REFLECTION PRINCIPLES FOR THE ITERATED HELMHOLTZ EQUATION

by

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Introduction.

The well known classical reflection principle for harmonic functions, due to H. A. Schwarz, has been extended, in the mathematical literature, in many directions.

The basic ingredients for such a reflection principle are:
(a) the partial differential equation, and (b) appropriate boundary conditions. In the classical Schwarz reflection principle, the partial differential equation is the Laplace equation

$$\Delta u = 0,$$

where $\Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian, and the boundary condition in (b) might be called the Dirichlet boundary condition, since the solution of the partial differential equation is required to vanish on the boundary. This terminology is convenient for describing quickly the results of the present paper, and their exact relation to the results of earlier papers.

It is to be kept in mind that the reflection considered here is only across «flat» boundaries. Using the terminology just introduced, the results of Diaz and Ludford [1], where references to the earlier literature are to be found, can be described by saying that the partial differential equation in (a) is the Helmholtz equation

$$\Delta u + \lambda u = 0,$$

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where λ is a real constant, and Δ is the Laplacian, while the boundary conditions in (b) are of three kinds:

(b_1) of Dirichlet type, that is, requiring that the function itself vanish on the boundary;

(b_2) involving a linear combination, with constant coefficients, of the normal derivative and the function;

(b_3) involving a linear combination, with constant coefficients, of a directional derivative, in a direction which is not tangential to the boundary, and the function.

The reflection principle (a), (b_1) of [1], for the Helmholtz equation, is obtained from what might be called the «classical» Schwarz reflection principle (that is to say, when (a) is the Laplace equation and (b) is the «Dirichlet» boundary condition) by the method of descent.

The reflection principle, in which (a) is the biharmonic partial differential equation and (b) is the Dirichlet boundary condition for the biharmonic equation (that is to say, vanishing of the function and of its normal derivative), is given in the work of Poritsky [2], Duffin [3], and Huber [4,5]. Their result was used by Diaz and Ram [6], using the method of descent, to obtain a reflection principle in which (a) is the iterated Helmholtz equation

$$(\Delta - \lambda^2)^2 u = 0,$$

where λ is a real constant, and Δ is the Laplacian, while (b) is the Dirichlet boundary condition, that is to say, the function and also its normal derivative vanish on the boundary. In the present paper, the partial differential equation in (a) is again the iterated Helmholtz equation, as in [6], while the boundary conditions in (b) are of two kinds:

(b_4) the simultaneous vanishing, on the boundary, of a linear combination, with constant coefficients, of the function and its normal derivative; and, also, of the normal derivative of the same linear combination;

(b_5) the simultaneous vanishing, on the boundary, of a linear combination, with constant coefficients, of the function and its directional derivative (in a fixed direction not tangential to the boundary); and, also, of the directional derivative, in the same fixed direction, of the same linear combination.

It is to be noticed that the present results give, as special cases, the reflection principles for the biharmonic equation, with boundary conditions (b_4) and (b_5), without having to prove them first for the biharmonic equation and then «descending» to the Helmholtz equation.

1. *Reflection principle for the iterated Helmholtz partial differential equation with Dirichlet boundary conditions.*

The following theorem, which is stated here without proof, and is the basis of the considerations of section 2, was proved by the authors in [6].

Theorem 1. Let D be a domain (i.e., an open, non-empty, connected set) in a real n dimensional $x_1,x_2, \ldots, x_{n-1}$, y Euclidean space, where $n \geq 2$ is a positive integer. Suppose that the domain D is symmetric about the $(n - 1)$ dimensional plane $y = 0$ (i. e., whenever the point $(x_1,x_2, \ldots, x_{n-1}, y)$ belongs to the domain D, then the point $(x_1, x_2, \ldots, \ldots, x_{n-1}, -y)$ also belongs to D; so that if d denotes the, supposed non-empty, intersection of the plane $y = 0$ and the domain D, while D^+ and D^- designate, respectively, the two open symmetric parts into which $D - d$ is divided by the plane $y = 0$, one has $D = D^+ + + d + D^-$). Suppose that $u(x_1,x_2, \ldots, x_{n-1}, y)$ is a real valued function of class $C^{(4)}$ in D^+, and satisfies, there, the fourth order λ partial differential equation

