THE DIVERGENT DISTRIBUTION PRODUCT $x_+^\lambda x_-^\mu$

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1. INTRODUCTION

In the following, the product of two distributions f and g is defined, as in [1], as the limit of the sequence $\{f_n g_n\}$, provided this sequence is regular, where

$$f_n = f * \delta_n, \qquad g_n = g * \delta_n, \qquad \delta_n(x) = n\,\varrho(nx)$$

for $n = 1, 2, \ldots$ and ϱ is a fixed infinitely differentiable function having the following properties:

$$\varrho(x) = 0 \quad \text{for} \quad |x| \geq 1, \tag{1}$$

$$\varrho(x) \geq 0, \tag{2}$$

$$\varrho(x) = \varrho(-x), \tag{3}$$

$$\int_{-1}^{1} \varrho(x)\,dx = 1. \tag{4}$$

It is obvious that $\{\delta_n\}$ is a regular sequence converging to the Dirac delta-function $\delta(x)$.

If the sequence $\{f_n g_n\}$ is regular the product fg is said to be convergent, otherwise it is said to be divergent.

We will now suppose that f, g, F and G are distributions for which the products fg and FG are both divergent. We will suppose further that the sequence $\{f_n g_n + F_n G_n\}$ is regular and converges to the distribution h. Then, as in [2], we will say that the sum of the products fg and FG exists and is equal to h and write

$$fg + FG = h,$$

even though the individual products fg and FG are divergent.

2. We will now consider the product $x_+^\lambda x_-^\mu$, where $\lambda + \mu > -2$ and $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$. It is obvious that if $\lambda + \mu > -1$ then

$$x_+^\lambda x_-^\mu = 0$$

and it was proved in [1] that

$$x_+^\lambda x_-^{-1-\lambda} = -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\,\delta(x)$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. We will therefore consider the product $x_+^\lambda \pi_-^\mu$ with $-2 < \lambda + \mu < -1$ and $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$.

We will first of all suppose that $\lambda, \mu > -1$. Then x_+^λ and x_-^μ are ordinary summable functions and

$$(x_+^\lambda)_n = x_+^\lambda * \delta_n(x)$$
$$= \begin{cases} \displaystyle\int_{-1/n}^{x} (x-t)^\lambda \,\delta_n(t)\,dt, & \text{for } x \geq -1/n \\ 0\quad, & \text{for } x \leq -1/n \end{cases}$$

$$(x_-^\mu)_n = x_-^\mu * \delta_n(x)$$
$$= \begin{cases} \displaystyle\int_{x}^{1/n} (s-x)^\mu \,\delta_n(s)\,ds, & \text{for } x \leq 1/n \\ 0\quad, & \text{for } x \geq 1/n. \end{cases}$$

It follows that the support of $(x_+^\lambda)_n (x_-^\mu)_n$ is contained in the interval $(-1/n, 1/n)$ and

$$\int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n\,dx = \int_{1/n}^{-1/n}\int_{-1/n}^{x}\int_{x}^{1/n} (x-t)^\lambda (s-x)^\mu \,\delta_n(t)\,\delta_n(s)\,ds\,dt\,dx$$
$$= \int_{-1/n}^{1/n} \delta_n(t) \int_{t}^{1/n} \delta_n(s) \int_{t}^{s} (x-t)^\lambda (s-x)^\mu \,dx\,ds\,dt$$

on changing the order of integration, which is permissible.

Making the substitution

$$x = t + (s-t)\,v$$

we have

$$\int_t^s (x-t)^\lambda (s-x)^\mu \, dx = \int_0^1 (s-t)^{\lambda+\mu+1} v^\lambda (1-v)^\mu \, dv$$
$$= (s-t)^{\lambda+\mu+1} B(\lambda+1, \mu+1),$$

where B denotes the beta function. Thus

$$\int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n \, dx$$
$$= B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (s-t)^{\lambda+\mu+1} \delta_n(s) \, ds \, dt$$
$$= B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} \delta_n(x) \int_x^{1/n} (s-x)^{\lambda+\mu+1} \delta_n(s) \, ds \, dx$$
$$= B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} (x_-^{\lambda+\mu+1})_n \delta_n(x) \, dx$$
$$= B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} \delta_n(x) \int_{-1/n}^{x} (x-t)^{\lambda+\mu+1} \delta_n(t) \, dt \, dx$$
$$= B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} (x_+^{\lambda+\mu+1})_n \delta_n(x) \, dx.$$

