NORMAL OPERATORS AND SPACES OF DISTRIBUTIONS

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This note is devoted to a problem raised by the late Professor J. Sebastião e Silva concerning his direct method for the construction of spaces of distributions. In his fundamental work [16], Sebastião e Silva considers the following problem:

> given is a set G with some algebraic structure (i.e. an object of some algebraic category) and a family Ψ of mappings from some sub-objects of G into itself which preserve the algebraic structure. The problem is to embedd G into an object $\widetilde{G}$ of the same category and to construct a family $\widetilde{\Psi}$ of morphisms of $\widetilde{G}$ which extends the family Ψ.

This problem is solved in a general setting in [16]. The motivation was the fact that the differentiation operator D on the space $C(I)$ of continuous, complex-valued functions on a compact interval I is not everywhere defined and that the space $C_\infty(I)$ of distributions is a solution of a problem of the above type. This simple idea has been used by Sebastião e Silva to give a natural development of the theory of distributions.

However, the algebraic structure of $C_\infty(I)$ is too weak for applications and it is important that $C_\infty(I)$ has a natural locally convex structure, in fact the structure of a Silva space (or (LN)-space in the terminology of [17]). The problem suggested to us by Sebastião e Silva was to find a fairly general situation in which this is true, i.e. to find a suitable category where the solution to the above extension problem automatically has a natural locally convex structure — and even, under certain conditions, that of a Silva space.

We present here what we believe to be a satisfactory and natural solution to this problem. The underlying category that we use is the

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category of Hilbert spaces, and the operators that we extend are normal unbounded operators. Under these conditions, we obtain a fairly rich theory for the constructed spaces — in particular, we can determine exactly when it will be a Silva space. Although this need not be the case, we do always obtain a member of a wider class of spaces which has recently been studied by de Wilde [26] and Komatsu [9] and which possesses many of the properties of the class of Silva spaces.

One of the features of modern distribution theory and, in particular, of its application to partial differential equations is the use of an extensive apparatus in the form of an array of distribution spaces (see, for example, Hörmander [6] and Wloka [25]). Most of these spaces can be obtained by the method given here. This leads to a certain economy since one can deduce a number of properties of these spaces (duality, completeness, nuclearity) from the theory given here without dealing with each space individually.

In § 1 we define the new spaces and give some properties which follow from the general theory of locally convex spaces. This construction is natural in the sense that we have actually constructed a functor between two categories and this fact seems important enough to us to justify a presentation within the framework of category theory although this leads, perhaps, to a rather heavy-handed treatment. In § 2 we give criteria for nuclearity. These involve the nuclearity of inverse mappings and are elementary in the sense that they do not use spectral theory. They are then expressed in terms of the spectrum of the operator. In § 3 we give results on the extension of linear operators on the original Hilbert space. Such results are important for defining operations on the constructed distributions. In § 4 we show that the dual spaces of our distribution spaces can be identified with a class of spaces introduced by A. Pietsch [14]). Inorder to ensure that the identifications obtained are linear (and not anti-linear), we have been careful to distinguish between a Hilbert space and its dual. In § 5 we show how the distribution spaces can be represented as spaces of measurable functions (in some cases as spaces of sequences). § 6 contains some remarks on generalisations.

We use, without special references, the standard results on Hilbert space (in particular, the theory of normal operators) and on locally convex spaces. See, for example:

F. Riesz and B. Sz.-Nagy, *Functional analysis* (New York, 1955).

G. Köthe, *Topologische lineare Räume* (Berlin, 1966).

We shall use the term «normal operator» in Hilbert space H to denote a not necessarily bounded normal operator from a dense subset of H into itself.

The space $F(H'; T')$ considered in § 4 has been introduced and studied by Pietsch [14]. In particular, because of the duality established in Proposition 4.5, Proposition 2.3 is equivalent to a result in [14]. Our proofs are different.

§ 1. THE CONSTRUCTION.

1.1. *Definition:* We introduce the category $HNOP$ whose object are pairs $(H; T)$ where H is a Hilbert space and T is a normal (not necessarily continuous) operator on H. If $(H; T)$ and $(H_1; T_2)$ are objects of $HNOP$, a morphism from $(H; T)$ into $(H_1; T_2)$ is a continuous, linear mapping S from H into H_1 which commutes with T and T_1 in the sense that $T_1 S \supseteq ST$ (i.e. is so that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad S \quad} & H_1 \\ T \downarrow & & \downarrow T_1 \\ H & \xrightarrow[\quad S \quad]{} & H_1 \end{array}$$

«commutes»).