$$(1.1) \qquad (\Delta - \lambda^2)^2 u(x_1,x_2,\ldots,x_{n-1}, y) = 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian, λ is real a constant, and that, further

$$(1.2) \qquad \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} u(x_1, x_2, \ldots, x_{n-1}, y) = 0,$$

$$(1.3) \qquad \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, y) = 0,$$

for $(x_1, x_2, \ldots, x_{n-1}, y)$ in D^+ and $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0)$ in d. Then, the real valued function U, defined in D by

$$U(x_1, x_2, \ldots, x_{n-1}, y)$$

$$= \begin{cases} u(x_1, x_2, \ldots, x_{n-1}, y), & \text{for } x_1, x_2, \ldots, x_{n-1}, y) \; \varepsilon \; D^+, \\ 0, & \text{for } (x_1, x_2, \ldots, x_{n-1}, 0) \; \varepsilon \; d, \\ - u(x_1, x_2, \ldots, x_{n-1}, -y) & \\ - 2y \dfrac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y) & \\ - y^2(\Delta - \lambda^2)\, u(x_1, x_2, \ldots, x_{n-1}, -y), & \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \; \varepsilon \; D^-, \end{cases}$$

where the point $(x_1, x_2, \ldots, x_{n-1}, -y)$ is the mirror image of the point $(x_1, x_2, \ldots, x_{n-1}, y)$ with respect to the hyperplane $y = 0$, is an analytic solution of the fourth order λ partial differential equation (1.1) throughout the domain D.

2. *Reflection principle for the iterated Helmoltz partial differential equation with «mixed Neumann boundary conditions».*

The purpose of this section is to prove the two reflection principles stated as theorems 2 and 3.

Theorem 2. Let D and u be as in theorem 1, except that now the domain D is further required to possess the geometric property that, if $(x_1, x_2, \ldots, x_{n-1}, y)$ is any point of D, then D also contains the entire closed straight line interval, on the perpendicular from $(x_1, x_2, \ldots, x_{n-1}, y)$ to d, which joins $(x_1, x_2, \ldots, x_{n-1}, y)$ to a point of d; and that the boundary conditions (1.2) and (1.3) on the function u are now replaced, respectively, by

$$\lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \left[\frac{\partial y}{\partial u}(x_1, x_2, \ldots, x_{n-1}, y) + \right.$$

(2.1)

$$\left. + ku(x_1, x_2, \ldots, x_{n-1}, y)\right] = 0,$$

$$\lim_{\substack{(x_1,x_2,\dots,x_{n-1},y)\to(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_{n-1},0)\\ y>0}}\left[\frac{\partial^2 u}{\partial y^2}(x_1,x_2,\dots,x_{n-1},y)+\right.$$

(2.2)

$$\left.+k\frac{\partial u}{\partial y}(x_1,x_2,\dots,x_{n-1},y)\right]=0,$$

for $(x_1, x_2, \dots, x_{n-1}, y)$ in D^+ and $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, 0)$ in d, where k is a real constant. Then, there is a uniquely determined real single-valued function U (for its definition, see equation (2.9) and the proof below), defined throughout D, which satisfies the fourth order λ partial differential equation (1.1) throughout D, and coincides with u in D^+.