Since $\lambda+\mu+1<0$ it follows that

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} (x_-^{\lambda+\mu+1})_n \delta_n(x) \, dx = \lim_{n\to\infty} \int_{-1/n}^{1/n} (x_+^{\lambda+\mu+1})_n \delta_n(x) \, dx = \infty$$

which implies that

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n \, dx = \infty.$$

It follows that the products, $x_+^\lambda x_-^\mu$, $x_+^{\lambda+\mu+1}\delta(x)$ and $x_-^{\lambda+\mu+1}\delta(x)$ are all divergent but we notice that

$$\int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n \, dx - B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} (x_-^{\lambda+\mu+1})_n \delta_n(x) \, dx$$
$$= \int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n \, dx - B(\lambda+1, \mu+1) \int_{-1/n}^{1/n} (x_+^{\lambda+\mu+1})_n \delta_n(x) \, dx = 0.$$

Next, since $(x_+^\lambda)_n$, $(x_-^\mu)_n$, $\delta_x(x) \geq 0$, we have

$$\int_{-1/n}^{1/n} |(x_+^\lambda)_n (x_-^\mu)_n x| \, dx \leq n^{-1} \int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n \, dx$$

$$= B(\lambda + 1, \ \mu + 1) \, n^{-1} \int_{-1/n}^{1/n} \delta_n(x) \int_{-1/n}^{x} (x - t)^{\lambda+\mu+1} \delta^\mu(t) \, dt \, dx$$

$$\leq B(\lambda + 1, \ \mu + 1) \, n^{-\lambda-\mu-2} \int_{-1}^{1} \varrho(v) \int_{-1}^{1} (v - u)^{\lambda+\mu+1} \varrho(u) \, du \, dv$$

$$\leq B(\lambda + 1, \ \mu + 1) \, n^{-\lambda-\mu-2} k \int_{-1}^{1} \int_{-1}^{1} (v - u)^{\lambda+\mu+1} \, du \, dv,$$

where $nx = v$, $nt = u$ and $k = \sup \varrho(x)$. Since $\lambda + \mu + 2 > 0$ it follows that

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} |(x_+^\lambda)_n (x_-^\mu)_n x| \, dx = 0.$$

Similarly we have

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} |(x_-^{\lambda+\mu+1})_n \delta_n(x) x| \, dx = \lim_{n\to\infty} \int_{-1/n}^{1/n} |(x_+^{\lambda+\mu+1})_n \delta_n(x) x| \, dx = 0.$$

Now let ϕ be an arbitrary test function in the space K of infinitely differentiable functions with compact support. We have

$$\phi(x) = \phi(0) + x \, \phi'(\xi x)$$

where $0 \leq \xi \leq 1$. Then by what we have proved above

$$|((x_+^\lambda)_n (x_-^\mu)_n - B(\lambda + 1, \ \mu + 1) (x_+^{\lambda+\mu+1})_n \delta_n(x), \ \phi)|$$

$$= \left| \int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^\mu)_n x \, \phi'(\xi x) \, dx \right.$$

$$\left. - B(\lambda + 1, \ \mu + 1) \int_{-1/n}^{1/n} (x_+^{\lambda+\mu+1})_n \delta_n(x) \, x \, \phi'(\xi x) \, dx \right|$$

$$\leq \sup |\phi'(x)| \left\{ \int_{-1/n}^{1/n} |(x_+^\lambda)_n (x_-^\mu)_n x| \, dx \right.$$

$$\left. + B(\lambda + 1, \ \mu + 1) \int_{-1/n}^{1/n} |(x_+^{\lambda+\mu+1})_n \delta_n(x) x| \, dx \right\}$$

and so

$$\lim_{n\to\infty} ((x_+^\lambda)_n (x_-^\mu)_n - B(\lambda+1, \mu+1)(x_+^{\lambda+\mu+1})_n \delta_n(x), \phi) = 0$$

or

$$x_+^\lambda x_-^\mu - B(\lambda+1, \mu+1) x_+^{\lambda+\mu+1} \delta(x) = 0 \tag{1}$$

for $\lambda, \mu > -1$ and $-2 < \lambda + \mu < -1$, even though the products $x_+^\lambda x_-^\mu$ and $x_+^{\lambda+\mu+1} \delta(x)$ are both divergent.

Similarly we have

$$x_+^\lambda x_-^\mu - B(\lambda+1, \mu+1) x_-^{\lambda+\mu+1} \delta(x) = 0 \tag{2}$$

for $\lambda, \mu > -1$ and $-2 < \lambda + \mu < -1$.

It follows from equations (1) and (2) that

$$(x_+^\lambda - x_-^\lambda) \delta(x) = (sgnx\,|x^\lambda|) \delta(x) = 0$$

for $\lambda > -1$, a result which was in fact given in [2].