It is easily seen that $HNOP$ is an additive category (we note that a subobject of $(H; T)$ is a pair $(K; T_K)$ where K is a closed, T-invariant subspace of H and T_K is the restriction of T to K).

1.2. *Notation:* Let $(H; T)$ be an object of $HNOP$. H^{n+1} denotes the $(n+1)$-fold product of H with itself and we write $(x_0, \ldots, x_n)$ for a typical element. We use the natural inner product

$$((x_0, \ldots, x_n), (y_0, \ldots, y_n)) \longrightarrow (x_0 \mid y_0) + \ldots + (x_n \mid y_n)$$

on H^{n+1}.

T^n denotes the n-fold product of T with itself (which is also a normal operator on H) and $D(T^n)$ denotes the domain of definition of T^n. We introduce the following subspaces of H^{n+1}:

$$D_n := \{(y_0, \ldots, y_n) \in H^{n+1} \; : \; y_k \in D(T^k) \; (k = 0, \ldots, n) \text{ and } y_0 + Ty_1 + \ldots + T^n y_n = 0\}.$$

$G_n := \overline{D}_n$ the closure of D_n in H^{n+1}.

$F_n := \{(x, T^* x, \ldots, (T^*)^n x) \; : \; x \in D((T^*)^n)\}$.

1.3. *Lema:* $F_n = D_n^{\perp}$ *and so* $G_n = F_n^{\perp}$.

Proof: It is clear that $F_n \subseteq D_n^{\perp}$. We show that the reverse inclusion holds. Suppose that $n = 1$. Then if $(x_0, x_1) \in D_1^{\perp}$,

$$(x_0 \mid y_0) + (x_1 \mid y_1) = 0$$

for every pair $(y_0, y_1) \in D_1$ i.e. such that $y_1 \in D(T)$ and $Ty_1 = -y_0$. Thus $x_0 \in D(T^*)$ and $x_1 = T^* x_0$.

Now suppose that $(x_0, \ldots, x_n) \in D_n^{\perp}$. By choosing those elements of D_n of the form $(y_0, 0, \ldots, 0, y_k, 0, \ldots, 0)$ and using a similar argument, one can show that $x_k = (T^*)^k x_0$ $(k = 1, 2, \ldots, n)$.

1.4. *Lemma: The mapping* $(y_0, \ldots, y_n) \longrightarrow (0, y_0, \ldots, y_d)$ *from* H^{n+1} *into* H^{n+2} *maps* G_n *into* G_{n+1}.

Proof: Consider the mapping

$$J : (x_0, \ldots, x_{n+1}) \longrightarrow (x_n, \ldots, x_{n+1})$$

from H^{n+2} into H^{n+1}. Then its adjoint is the above mapping. Now J maps F_{n+1} into F_n and so its adjoint maps $F_n^{\perp}$ into $F_{n+1}^{\perp}$ i.e. G_n into G_{n+1}.

1.5. *Proposition: Let* $(H; T)$ *be an object of HNOP. Then there exists a locally convex space* $E(H; T)$*, a continuous, linear mapping* $\tilde{T}$ *from* $E(H; T)$ *into itself and a continuous injection* i *from* H *into* $E(H; T)$ *so that*

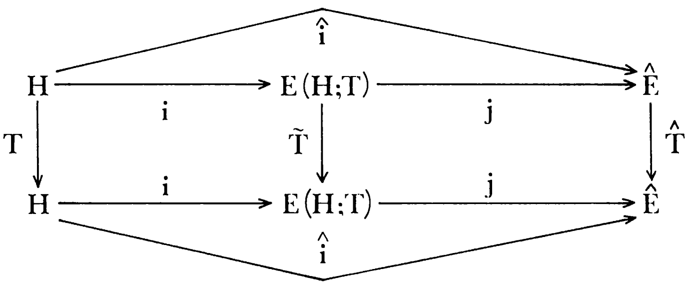
(i) $\tilde{T}$ *is an extension of* T i.e. *the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\quad i \quad} & E(H; T) \\ T \downarrow & & \downarrow \tilde{T} \\ H & \xrightarrow[\quad i \quad]{} & E(H; T) \end{array}$$

commutes;

(ii) $E(H; T)$ *is minimal with respect to this property. That is, if* $\hat{E}$, $\hat{i}$, $\hat{T}$ *is a triple with the same property, then there is a conti-*

nuous injection j from $E(H; T)$ into $\widehat{E}$ so that the following diagram commutes:



(iii) *if $S : (H; T) \longrightarrow (H_1; T_1)$ is a morphism, then S lifts to a continuous linear mapping $\widetilde{S}$ from $E(H; T)$ into $E(H_1; T_1)$ so that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\quad i \quad} & E(H; T) \\ S \downarrow & & \downarrow \widetilde{S} \\ H_1 & \xrightarrow[\quad i_1 \quad]{} & E(H_1; T_1) \end{array}$$

commutes;

(iv) *the correspondences*

$$\begin{array}{ccc} (H; T) & \longrightarrow & E(H; T) \\ S & \longrightarrow & \widetilde{S} \end{array}$$

form a covariant functor from HNOP into the category LOCCONV of locally convex spaces.

Proof: We define the space $E_n(H; T)$ to be the quotient space H^{n+1}/G_n. The natural injection

$$(x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_n, 0)$$

from H^{n+1} into H^{n+1} lifts to an injection from $E_n(H; T)$ into $E_{n+1}(H; T)$ (since $G_{n+1} \cap H^{n+1} = G_n$). Hence we have an inductive sequence $\{i_n : E_n(H; T) \longrightarrow E_{n+1}(H; T)\}$ of Hilbert spaces. We let $E(H; T)$ be its inductive limit (in the category of locally convex spaces) i.e. the union of the spaces $E_n(H; T)$ with the finest locally convex structure which induces a coarser structure on each $E_n(H; T)$ than that defined by the norm.

We let i be the natural injection from $H(=E_0(H; T))$ into $E(H; T)$. For each n, the mapping

$$(x_0, \ldots, x_n) \longrightarrow (0, x_0, \ldots, x_n)$$

from H^{n+1} into H^{n+2} lifts to a continuous, linear mapping from $E_n(H; T)$ into $E_{n+1}(H; T)$ (1.4). These mappings are all compatible with the inclusions from $E_n(H; T)$ into $E_{n+1}(H; T)$ and so can be used to define a continuous linear mapping $\widetilde{T}$ from $E(H; T)$ into itself.

(ii) Suppose that $\widehat{E}, \widehat{i}, \widehat{T}$ is such a triple. We define an injection from $E_n(H; T)$ into $\widehat{E}$ as follows:
consider the mapping

$$(x_0, \ldots, x_n) \longrightarrow \widehat{i}(x_0) + \widehat{T}(\widehat{i}x_1) + \ldots + \widehat{T}^n(\widehat{i}x_n).$$

Then this defines a linear mapping from H^{n+1} into $\widehat{E}$ whose kernel is just G_n. Hence it lifts to an injection from $E_n(H; T)$ into $\widehat{E}$. These injections are all compatible and so define an injection from $E(H; T)$ into $\widehat{E}$ with the required properties.

(iii) Consider the mapping

$$(x_0, \ldots, x_n) \longrightarrow (Sx_0, \ldots, Sx_n)$$

from H^{n+1} into H^{n+1}. Its adjoint is the mapping

$$(y_0, \ldots, y_n) \longrightarrow (S^*y_0, \ldots, S^*y_n)$$

and this maps F_n into F_n (Since S^* commutes with T^*). Hence the former mapping maps G_n into G_n and so lifts to a continuous linear mapping from $E_n(H; T)$ into $E_n(H; T)$. These mappings can be combined to form the required mapping $\widehat{S}$.

(iv) follows directly from (i)-(iii).

1.6. *Proposition:* (i) *The space* $E(H; T)$ *with the above structure is a separated, complete, semi-reflexive* (DF)*-space;*

(ii) *a subset $B \subseteq E(H; T)$ is bounded if and only if it is contained in some $E_n(H; T)$ and bounded there;*

(iii) *A subset A of $E(H; T)$ is weakly closed if and only if its intersection with each $E_n(H; T)$ is weakly closed in $E_n(H; T)$. In particular, an absolutely convex subset A of $E(H; T)$ is closed if and only if $A \cap E_n(H; T)$ is closed each n;*

(iv) *a sequence (x_k) in $E(H; T)$ is weakly convergent if and only if it is contained in some $E_n(H; T)$ and weakly convergent there.*

Proof: Since $E_n(H; T)$ is a Hilbert space and so semi-reflexive the injection $E_n(H; T) \longrightarrow E_{n+1}(H; T)$ is weakly compact and the results of [9] and [26] can be applied.