Proof. Since u is a real valued function of class $C^{(4)}$ in D^+, and satisfies, there, the fourth order λ partial differential equation (1.1), it follows (from the theory of the biharmonic equation, without appeal to any general theorem on the analyticity of solutions of elliptic partial differential equations) that the function u is analytic in $(x_1, x_2, \dots, x_{n-1}, y)$ on D^+. Hence, if one defines (compare the boundary condition (2.1))

$$v=\frac{\partial u}{\partial y}+ku$$

in D^+, then

$$\begin{aligned}(\Delta-\lambda^2)^2 v &= (\Delta-\lambda^2)^2\left(\frac{\partial u}{\partial y}+ku\right)=\\ &=(\Delta-\lambda^2)^2\left(\frac{\partial u}{\partial y}\right)+(\Delta-\lambda^2)^2(ku)\\ &=\frac{\partial}{\partial y}[(\Delta-\lambda^2)^2u]+k[(\Delta-\lambda^2)^2u]\\ &=0;\end{aligned}$$

and, clearly, from the boundary conditions (2.1) and (2.2), v satisfies the boundary conditions (1.2) and (1.3). It follows that the function v satisfies the hypotheses required of the function u of theorem 1, and, hence, the function V defined on D by

$$V(x_1, x_2, \ldots, x_{n-1}, y)$$

$$(2.3) = \begin{cases} v(x_1, x_2, \ldots, x_{n-1}, y), & \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \text{ in } D^+, \\ 0, & \text{for } (x_1, x_2, \ldots, x_{n-1}, 0) \text{ in } d, \\ -v(x_1, x_2, \ldots, x_{n-1}, -y) \\ -2y \dfrac{\partial v}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y) \\ -y^2(\Delta - \lambda^2)\, v\, (x_1, x_2, \ldots, x_{n-1}, -y), \\ & \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \text{ in } D^-, \end{cases}$$

where $(x_1, x_2, \ldots, x_{n-1}, -y)$ is the mirror image of $(x_1, x_2, \ldots, x_{n-1}, y)$ with respect to the hyperplane $y = 0$, is analytic in $(x_1, x_2, \ldots, x_{n-1}, y)$, and satisfies the fourth order λ partial differential equation

$$(\Delta - \lambda^2)^2 V = 0$$

throughout D.

The geometric restriction placed upon D means simply that if $(x_1, x_2, \ldots, x_{n-1}, y)$ is a point of D^+, then the set of all points $(x_1, x_2, \ldots, x_{n-1}, s)$, where $0 \leq s \leq y$, is a subset of D^+; while, if $(x_1, x_2, \ldots, x_{n-1}, y)$ is a point of D^-, then the set of all points $(x_1, x_2, \ldots, x_{n-1}, s)$, where $y \leq s \leq 0$, is a subset of D^-. The desired extension U of u will be obtained, from the function V, upon making use of this additional geometric property of D just mentioned, by integrating V along straight line segments which are perpendicular to the plane $y = 0$.

A priori, the hypotheses of the theorem (and, in particular, the boundary conditions (2.1) and (2.2)) do not seem to guarantee the existence of the limits

$$(2.4) \qquad \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} u(x_1, x_2, \ldots, x_{n-1}, y),$$

and

$$(2.5) \qquad \lim_{\substack{(x_1, x_2 \ldots x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, y),$$

where $(x_1, x_2, ..., x_{n-1}, y)$ is in D^+ and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ is in d. It will be shown, however, that these limits do indeed exist, and that the function U, which will at first be defined only locally in D, is actually single valued throughout D.

For each point $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ in d, choose a number $h = h(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}) > 0$ so small, that all points $(x_1, x_2, ..., x_{n-1}, y)$ satisfying both $|y| < h$ and $|x_i - \bar{x}_i| < h$, for $i = 1, 2, ..., n-1$, are points of D. Such a choice of h is always possible, since $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ is an interior point of D. Now, let $T_h = T_h(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1})$ denote the open cylindrical tube consisting of all $(x_1, x_2, ..., x_{n-1}, y)$ in D such that the perpendicular projection on d, i.e., the point $(x_1, x_2, ..., x_{n-1}, 0)$, satisfies $|x_i - \bar{x}_i| < h$, for $i = 1, 2, ..., n-1$. Each T_h is an open subset of D. A function U_h will be defined on each $T_h(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1})$ by means of the formula (compare the boundary condition (2.1)):

$$(2.6) \qquad U_h(x_1, x_2, ..., x_{n-1}, y)$$

$$= e^{-ky}\left[\int_{t=h}^{y} e^{kt}V(x_1, x_2, ..., x_{n-1}, t)\, dt + e^{kh}u(x_1, x_2, ..., x_{n-1}, h)\right].$$

