prime}(r)}{g_k^*(r)} < \frac{G'(r)}{G(r)} (1 + 0(1)),$$

for all sufficiently large values of r.

Hence

(4.6) $$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \{r g_k^{*\prime}(r) / g_k^*(r)\}}{\log \log r} \leqslant \begin{matrix} \varrho_1^* \\ \lambda_1^* \end{matrix} .$$

Again from (4.5), for $\eta > 1$, we notice that

$$\frac{g_k^{*\prime}(r^\eta)}{g_k^*(r^\eta)} = \frac{k+1}{(\log r^\eta)^{k+1}} \lim_{\epsilon \to 0} \frac{1}{r^\eta - r^{\eta(1-\epsilon)}} \left[\int_1^{r^\eta} \frac{\log G(x) - \log G(x^{1-\epsilon})}{x - x^{1-\epsilon}} \cdot (1 - x^{-\epsilon}) (\log x)^k \, dx \right]$$

$$> \frac{k+1}{(\log r^\eta)^{k+1}} \lim_{\epsilon \to 0} \frac{1}{r^\eta - r^{\eta(1-\epsilon)}} \left[\int_r^{r^\eta} \frac{\log G(x) - \log G(x^{1-\epsilon})}{x - x^{1-\epsilon}} \cdot (1 - x^{-\epsilon}) (\log x)^k \, dx \right]$$

$$> \frac{k+1}{r^\eta . \eta^{k+1} (\log r)^{k+1}} \left(\lim_{\epsilon \to 0} \frac{1 - r^{-\epsilon}}{1 - r^{-\eta\epsilon}} \right) \left[\int_r^{r^\eta} \lim_{\epsilon \to 0} \left\{ \frac{\log G(x) - \log G(x^{1-\epsilon})}{x - x^{1-\epsilon}} \right\} \cdot (\log x)^k \, dx \right]$$

$$= \frac{k+1}{r^{\eta}.\ \eta^{k+2} (\log r)^{k+1}} \int_{r}^{r^{\eta}} \frac{G'(x)}{G(x)} (\log x)^k \, dx$$

$$> \frac{k+1}{r^{\eta-1}.\ \eta^{k+2} (\log r)^{k+1}} \frac{G'(r)}{G(r)} \int_{r}^{r^{\eta}} (\log x)^k \frac{dx}{x}$$

$$= \frac{\eta^{k+1}-1}{\eta^{k+2}} \frac{G'(r)}{r^{\eta-1}\, G(r)},$$

which implies

$$\log \frac{r^{\eta}.\ g_k^{*\prime}(r^{\eta})}{g_k^{*}(r^{\eta})} > \log \left\{ \frac{r\, G'(r)}{G(r)} \right\} + 0(1).$$

Hence

$$\lim_{r\to\infty} \frac{\log \{r g_k^{*\prime}(r) / g_k^{*}(r)\}}{\log \log r} \geqslant \frac{\varrho^*_1}{\lambda^*_1}. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain (4.2).

Now, proceeding on the above lines, we find that

$$\frac{g_k'(r)}{g_k(r)} = \lim_{\epsilon\to 0} \left\{ \frac{\log g_k(r) - \log g_k(r - \epsilon r)}{\epsilon r} \right\}$$

$$< \frac{G'(r)}{G(r)} + 0(1), \tag{4.8}$$

and

$$\frac{g_k'(\beta r)}{g_k(\beta r)} > \frac{\beta^{k+1}-1}{\beta^{k+2}} \frac{G'(r)}{G(r)},\ (\beta > 1), \tag{4.9}$$

for all sufficiently large values of r. The inequalities (4.8). and (4.9) coupled with (4.1) imply the relation (4.3).

NOTE: For completeness sake the ontlines of the proofs of theorems 1 and 2 above have also been given elsewhere by the authors.

5. COMPARATIVE GROWTH OF $G(r)$ AND $g^*(r)$.

THEOREM 3. *If*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{g_k^*(r)}{G(r)} \right\}^{\frac{1}{N(r)}} = \frac{L}{M},$$

then

$$\exp\left(-\frac{\lambda_1^*+1}{k+2}\right) \leqslant L, \tag{5.1}$$

and

$$e^{-1} \leqslant M \leqslant L \leqslant 1, \tag{5.2}$$

provided $\lambda_1^* < \infty$.