1.6. *Remark:* It is easy to see that the properties (i) and (ii) of 1.5 determine $E(H; T)$ uniquely in the sense that if $\widehat{E}, \widehat{i}, \widehat{T}$ is a triple with the same properties then there is an isomorphism j from $E(H; T)$ onto $\widehat{E}$ so that the restriction of j to H is the identity. It follows from this that if T is a normal operator on H and λ is an element of the resolvent set $\varrho(T)$ of T (e.g. if T is selfadjoint and $Im\, \lambda \neq 0$) then $E(H; T)$ is naturally isomorphic to $E(H; \lambda I - T)$ (since a solution of the extension problem for T is clearly a solution of the corresponding problem for $(\lambda I - T)$ and vice versa). Hence it can often be assumed, without loss of generality, that T hasa continuous inverse. In the latter case, the theory has an especially simple form.

1.8. *Proposition: Suppose that T has a continuous inverse S. Then.*

(i) *$\widetilde{T}$ has a continuous inverse on $E(H; T)$;*

(ii) *every bounded subset of $E(H; T)$ has the form $\widetilde{T}^n(B_1)$where B_1 is a bounded subset of H and n is a positive integer. In particular, every element of $E(H; T)$ has the form $\widetilde{T}^n x$ where $x \in H$ and n is a positive integer;*

(iii) *if (x_k) is a sequence in $E(H; T)$ which converges weakly to zero, then there is a sequence (y_k) in H which converges weakly to zero and a positive integer n so that $x_k = \widetilde{T}^n y_k$ for each k.*

Proof: It is clear that the extension $\widetilde{S}$ of S is an inverse for $\widetilde{T}$. We denote by π_n the natural projection from H^{n+1} onto $E_n(H; T)$. If $y_0, \ldots, y_n \in H$, then

$$\pi_n(y_0, \ldots, y_n) = \pi_n(0, \ldots, 0, S^n y_0 + S^{n-1} y_1 + \ldots + y_n)$$

i.e. $(y_0, y_1, \ldots, -(S^n y_0 + S^{n-1} y_1 + \ldots + S y_{n-1})) \in G_n$. This means that the element $x = \pi_n(y_0, \ldots, y_n)$ of $E_n(H; T)$ is equal to $\widetilde{T}^n y$ where $y = S^n y_0 + S^{n-1} y_1 + \ldots + y_n$. Hence the mapping

$$\widetilde{T}^n : H \longrightarrow E_n(H; T)$$

is onto and so is a homomorphism. It is even an isomorphism since $\widetilde{T}$ is injective. (ii) and (iii) follow immediately from this.

1.9. *Corollary: Let T be self-adjoint. Then if B is bounded in $E(H; T)$, there is a positive integer n and bounded set $B_0, B_1, \ldots, B_n$ in H so that*

$$B \subseteq B + \widetilde{T} B_1 + \ldots + \widetilde{T}^n B_n.$$

A similar statement holds for sequences which converge weakly to zero in $E(H; T)$.

Proof: Apply 1.8 to $(i + T)$.

§ 2. SILVA SPACE AND NUCLEAR SPACES OF TYPE $E(H; T)$

2.1. *Proposition: Let T have a continuous inverse S. Then $E(H; T)$ is a Silva space if and only if S is compact.*

Proof: Suppose that S is compact. If B is the unit ball of $E_n(H; T)$, then, by 1.8, $B \subseteq \widetilde{T}^n B_1$ where B_1 is bounded in H. Then $B \subseteq \widetilde{S}(\widetilde{T}^{n+1} B_1)$, $\widetilde{S}$ is compact from $E_{n+1}(H; T)$ into $E_{n+1}(H; T)$ and $\widetilde{T}^{n+1}(B_1)$ is bounded in $E_{n+1}(H; T)$ so that B is relatively compact in $E_{n+1}(H; T)$.

If $E(H; T)$ is Silva, then there is an integer n so that the inclusion $H \longrightarrow E_n(H; T)$ is compact. We can write this inclusion as the product $\widetilde{T}^n S^n$. Since $\widetilde{T}^n$ is an isomorphism, S^n is compact and so S is compact.

2.2. *Proposition: Let T have a continuous inverse S. Then $E(H; T)$ is nuclear if and only if there is an integer n so that S^n is nuclear.*

Proof: Similar to that of 2.1.