Since $V = \frac{\partial u}{\partial y} + \text{ku}$ in D^+, for $(x_1, x_2 \ldots x_{n-1}, y)$ in T_h and $y > 0$,

$$U_h(x_1, x_2, ..., x_{n-1}, y)$$

$$= e^{-ky}\left[\int_{t=h}^{y} e^{kt}\{\frac{\partial u}{\partial t}(x_1, x_2, ..., x_{n-1}, t) + ku(x_1, x_2, ..., x_{n-1}, t)\}\, dt + \right.$$

$$\left. + e^{kh}u(x_1, x_2, ..., x_{n-1}, h)\right],$$

$$= e^{-ky}\left[\int_{t=h}^{y} \frac{\partial}{\partial t}\{e^{kt}u(x_1, x_2, ..., x_{n-1}, t)\}\, dt + e^{kh}u(x_1, x_2, ..., x_{n-1}, h)\right],$$

$$= u(x_1, x_2, ..., x_{n-1}, y).$$

Hence, without any computation, from (1.1), it follows that $(\Delta - \lambda^2)^2 U_h(x_1, x_2, ..., x_{n-1}, y) = 0$ for $(x_1, x_2, ..., x_{n-1}, y)$ in T_h and $y > 0$. But, from (2.6), since the integrand is analytic in $(x_1, x_2, ..., x_{n-1}, t)$ and the second term inside the square bracket is certainly analytic for $(x_1, x_2, ..., x_{n-1}, y)$ on all of T_h, it follows that U_h is analytic in $(x_1, x_2, ..., x_{n-1}, y)$ throughout T_h, and this means that the function

$(\Delta - \lambda^2)^2 U_h$ is also analytic throughout T_h. Since it has just been shown that this function $(\Delta - \lambda^2)^2 U_h$ is zero for $(x_1, x_2, \ldots, x_{n-1}, y)$ in T_h and $y > 0$, it follows, from its analyticity throughout T_h, again without any computation, that one must have

$$(\Delta - \lambda^2)^2 U_h = 0$$

throughout T_h.

So far, the desired analytic extension of u has, apparently, been obtained only locally, on each tube T_h. However, it is readily seen that formula (2.6), upon varying $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0)$ in d, may be used to define a single-valued function U throughout D, this function U being the sought extensión of u to all of D. The function U is single-valued in D, because, whenever the tubes $T_h(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1})$ and $T_h(\bar{x}'_1, \bar{x}'_2, \ldots, \bar{x}'_{n-1})$ have a non-empty intersection, then their corresponding analytic functions U_h and U'_h must coincide with u for all points of T_h and T'_h lying in D^+, and hence U_h and U'_h must be identical throughout the entire common part of T_h and T'_h. This completes the proof of theorem 2.

Since U and $\dfrac{\partial U}{\partial y}$ are continuous in D, the limits appearing in (2.4), (2.5) exist, and are given by

$$\lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} u(x_1, x_2, \ldots, x_{n-1}, y)$$

$$= \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} U(x_1, x_2, \ldots, x_{n-1}, y)$$

$$= U(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0),$$

and

$$\lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, y)$$

$$= \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \frac{\partial U}{\partial y}(x_1, x_2, \ldots, x_{n-1}, y)$$

$$= \frac{\partial U}{\partial y}(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0),$$

for $(x_1, x_2, ..., x_{n-1}, y)$ in D^+ and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ in d, and will be denoted, for simplicity, by $u(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ and $\frac{\partial u}{\partial y}(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ respectively.