For this, we need the following lemma:

LEMMA 3. *This is*

$$\liminf_{r\to\infty} \frac{n(r)\log r}{N(r)} \leqslant \lambda_1^* + 1. \tag{5.3}$$

Proof of lemma 3. Suppose (5.3) is not true; then we have

$$n(r)\log r < (\lambda_1^* + 1 + 2\epsilon)\, N(r),\ r \geqslant r_0(\epsilon),\ \epsilon > 0.$$

Therefore, for $R > r_0$, we find that

$$(\lambda^*_1 + 1 + 2\epsilon)\int_{r_0}^{R} N(x)(\log x)^{-\lambda_1^*-2-\epsilon}\frac{dx}{x} < \int_{r_0}^{R} n(x)(\log x)^{-\lambda_1^*-1-\epsilon}\frac{dx}{x}$$

$$= \int_{r_0}^{R} (\log x)^{-\lambda_1^*-1-\epsilon}\, dN(x)$$

$$= \left| N(x)(\log x)^{-\lambda^*_1-1-\epsilon}\right|_{x=r_0}^{x=R}$$

$$+ (\lambda_1^* + 1 + \epsilon)\int_{r_0}^{R} N(x)(\log x)^{-\lambda_1^*-2-\epsilon}\frac{dx}{x},$$

which implies

$$\epsilon\int_{r_0}^{R} (\log x)^{-\lambda^*_1-2-\epsilon} N(x)\frac{dx}{x} < N(R)(\log R)^{-\lambda_1^*-1-\epsilon}$$

$$< \frac{n(R)(\log R)^{-\lambda_1^*-\epsilon}}{(\lambda_1^* + 1 + 2\epsilon)}. \tag{5.4}$$

Also, by (2.2), we have

$$n(R)(\log R)^{-\lambda_1^*-\epsilon} \to 0,$$

for a sequence of the values of R tending to infinity. However, the integrand in (5.4) is positive and R is independent of r_0, so this makes (5.4) impossible. Hence (5.3) must be true.

Proof of theorem 3. By (2.3) and (3.5) we have

$$(5.5)\quad \left\{\frac{g_k^*(r)}{G(r)}\right\}^{\frac{1}{N(r)}} = \exp\left[\frac{1}{N(r)}\left\{\frac{k+1}{(\log r)^{k+1}}\int_1^r N(x)(\log x)^k \frac{dx}{x} - N(r)\right\}\right]$$

$$= \exp\left[-\frac{1}{N(r)(\log r)^{k+1}}\int_1^r n(x)(\log x)^{k+1}\frac{dx}{x}\right]$$

$$\geqslant \exp\left[-\frac{n(r)\log r}{(k+2)N(r)}\right].$$

Hence (5.1) follows when lemma 3 is used in the above inequality.

Again, from (5.5), we have

$$(5.6)\qquad \left\{\frac{g_k^*(r)}{G(r)}\right\}^{\frac{1}{N(r)}} \geqslant \exp\left[\frac{1}{N(r)}(N(1)-N(r))\right] \sim e^{-1},$$

and

$$(5.7)\qquad \left\{\frac{g_k^*(r)}{G(r)}\right\}^{\frac{1}{N(r)}} \leqslant \exp\left[\frac{1}{N(r)}(N(r)-N(r))\right] = 1.$$

So, (5.2) is proved.

THEOREM 4. *For a class of entire functions for which* $\log\log G(r)$ *is a convex function of* $\log\log r$, *we have*

$$(5.8)\qquad \lim_{r\to\infty}\ {\sup \atop \inf}\left\{\frac{\log_3 g_k^*(r)}{\log\log r}\right\} = \frac{\log A}{\log B},$$

where

$$(5.9)\qquad \lim_{r\to\infty}\ {\sup \atop \inf}\left\{\frac{\log G(r)}{\log g_k^*(r)}\right\}^{\frac{1}{\log\log r}} = \frac{A}{B},$$

and

$$\log_3 X = \log \log \log X.$$

To prove this theorem we require:

LEMMA 4. *Under the hypothesis of theorem* 4, $(\log r)^{k+1} \log G(r)$ *is a convex function of* $(\log r)^{k+1} \log g_k^*(r)$.

Proof. Since $\log \log G(r)$ is a convex function of $\log \log r$, we have (c.f.[5], equation (4), p. 73.)

$$\log \log G(r) = \log \log G(r_0) = \log \log G(r_0) + \int_{r_0}^{r} w(x).\ d(\log \log x),$$

where, $w(x)$ is non-decreasing and tends to infinity with x. Also.

$$\frac{d\,[(\log r)^{k+1} \log G(r)]}{d\,[(\log r)^{k+1} \log g_k^*(r)]} = \frac{d/dr\,[(\log r)^{k+1} \log G(r)]}{d/dr\,[(\log r)^{k+1} \log g_k^*(r)]}$$

$$= 1 + \frac{r\,G'(r) \log r}{(k+1)\,G(r) \log G(r)}$$

$$= 1 + \frac{w(r)}{k+1}.$$

Therefore

$$(\log r)^{k+1} \log G(r) = 0(1) + \int_{r_0}^{r} w^*(x).\ d\,((\log x)^{k+1} \log g^*_k(x)),$$

where

$$w^*(x) = 1 + \frac{w(x)}{k+1}.$$

Hence the lemma follows.