2.3. *Proposition: Suppose that T is self-adjoint. Then*

(i) *$E(H; T)$ is a Silvà space if and only if the spectrum of T consists of a sequence (λ_n) of eigenvalues of finite multiplicity so that $|\lambda_n| \to \infty$;*

(ii) *$E(H; T)$ is nuclear if and only if there is a positive number ε so that*

$$\sum_n \frac{k_n}{|(i + \lambda_n)|^\varepsilon} < \infty$$

(k_n is the multiplicity of the eigenvalue λ_n).

Proof: We apply 2.1 and 2.2 to $(i - T)$ and use the fact that a normal A (in our case $A := (i - T)$) is compact if and only if its spectrum consists of a sequences of eigenvalues (μ_n) of finite multiplicity which converges to zero. A is nuclear if and only if $\sum_n k_n |\mu_n| < \infty$ (k_n is the multiplicity of μ_n).

§ 3. EXTENSION PROPERTIES

3.1. *Proposition: Let $(H; T)$ and $(H_1; T_2)$ be two objects of HNOP and let M be a norm bounded collection of continuous linear mappings from H into H_1 so that for each $U \in M$, $T_1 U \supseteq UT$. Then M can be extended to an equicontinuous family of linear mappings from $E(H; T)$ into $E(H_1; T_1)$.*

Proof: Similar to 1.5 (iii).

3.2. *Corollary: Let T be self-adjoint operator so that H possesses an orthonormal basis (x_k) composed of eigenvectors of T. Then (x_k) is a Schauder basis for $E(H; T)$.*

Proof: Let P_n be the orthogonal projection on the subspace of H spanned by the first n elements of the basis. Then each P_n commutes with T and so the P_n's lift to an equicontinuous family of projections in $E(H; T)$ and it can easily be deduced from this that (x_k) is a Schauder basis for $E(H; T)$.

3.3. *Definition:* Let M be a measurable subset of K. A measurable complex-valued function x on M is said to be *of polynomial growth on* M if there is a positive integer n so that

$$|x(t)| \leq K(1+t^2)^n \qquad (K > 0)$$

for almost every $t \in M$. If T is a self-adjoint operator in a Hilbert space H, then a normal operator A on H is said to be *of polynomial growth with respect to* T if $A = x(T)$ where x is a function of polynomial growth on the spectrum of T.

3.4. *Proposition: Let T be a self-adjoint operator on H, A a normal operator of polynomial growth with respect to T. Then A can be lifted to a continuous linear operator from $E(H; T)$ into itself.*

Proof: We suppose that $A = x(T)$ where $|x(t)| \leq K(1+t^2)^n$ almost everywhere on the spectrum of T. Let y be the function

$$t \longrightarrow x(t)(1+t^2)^{-n}$$

Then y is bounded (a.e.) on the spectrum of T and so $B := y(T)$ is a bounded operator on H which commutes with T. Hence it can be lifted to a continuous linear operator $\tilde{B}$ from $E(H; T)$ into itself. We define $\tilde{A}$ to be $\tilde{B}(I + \tilde{T}^2)^n$. Then $\tilde{A}$ is the required extension.

§ 4. DUALITY THEORY

4.1. *Definition:* Let $(H; T)$ be an object of $HNOP$. We write H' for the dual space of H (as a Banach space) and J_H for the mapping

$$x \longrightarrow (y \longrightarrow (y \mid x))$$

from H into H'. Then, by the Riesz representation theorem, J_H is an antilinear isomorphism from H onto H'. We give H' a Hilbert space structure by defining

$$(J_H x \mid J_H y) := (y \mid x) \qquad (x, y \in H).$$

Then this scalar product defines the dual norm on H'. If T is an operator on H, then its transpose is $T' = J_H T^* J_H^{-1}$.

$$\begin{array}{ccc} H & \xrightarrow{\quad T^* \quad} & H \\ {\scriptstyle J_H}\downarrow & & \downarrow{\scriptstyle J_H} \\ H' & \xrightarrow{\quad T' \quad} & H' \end{array}$$

We define $F_n(H'; T)$ to be $D((T')^n)$ with the norm

$$y \longrightarrow \sqrt{\|y\|^2 + \|T'y\|^2 + \dots + \|(T')^n y\|^2}$$

4.2. *Proposition:* $E_n(H; T)$ *is isomorphic to the dual space of* $F_n(H'; T')$ *under the bilinear form*

$$(y, \pi_n(x_0, \dots, x_n)) \longrightarrow (y \mid J_H x_0) + \dots + ((T')^n y \mid J_H x)_n.$$