Thus, a posteriori, and, with this agreement as to the meanings of $u(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ and $\frac{\partial u}{\partial y}(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$, which was not initially included in the hypotheses, one may put $h = 0$ in the definition (2.6), to obtain

$$U(x_1, x_2, ..., x_{n-1}, y)$$

(2.7)

$$= e^{-ky}\left[\int_{t=0}^{y} e^{kt}\, V(x_1, x_2, ..., x_{n-1}, t)\, dt + u(x_1, x_2, ..., x_{n-1}, 0)\right].$$

For $y < 0$, the definite integral in (2.7) may be rewritten, using the definition (2.3) of V:

$$\int_{t=0}^{y} e^{kt}\left[-v(x_1, x_2, ..., x_{n-1}, -t) - 2t\frac{\partial v}{\partial t}(x_1, x_2, ..., x_{n-1}, -t) - \right.$$

$$\left. - t^2(\Delta - \lambda^2)\, v(x_1, x_2, ..., x_{n-1}, -t)\right] dt$$

(2.8)

$$= \int_{s=0}^{-y} e^{-ks}[v(x_1, x_2, ..., x_{n-1}, s) - 2s\frac{\partial v}{\partial s}(x_1, x_2, ..., x_{n-1}, s) +$$

$$+ s^2(\Delta - \lambda^2)\, v(x_1, x_2, ..., x_{n-1}, s)]\, ds.$$

Integrating by parts the second integrand, $-2\,s\,e^{-ks}\frac{\partial v}{\partial s}$, on the right hand side of (2.8), this equation becomes

$$= 2ye^{ky}v(x_1, x_2, ..., x_{n-1}, -y) + 3\int_{s=0}^{-y} e^{-ks}\, v(x_1, x_2, ..., x_{n-1}, s)\, ds$$

$$- 2k\int_{s=0}^{-y} se^{-ks}\, v(x_1, x_2, ..., x_{n-1}, s)\, ds$$

$$+\int_{s=0}^{-y} s^2 e^{-ks} (\varDelta - \lambda^2)\, v(x_1, x_2, \ldots, x_{n-1}, s)\, ds$$

$$= 2ye^{ky}\left[\frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y) + ku(x_1, x_2, \ldots, x_{n-1}, -y)\right]$$

$$+ 3\int_{s=0}^{-y} e^{-ks}\left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right] ds$$

$$- 2k\int_{s=0}^{-y} se^{-ks}\left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right]ds$$

$$+ \int_{s=0}^{-y} s^2 e^{-ks}(\varDelta - \lambda^2)\left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right] ds.$$

Integrating by parts the first and the second integrals, one has

$$= e^{ky}[(3 + 4ky)\, u\, (x_1, x_2, \ldots, x_{n-1}, -y) + 2y\frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y)]$$

$$- 3u(x_1, x_2, \ldots, x_{n-1}, 0) + \int_0^{-y} e^{-ks}\,\{8ku - 4k^2 su$$

$$+ s^2(\varDelta - \lambda^2)\left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right]\}\, ds.$$

Substituting this back in (2.7), one has, for $y < 0$,

$U(x_1, x_2, \ldots, x_{n-1}, y)$

$$= (3 + 4ky)\, u(x_1, x_2, \ldots, x_{n-1}, -y) + 2y\frac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y)$$

$$- 2e^{-ky}u(x_1, x_2, \ldots, x_{n-1}, 0) + e^{-ky}\int_0^{-y} e^{-ks}\{8ku - 4k^2 su$$

$$+ s^2(\varDelta - \lambda^2)\left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right]\}\, ds.$$