Proof of the theorem. It is readily seen from the definition of $g_k^*(r)$ that

$$\log \{(\log r)^{k+1} \log g_k^*(r)\} = 0(1) + (k+1) \int_{r_0}^{r} \frac{\log G(x)}{\log g_k^*(x)} \frac{dx}{x \log x}\ (r_0 > 1),$$

$$< 0(1) + (k+1)\int_{r_0}^{r} (A+\epsilon)^{\log\log x} \frac{dx}{x \log x}$$

$$= \frac{(k+1)(A+\epsilon)^{\log\log r}}{\log (A+\epsilon)} + 0(1),$$

and so

$$\limsup_{r\to\infty} \frac{\log_3 g_k^*(r)}{\log\log r} \leqslant \log A, \tag{5.10}$$

since, using the hypothesis in theorem 1, $\lambda_1^* = \infty$.

Again, for $\beta > 1$

$$\log [(\log r^{\beta})^{k+1} \log g_k^*(r^{\beta})] = (k+1)\int_{1}^{r^{\beta}} \frac{\log G(x)}{\log g_k^*(x)} \frac{dx}{x\log x}$$

$$< (k+1)\int_{r}^{r^{\beta}} \frac{\log G(x)}{\log g_k^*(x)} \frac{dx}{x\log x}.$$

By lemma 4 and the relation (5.9), the above inequality reduces to

$$\log [(\log r^{\beta})^{k+1} \log g_k^*(r^{\beta})] > (k+1)(A-\epsilon)^{\log\log r} \log \beta,$$

for a sequence of values of r which tend to infinity.

Consequently

$$\limsup_{r\to\infty} \frac{\log_3 g_k^*(r)}{\log\log r} \geqslant \log A. \tag{5.11}$$

Thus the proof for the limit superior in (5.8) follows from (5.10) and (5.11). The proof for the limit inferior in (5.8) can similarly be disposed of.

6. Comparative Growth of $G(r)$ and $g_k(r)$.

THEOREM 5. *If*

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \left\{ \frac{g_k(r)}{G(r)} \right\}^{\frac{1}{N(r)}} = \begin{matrix} A \\ B \end{matrix},$$

then

$$e^{-1} \leqslant B \leqslant A = 1, \tag{6.1}$$

Proof. Using (2.3) in (3.5) we notice that

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{1}{N(r)}} = \exp\left[\frac{1}{N(r)}\left\{\frac{k+1}{r^{k+1}}\int_0^r x^k N(x)\,dx - N(r)\right\}\right] \tag{6.2}$$

$$= \exp\left[-\frac{1}{N(r)\,r^{k+1}}\int_0^r x^k \,.\, n(x)\,dx\right]$$

$$\geqslant \exp\left[-\frac{n(r)}{(k+1)\,N(r)}\right].$$

Now, applying lemma 3 in the above inequality, we get

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{1}{N(r)}} \geqslant \exp\left[-\frac{\lambda_1{}^* + 1 + \epsilon}{(k+1)\log r}\right]$$

which implies

$$A \geqslant 1. \tag{6.3}$$

Also, from (6.2), we have

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{1}{N(r)}} \geqslant \exp\left[\frac{1}{N(r)}\left\{N(0) - N(r)\right\}\right] \tag{6.4}$$

$$\sim e^{-1},$$

and

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{1}{N(r)}} \geqslant \exp\left[\frac{1}{N(r)}\left\{N(r) - N(r)\right\}\right] = 1 \tag{6.5}$$

Therefore

$$e^{-1} \leqslant B \leqslant A \leqslant 1. \tag{6.6}$$

Hence (6.1) follows from (6.3) and (6.6).

THEOREM 6. *If*

$$\limsup_{r\to\infty} \left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{\log r}{N(r)}} = H,$$

then

$$\exp.\left(-\frac{\lambda_1^* + 1}{k+1}\right) \leqslant H, \tag{6.7}$$

provided $\lambda_1^* < \infty$.

Proof. Following the earlier part of the proof of theorem 5, we see that

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{\frac{\log r}{N(r)}} \geqslant \exp\left\{-\frac{n(r)\log r}{(k+1)N(r)}\right\}.$$

In view of lemma 3, the above inequality implies (6.7).

REMARK. It is worth-noting that some of the results for the function $g_k^*(r)$ and $g_k(r)$ are analogous but at the same time quite different from each other. Compare, for instance, (5.2) with (6.1).

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