Proof: It follows immediately from the definition that

$$y \longrightarrow (y, T'y, \dots, (T')^n y)$$

is an isometry from $F_n(H; T')$ onto a subspace of $(H')^{n+1}$ and this subspace is mapped by the anti-linear isometry $J_H \times \dots \times J_H$ onto $F_n \subseteq H^{n+1}$. (Note that this implies that $F_n(H'; T')$ is a Banach space since F_n is a closed subspace of H^{n+1}). Now if we regard $F_n(H'; T')$ as a subspace of $(H')^{n+1}$, then its dual space is the quotient space of the dual of $(H')^{n+1}$ by the polar of $F_n(H'; T')$. But the dual of $(H')^{n+1}$ is anti-linearly isomorphic (under $J_H^{-1} \times \dots \times J_H^{-1}$) to H^{n+1} and under this mapping the polar of $F_n(H'; T')$ is mapped onto $F_n^{\perp} = G_n$. Hence the dual of $F_n(H'; T')$ is isomorphic to $H^{n+1}/G_n = E_n(H; T)$ and this is precisely the statement of the Proposition.

4.4 *Definition:* We write $F(H'; T')$ for $\bigcap D((T')^n)$ with its natural locally convex structure as the intersection of a decreasing sequence of Banach spaces. Then $F(H'; T')$ is a Fréchet space.

4.5. *Proposition:* $E(H; T)$ *is naturally isomorphic (as a locally convex space) to the dual of* $F(H'; T')$.

Proof: This follows from 4.3 and the general theory of locally convex spaces (note that $F(H'; T')$ is dense in each $F_n(H'; T')$-spectral theorem).

4.6. *Remark:* We have constructed a Hilbert triad («Gelfandsches Raumtripel»)

$$E(H; T) \supseteq H \xrightarrow{\quad J_H \quad} H' \supseteq F(H|; T').$$

in the terminology of [3]. Thus we can restate our result as follows:

Let T be a normal (unbounded) operator in a Hilbert space. Then there exists a Hilbert triad as above so that T extends to a continuous linear mapping mapping on E (H; T) (resp. T' restricts to a continuous linear mapping from F (H'; T') into itself) and this triad is natural in the sense that E (H; T) is minimal with respect to this property (resp. F (H'; T') is maximal).

4.7. *Proposition: H is dense in E (H; T).*

Proof: The dual of $E(H; T)$ is $F(H'; T')$ since $E(H; T)$ isreflexive. The result now follows from the Hahn-Banach theorem since if an element of $F(H'; T')$ $(\subseteq H')$ vanishes on H then it is zero.

§ 5. A REPRESENTATION THEOREM

5.1. *Definition:* Let M be a locally compact space, μ a Radon measure on M and x a measurable complex-valued function on M. Then a measurable complex-valued function y on M is *of L^2-polynomial growth with respect to x* if there is a positive integer n so that the function $y(1 + |x|^2)^{-n} \in L^2(M; \mu)$. We denote by $\mathrm{Pol}_2(x)$ the set of all such functions. Then

$$\mathrm{Pol}_2(x) = \bigcup_{n=1}^{\infty} L^2(M; (1 + |x|^2)^{-n} \mu).$$

We regard $\mathrm{Pol}_2(x)$ as a locally convex space with the natural inductive limit structure as the union of the above spaces (note that $\mathrm{Pol}_2(x)$ is separated since this structure defines a topology which is finer than that of convergence in measure).

5.2. *Proposition: Let T be a normal operator in H. Then there is a locally compact space M, a positive Radon measure μ on M and a* *measurable complex-valued function* x *on* M *so that* $E(H; T)$ *is isomorphic* (*as a locally convex space*) to $\mathrm{Pol}_2(x)$.

Proof: This follows from the fact that the pair $(H; T)$ is unitarily equivalent to a pair $(L^2(M; \mu); T_x)$ where M, μ, x are as above and T_x is the operation of multiplication by x (Segal and Kunze [20]). It can easily be checked that $\mathrm{Pol}_2(x)$ is $E(L^2(M; \mu); T_x)$.