We will now suppose that equation (1) holds for $-r < \lambda < -r+1$, $r-2 < \mu < r-1$, and $-2 < \lambda + \mu < -1$. For such λ, μ we then have

$$\lim_{n\to\infty} (x_+^\lambda)_n (x_-^{\mu+1})_n = x_+^\lambda x_-^{\mu+1} = 0$$

since $\lambda + \mu + 1 > -1$. Differentiating this equation we see that

$$\lambda x_+^{\lambda-1} x_-^{\mu+1} - (\mu+1) x_+^\lambda x_-^\mu = 0.$$

It follows from equation (1) that

$$\begin{aligned} x_+^{\lambda-1} x_-^{\mu+1} &- (\mu+1)\lambda^{-1} B(\lambda+1, \mu+1) x^{\lambda+\mu+1} \delta(x) = 0 \\ &= x_+^{\lambda-1} x_-^{\mu+1} - B(\lambda, \mu+2) x_+^{\lambda+\mu+1} \delta(x) \end{aligned}$$

and so equation (1) holds by our assumption for $-r-1 < \lambda < -r$, $r-1 < \mu < r$ and $-2 < \lambda + \mu < -1$. It now follows by induction that equation (1) holds for $\lambda < 0$, $\lambda \neq -1, -2, \ldots$, $\mu > -1$, $\mu \neq 0, 1, 2, \ldots$ and $-2 < \lambda + \mu < -1$.

By a similar induction argument it follows that equation (1) holds for $\lambda > -1$, $\lambda \neq 0, 1, 2, \dots$, $\mu < 0$, $\mu \neq -1, -2, \dots$ and $-2 < \lambda + \mu < -1$. It follows that we have proved that equation (1) holds for λ, $\mu \neq 0, \pm 1, \pm 2, \dots$ and $-2 < \lambda + \mu < -1$.

Similarly, equation (2) holds for λ, $\mu \neq 0, \pm 1, \pm 2, \dots$ and $-2 < \lambda + \mu < -1$.

By the symmetry it also follows that

$$x_+^\lambda x_-^\mu - x_+^\mu x_-^\lambda = 0$$

for λ, $\mu \neq 0, \pm 1, \pm 2, \dots$ and $-2 < \lambda + \mu < -1$.

3. We can now prove that

$$x_+^\lambda x_-^\mu - (-1)^r \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+r+2)} x_+^{\lambda+\mu+r+1} \delta^{(r)}(x) = 0 \tag{3}$$

for λ, $\mu \neq 0, \pm 1, \pm 2, \dots, -2 < \lambda + \mu < -1$ and $r = 0, 1, 2, \dots$. The particular case $r = 0$ is of course equation (1).

We will assume that equation (3) holds for some r. Then differentiating the equation

$$\lim_{n \to \infty} (x_+^{\lambda+\mu+r+2})_n \, \delta_n^{(r)}(x) = x_+^{\lambda+\mu+r+2} \delta^{(r)}(x) = 0$$

we get

$$(\lambda + \mu + r + 2)\, x_+^{\lambda+\mu+r+1} \delta^{(r)}(x) + x_+^{\lambda+\mu+r+2} \delta^{(r+1)}(x) = 0.$$

It follows from our assumption that

$$x_+^\lambda x_-^\mu + (-1)^r \frac{\Gamma(\lambda+1)\Gamma(\mu+1}{(\lambda+\mu+r+2)\Gamma(\lambda+\mu+r+2)} x_+^{\lambda+\mu+r+2} \delta^{(r+1)}(x) = 0$$

$$= x_+^\lambda x_-^\mu - (-1)^{r+1} \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+r+3)} x_+^{\lambda+\mu+r+2} \delta^{(r+1)}(x)$$

and so equation (3) holds for $r + 1$. Our result now follows by induction.

Replacing x by $-x$ and interchanging λ and μ in equation (3), it follows that

$$x_+^\lambda x_-^\mu - \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+r+2)} x_+^{\lambda+\mu+r+2} \delta^{(r)}(x) = 0 \tag{4}$$

for λ, $\mu \neq 0, \pm 1, \pm 2, \dots, -2 < \lambda + \mu < -1$ and $r = 0, 1, 2, \dots$.