Finally, the sought extension U of u to all of D is given by

$U(x_1, x_2, \ldots, x_{n-1}, y)$

$$
(2.9) = \begin{cases}
u(x_1, x_2, \ldots, x_{n-1}, y) & \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \text{ in } D^+, \\
\lim\limits_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} u(x_1, x_2, \ldots, x_{n-1}, y), & \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \text{ in } D^+ \\
& \text{and } (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \text{ in } d, \\
(3 + 4\,ky)\, u\,(x_1, x_2, \ldots, x_{n-1}, -y) + 2y \dfrac{\partial u}{\partial y}(x_1, x_2, \ldots, x_{n-1}, -y) \\
\quad - 2e^{-ky}\, u\,(x_1, x_2, \ldots, x_{n-1}, 0) + e^{-ky} \displaystyle\int_0^{-y} e^{-ks} \{8\,ku - 4k^2\, su \\
\quad + s^2(\Delta - \lambda^2) \left[\dfrac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, s) + ku(x_1, x_2, \ldots, x_{n-1}, s)\right]\}\, ds, \\
& \text{for } (x_1, x_2, \ldots, x_{n-1}, y) \text{ in } D^-.
\end{cases}
$$

Theorem 3. Let D and u be as in theorem 1, except that now the domain D is further required to possess the geometric property that, if $(x_1, x_2, \ldots, x_{n-1}, y)$ is any point of D, then D also contains the entire closed straight line interval, in a fixed direction specified by the unit vector $s = (s_1, s_2, \ldots, s_n)$, with $s_n \neq 0$, from $(x_1, x_2, \ldots, x_{n-1}, y)$ to d, which joins $(x_1, x_2, \ldots, x_{n-1}, y)$ to a point $(x_1 - \frac{y}{s_n} s_1, x_2 - \frac{y}{s_n} s_2, \ldots, x_{n-1} - \frac{y}{s_n} s_{n-1}, 0)$ of d; and that the boundary conditions (1.2) and (1.3) on the function u are now replaced, respectively, by

$$
(3.1) \quad \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \left[\frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, y) + ku(x_1, x_2, \ldots, x_{n-1}, y)\right] = 0,
$$

$$
(3.2) \quad \lim_{\substack{(x_1, x_2, \ldots, x_{n-1}, y) \to (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, 0) \\ y > 0}} \left[\frac{\partial^2 u}{\partial s^2}(x_1, x_2, \ldots, x_{n-1}, y) + k \frac{\partial u}{\partial s}(x_1, x_2, \ldots, x_{n-1}, y)\right] = 0,
$$

for $(x_1, x_2, ..., x_{n-1}, y)$ in D^+ and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{n-1}, 0)$ in d, where k is a real constant and $\frac{\partial}{\partial s}$ denotes differentiation in the fixed direction of the unit vector s, (note that this direction is *not* tangential to the plane $y = 0$). Then, there is a uniquely determined real single-valued function U_s(for its definition, see equation (3.3) and the proof below), defined throughout D, which satisfies fourth order λ partial differential equation (1.1) throughout D, and coincides with u in D^+.

Proof. The proof of this theorem is similar to that of theorem 2, with the following modifications: the function v of theorem 2 is replaced by

$$v_s = \frac{\partial u}{\partial s} + ku$$

defined again in D^+, the function V is replaced by V_s, the analytic continuation of v_s to all of D, and, finally, the function U is replaced by the single-valued function U_s, defined in D and obtained from V_s by line integration in the s direction. The final formula for U_s (cf. the final formula (2.7) for U of theorem 2) is

$$U_s(x_1, x_2, ..., x_{n-1}, y)$$

$$(3.3) \quad = e^{-k\frac{y}{s_n}} \left[\int_0^{\frac{y}{s_n}} e^{kt} V_s(x_1 + \bar{t}s_1, x_2 + \bar{t}s_2, ..., x_{n-1} + \bar{t}s_{n-1}, ts_n) dt \right.$$
$$\left. + u(x_1 - \frac{y}{s_n} s_1, x_2 - \frac{y}{s_n} s_2, ..., x_{n-1} - \frac{y}{s_n} s_{n-1}, 0) \right],$$

where $\bar{t} = t - \frac{y}{s_n}$. In this case, there is no formula analogous to (2.8), involving u and its derivatives alone.