5.3. *Notation:* Let $\mathbf{N}$ denote the set of positive integers. We regard this as a locally compact space with the discrete topology. If ν denotes the counting measure on $\mathbf{N}$ we write, as is customary, $l^\infty(\mathbf{N})$ (resp. $l^2(\mathbf{N})$) for $L^\infty(\mathbf{N}; \nu)$ (resp. $L^2(\mathbf{N}; \nu)$). If x is a complex-valued sequence, we define $\mathrm{Pol}_2(x)$ as above and $\mathrm{Pol}_\infty(x)$ to be $\bigcup_{n=1}^{\infty} L^\infty(\mathbf{N}; (1+|x|^2)^{-n}\nu)$ (so that $y \in \mathrm{Pol}_\infty(x)$ if and only if there is a positive integer n so that $y(1+|x|^2)^{-n}$ is bounded).

5.4. *Proposition: Let* $(H; T)$ *be an object of HNOP so that* $E(H; T)$ *is a Silva space. Then*

(i) *there is a sequence* x *so that* $E(H; T)$ *is isomorphic to* $\mathrm{Pol}_2(x)$;

(ii) $E(H; T)$ *is nuclear if and only if* $\mathrm{Pol}_2(x) = Pol_\infty(x)$ (x *as in* (i)) *and so in this case* $E(H; T)$ *is isomorphic to* $\mathrm{Pol}_\infty(x)$.

Proof: (i) This follows from 5.2 and the fact that if $E(H; T)$ is a Silva space, then, by 2.3 (i), we can choose the space $(\mathbf{N}; \nu)$ for $(M; \mu)$ and the sequence (λ_n) of eigenvalues of T (repeated according to multiplicity) for x.

(ii) if $E(H; T)$ is nuclear, then there is a positive integer n so that $\sum_{k=1}^{\infty}(1+|\lambda_k|^2)^{-n} < \infty$. Then $L^\infty(\mathbf{N}; (1+|x|^2)^{-n}\nu)$ is contained in $l^2(\mathbf{N})$ and so $\mathrm{Pol}_\infty(x) \subseteq \mathrm{Pol}_2(x)$. Since the reverse inclusion is always valid, $\mathrm{Pol}_\infty(x) = \mathrm{Pol}_2(x)$. Now suppose that $\mathrm{Pol}_\infty(x) = \mathrm{Pol}_2(x)$. Then since the embedding $l^\infty(\mathbf{N}) \subseteq \mathrm{Pol}_2(x)$ has a closed graph, there is an integer n so that $l^\infty(\mathbf{N}) \subseteq L^2(\mathbf{N}; (1+|x|^2)^{-n}\nu)$ (Grothendieck [4] Introduction IV, Théoreme B). Then $\sum_{k=1}^{\infty}(1+|\lambda_k|^2)^{-2n} < \infty$ and so $E(H; T)$ is nuclear (2. 3(ii)).

§ 6. REMARKS

1. The above construction has a natural generalisation as follows: we suppose that $\mathcal{E}$ is a generalised semigroup of unbounded operators (closed and densely defined) on a Hilbert space, that is, there is a dense subset D of H which lies in the domain of definition of each $S \in \mathcal{E}$ and if $S, T \in \mathcal{E}$, ST is closeable and its closure lies in $\mathcal{E}$. We write $S \cdot T$ for this closure. If $J = (S_1, \ldots, S_n)$ is an ordered n-tuple of elements of $\mathcal{E}$ we define the spaces:

$$D_J = \{(y_0, \ldots, y_n) \in H^{n+1} : y_k \in D(S_k) \quad \text{and} \quad y_0 + S_1 y_1 + \ldots + S_n y_n = 0\}$$

$$G_J := \overline{D}_J \quad \text{the closure of} \quad D_J \quad \text{in} \quad H^{n+1};$$

$$F_J := \{(x, S_1 x, \ldots, S_n x) : x \in \bigcap_{k-1}^{n} D(S_k)\}$$

Then, as in 1.3, it can be shown that $F_J = D_J^{\perp}$ and $G_J = F_J^{\perp}$. We define the space $E_J(H; \mathcal{E})$ to be the quotient space H^{n+1}/G_J. If $\mathcal{J}_n$ denotes the set of all ordered n-tuples of elements of $\mathcal{E}$ and $\mathcal{J} := \bigcup_{n=1}^{\infty} \mathcal{J}_n$ then we can order $\mathcal{J}$ by defining J to be smaller than J_1 (written $J \leq J_1$) if the elements of J occur in the same order in J_1 (but not necessarily directly following each other). Then $(\mathcal{J}, \leq)$ is a directed system. If $J \leq J_1$ then there is a natural injection i_{J,J_1} from $E_J(H; \mathcal{E})$ into $E_{J_1}(H; \mathcal{E})$. Hence we have an inductive system