It also follows immediately from equation (3) that

$$(-1)^r \Gamma(\lambda+\mu+s+2)\, x_+^{\lambda+\mu+r+1}\, \delta^{(r)}(x)$$
$$-(-1)^s \Gamma(\lambda+\mu+r+2)\, x_+^{\lambda+\mu+s+1}\, \delta^{(s)}(x) = 0$$

and from equation (4) that

$$\Gamma(\lambda+\mu+s+2)\, x_-^{\lambda+\mu+\ +1}\, \delta^{(r)}(x)$$
$$-\Gamma(\lambda+\mu+r+2)\, x^{\lambda}{}_-^{+\mu+s+1}\, \delta^{(s)}(x) = 0$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots, -2 < \lambda + \mu < -1$ and $r, s = 0, 1, 2, \ldots$.

4. We next note that

$$\lim_{n\to\infty} (x_+^{\lambda+\mu+1})_n H_n(x) = x_+^{\lambda+\ +\mu+1} H(x) = x_+^{\lambda+\mu+1}$$

for $-2 < \lambda + \mu < -1$, where $H(x)$ denotes Heaviside's function. Differentiating this equation we get

$$(\lambda+\mu+1)\, x_+^{\lambda+\mu} H(x) + x_+^{\lambda+\mu+1}\, \delta(x) = (\lambda+\mu+1)\, x^{\lambda+\mu}.$$

It follows from equation (3) that

$$x_+^\lambda x_-^\mu + B(\lambda+1, \mu+1)\,(\lambda+\mu+1)\, x_+^{\lambda+\mu} H(x)$$
$$= B(\lambda+1, \mu+1)\,(\lambda+\mu+1)\, x_+^{\lambda+\mu}$$

or

$$x_+^\lambda x_-^\mu + \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+1)}\, x_+^{\lambda+\mu} H(x) = \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+1)}\, x_+^{\lambda+\mu} \tag{5}$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$ and $-2 < \lambda + \mu < -1$.

We can now prove by induction that

$$x_+^\lambda x_-^\mu + (-1)^r \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{r!\,\Gamma(\lambda+\mu-r+1)}\, x_r^{\lambda+\mu-r}\, x_+^r$$
$$= (-1)^r \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{r!\,\Gamma(\lambda+\mu-r+1)}\, x_+^{\lambda+\mu} \tag{6}$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots, -2 < \lambda + \mu < -1$ and $r = 0, 1, 2, \ldots$.

We note first of all that the particular case $r = 0$ is equation (5) and so equation (6) certainly holds when $r = 0$. Now suppose equation (6) holds for some r. Then differentiating the equation

$$\lim_{n\to\infty} (x_+^{\lambda+\mu-r})_n (x_+^{r+1})_n = x_+^{\lambda+\mu-r} x_+^{r+1} = x_+^{\lambda+\mu+1}$$

we get

$$(\lambda + \mu - r) x_+^{\lambda+\mu-r-1} x_+^{r+1} + (r + 1) x_+^{\lambda+\mu-r} x_+^{r} = (\lambda + \mu + 1) x_+^{\lambda+\mu}.$$

It follows from our assumption that

$$x_+^{\lambda} x_-^{\mu} - (-1)^r \frac{(\lambda + \mu - r)\,\Gamma(\lambda + 1)\,\Gamma(\mu + 1)}{(r + 1)!\,\Gamma(\lambda + \mu - r + 1)} x_+^{\lambda+\mu-r-1} x_+^{r+1}$$

$$= (-1)^r \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{r!\,\Gamma(\lambda+\mu-r+1)} x_+^{\lambda+\mu} - (-1)^r \frac{(\lambda+\mu+1)\Gamma(\lambda+1)\Gamma(\mu+1)}{(r+1)!\,\Gamma(\lambda+\mu-r+1)} x_+^{\lambda+\mu}$$

or

$$x_+^{\lambda} x_-^{\mu} + (-1)^{r+1} \frac{\Gamma(\lambda + 1)\,\Gamma(\mu + 1)}{(r + 1)!\,\Gamma(\lambda + \mu - r)} x_+^{\lambda+\mu-r-1} x_+^{r-1}$$

$$= (-1)^{1r+1} \frac{\Gamma(\lambda + 1)\,\Gamma(\mu + 1)}{(r + 1)!\,\Gamma(\lambda + \mu - r)} x_+^{\lambda+\mu}$$

and so equation (6) holds for $r + 1$. Our result now follows by induction.

Replacing x by $-x$ and interchanging λ and μ in equation (6) we get

$$x_+^{\lambda} x_-^{\mu} + (-1)^r \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{r!\,\Gamma(\lambda+\mu-r+1)} x_-^{\lambda+\mu-r} x_-^{r} = (-1)^r \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{r!\,\Gamma(\lambda+\mu-r+1)} x_+^{\lambda+\mu}$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots, -2 < \lambda + \mu < -1$ and $r = 0, 1, 2, \ldots$.

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