EXAMPLE. The following simple example shows that reflection need not be possible, at least when $n = 3$, when the differentiation in the boundary conditions (3.1), (3.2) are actually tangential to the plane $y = 0$. For definiteness, take $n = 3$, and use the notation (x, y, z) for (x_1, x_2, y). Let $f(x, y)$ be a real valued function, defined for $x \geq 0$, $-\infty < y < \infty$, which is of class $C(^4)$ in this closed half plane, and satisfies the biharmonic partial differential equation in the open half plane $x > 0$, $-\infty < y < \infty$. Further, suppose that the function $f(0, y)$ and $\frac{\partial f}{\partial x}(0, y)$ are not analytic functions of y. (Such a function f can be easily obtained, by means of Poisson's type integral formula for the solution of the (Dirichlet) boundary value problem for the biharmonic equation $f_{xxxx} + 2f_{xxyy} + f_{yyyy} + 0$ for the half plane $x > 0$; see, for example, pp. 28-29 in [7]). Now define $u(x, y, z) \equiv f(x, y)$ for $x > 0$, $-\infty < y, z < \infty$. One has, then, that $\frac{\partial u}{\partial z}(x, y, z) = 0$, $\frac{\partial^2 u}{\partial z^2}(x, y, z) = 0$ there, and, consequently,

$$(3.4)\quad \begin{cases} u_{xxxx} + u_{yyyy} + u_{zzzz} + 2(u_{xxyy} + u_{yyzz} + u_{zzxx}) = 0,\ x > 0, \\ \lim\limits_{\substack{(x, y, z) \to (0, \bar{y}, \bar{z}) \\ x > 0}} \frac{\partial u}{\partial z}(x, y, z) = 0, \\ \lim\limits_{\substack{(x, y, z) \to (0, \bar{y}, \bar{z}) \\ x > 0}} \frac{\partial^2 u}{\partial z^2}(x, y, z) = 0\,. \end{cases}$$

In this exemple, D^+ is the half space $x > 0$, the plane $x = 0$ corresponds to the plane $y = 0$ of theorem 3, and the boundary conditions in (3.4) involve the tangential derivative $\frac{\partial}{\partial z}$. But, since the functions $f(0, y)$ and $\frac{\partial f}{\partial x}(0, y)$ are not analytic in y, the function $u(x, y, z)$ cannot be continued analytically across the plane $x = 0$.

Concluding Remarks.

In theorem 2, a reflection principle was obtained for solutions of the iterated Helmholtz equation obeying mixed Neumann boundary conditions (2.1), (2.2). Using integration by parts, this reflection principle was finally written in the form (2.9). Since the expression in formula (2.9), which gives the value of the function U for $y < 0$, turns out to be a bit complicated, it is desirable to have at least a partial check on this formula. This partial verification was obtained by using solutions, of the twice iterated Helmholtz equation, which depend only on y. Notice that the function $Ae^{-\lambda y} + Bye^{-\lambda y} + Ce^{\lambda y} +$

$+ Dye^{\lambda y}$ is a solution of the twice iterated Helmholtz equation, for arbitrary real values of the constants A, B, C, D. Further, the function

$e^{-\lambda y} + \frac{1}{2}(\lambda - 2k - \frac{k2}{\lambda})ye^{-\lambda y} + e^{\lambda y} - \frac{1}{2}(\lambda + 2k - \frac{k2}{\lambda})ye^{\lambda y}$ satisfies, not only the iterated Helmholtz equation, but also the boundary conditions (2.1) and (2.2) of theorem 2. It has been verified, by direct computation, that the value of this particular function, for $y < 0$, actually coincides with its value for $y < 0$, as «predicted» by formula (2.9), thus giving the desired partial check on formula (2.9).

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