$$\{i_{J,J_1} : E_J(H; \mathcal{E}) \longrightarrow E_{J_1}(H; \mathcal{E}); \ J \leq J_1\}$$

of Hilbert spaces. We denote its inductive limit by $E(H; \mathcal{E})$. Similarly we define $F_J(H'; \mathcal{E}')$ ($J = (S_1, \ldots, S_n) \in \mathcal{J}$, H' the dual of H, $\mathcal{E}'$ the semigroup of operators of the form $S' = J_H S^* J_H^{-1}$ ($S \in \mathcal{E}$) on H') to be $\bigcap_{k=1}^{n} D(S_k')$ under the norm

$$y \longrightarrow \{||y||^2 + ||S_1' y||^2 + \ldots + ||S_n' y||^2\}$$

The spaces $F_J(H'; \mathcal{E}')$ form a projective system and their projective limit $F(H'; \mathcal{E}')$ is a complete, semi-reflexive locally convex space (not necessarily a Fréchet space). Then

(i) each $T \in \mathcal{E}$ has a unique extension $\widetilde{T}$ to a continuous linear operator on $E(H; \mathcal{E})$ and if $S, T \in \mathcal{E}$, then $\widetilde{S \cdot T} = \widetilde{S}\widetilde{T}$;

(ii) $E(H; \mathcal{E})$ is naturally isomorphic to the dual of $F(H'; \mathcal{E}')$.

2. In the construction of the space $E(H; T)$ it is often possible to assume that the operator T is positive (and so self-adjoint). For if T is a normal operator, T can be expressed in the form UA where U is unitary, A is positive and A and U commute. It can be easily checked (cf. 1.6) that $E(H; T)$ and $E(H : A)$ are naturally isomorphic.

3. A similar theory can be developed for closed, densely defined operators in Banach spaces. However, the lack of a spectral theory leads to some difficulties and to obtain reasonable results, it is necessary to introduce a number of ad hoc assumptions (for example, to ensure that the powers of T are closed, densely defined operators).

4. The spaces $E_n(H; T)$ and $F_n(H; T)$ used to construct $E(H; T)$ and $F(H; T)$ can be regarded as abstract Sobolew spaces. Sobolew spaces with non-integral indexes can be obtained by applying interpolation methods to pairs $(E_n(H; T), E_{n+1}(H; T))$ or by assuming that T is positive (cf. Remark 3) and constructing spaces $E_\alpha(H; T) := E_n(H; T^{\alpha/n})$ (α a positive number).

5. If $(H_n; T_1)$ and $(H_2; T_2)$ are objects of $HNOP$ and $H_1 \otimes H_2$ denotes the Hilbert space tensor product of H_1 and H_2, then the closure of the operator $T_1 \otimes T_2$ on the algebraic tensor product of H_1 and H_2 is a normal operator on $H_1 \otimes H_2$. We denote this operator also by $T_1 \otimes T_2$. If one of the operators T_1, T_2 satisfies the conditions of 2.3 (ii), then $E(H_1 \otimes H_2; T_1 \otimes T_2)$ is naturally isomorphic to $E(H_1; T_1) \widehat{\otimes} E(H_2; T_2)$, the projective tensor product of $E(H_1; T_1)$ and $E(H_2; T_2)$. This can be used to deduce a number of representations of spaces of distributions on higher-dimensional euclidean space as tensor products of one-dimensional spaces. It also suggests the definition of «mixed» distribution spaces by choosing T_1 and T_2 to be distinct operators.

6. Suppose that T_1 and T_2 satisfy the conditions of 2.3 (i) and let (λ_n) (resp. (μ_n)) be the non-zero eigenvalues of T_1 and T_2 resp. (we suppose for convenience that T_1 and T_2 are strictly positive and that the eigenvalues are arranged so that $\lambda_n \leq \lambda_{n+1}$ for each n (resp. $\mu_n \leq \mu_{n+1}$)-eigenvalues are repeated according to multiplicity). Then $E(H; T_2)$ and $E(H_1; T_2)$ are isomorphic if and only if there exist positive integers k and 1 so that $\left\{\frac{\lambda_n}{\mu_n^k}\right\}_{n+1}^{\infty}$ and $\left\{\frac{\mu_n}{\lambda_n^l}\right\}_{n+1}^{\infty}$ are bounded. This fact can be used to establish the isomorphicity (resp. non-isomorphicity) of various spaces of distributions.

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