BOUNDED VARIATION AND DIFFERENTIABILITY OF FUNCTIONS

by

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[0] This paper is, in part, an attempt to provide an updated version of Chapter VI of «Theory of the integral» (Saks 1937) and, in part, an introduction to some problems posed by A. Zygmund [9].

In Chapter I we define the concept of area of a non-parametric surface and we introduce adequate machinery to prove Tonelli's theorem, that is: the characterization of the surfaces with finite area. Chapter II is devoted to the study of the existence of tangent planes to a surface of finite area. We present two observations:

(1°) R. CACCIOPOLI. There exists a surface of finite area that fails to have a tangent plane on a set of positive measure.

(2°) CALDERON-ZYGMUND. The condition of finite area is sufficient to imply the existence of a tangent plane, in the sense of $L^2(\mathbf{R}^2)$, for almost every point.
Finally we present some remarks of J. P. Kahane and A. Cordoba about Cacciopoli's example.

[1] JORDAN LENGTH.

Suppose that $x = f(t)$ is a continuous curve $(0 \leq t \leq 1)$ and for any partition $0 = t_0 < t_1 < \dots < t_n = 1$ we denote by P the polygonal line inscribed in the curve and whose vertices are the points $\{(t_i, f(t_i))\}$. Let $\sigma(P) = \max\{|t_i - t_j|\}$, then we can define the length of the curve f as

$$L(f) = \lim_{\sigma(P) \to 0} L(P)$$

where $L(P) = \Sigma\,(|t_{i+1} - t_i|^2 + |f(t_{i+1}) - f(t_i)|^2)^{1/2}$.

Therefore we are tempted to adopt the following definition for area of a continuous surface: «Area is the limit of the elementary areas of the inscribed polyhedral surfaces P».

This definition was proved to be wrong by Schwarz and Peano (1881). They observed that, even for elementary surfaces S, it is possible to define sequences $\{P_n\}$ of inscribed polyhedral surfaces, whose elementary areas approach any given number bigger than or equal to the «usual area» of S.

EXAMPLE OF SCHWARZ AND PEANO.

Let the surface S be given by $x = r \cos u, y = r \sin u, z = v$, $0 \leq u \leq 2\pi, 0 \leq v \leq h$. Therefore S is a right cylinder of height h and radius r, whose area is $2\pi rh$.

Let $P_{m,n}$ be the inscribed polyhedral surfaces of $4mn$ triangular faces T, whose vertices are the points (x, y, z) images of the points $w = (u', v')$ or (u'', v''), where

$$\begin{cases} u' = 2\pi m^{-1} \mu, & \mu = 0, \ldots, m-1. \\ v' = hn^{-1} \nu, & \nu = 0, \ldots, n \end{cases}$$

$$\begin{cases} u'' = 2\pi(2m)^{-1}(2\mu + 1), & \mu = 0, \ldots, m-1 \\ v'' = h(2n)^{-1}(2\nu + 1), & \nu = 0, \ldots, n-1 \end{cases}$$

Then the $4mn$ faces of $P_{m,n}$ are isosceles and congruent triangles T of base $b = 2r \sin(m^{-1}\pi)$ and height
$h = [r^2(1\text{-}\cos(m^{-1}\pi))^2 + (2^{-1}hm^{-1})^2]^{1/2}$.

Thus, the area of $P_{m,n}$ is $4mn\ r \sin(m^{-1}\pi)\ [(2^{-1}hn^{-1})^2 + 4r^2 \sin^4(2^{-1}\pi\, m^{-1})]^{1/2}$. In particular if $n = m^4$ we have that $S(P_{m,n}) \geq$ $\geq 8r^2m^5\sin^3(2^{-1}\pi\, m^{-1}) \to \infty$ as $m \to \infty$.

The geometric interpretation is given by the fact that, if the ratio n/m is big, then the triangles of the partition $P_{m,n}$ are «almost perpendicular» to the surface.

It was Lebesgue who first modified the wrong definition, in a form that may be roughly described as follows: «The area of a surface is the lower limit of the areas of polyhedra tending uniformly to the surface». (In the following we shall restrict ourselves to the non-parametric case i.e. continuous surfaces of the form $z = F(x, y)$ where F is a continuous function on the unit square Q).

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Definition 1 (Saks). A continuous surface $z = P(x, y)$ on the unit square Q is called polyhedron if there exists a decomposition of Q into a finite number of non-overlapping triangles $T_1, \ldots, T_n$ such that the function P is linear on each of these triangles *i.e* such that $P(x, y) = a_i x + b_i y + c_i$ for $(x, y) \varepsilon T_i$.

The sum of the areas of the faces i.e. the number

$$\sum_i |T_i|(a_i^2 + b_i^2 + 1)^{1/2} = \iint_Q \sqrt{1 + \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2}\, dx\, dy$$

will be called elementary area of the polyhedron P and denote by $A(P)$.

Definition 2. Let $z = F(x, y)$ be a continuous surface on Q, then $S(F) \equiv$ Lebesgue area of F is the following number

$$S(F) = \inf_{\substack{P_n \to F \\ \text{uniformly}}} \{\liminf A(P_n\}$$

Now we have for curves the theorem of Jordan: «in order that the length of $x = f(t)$ be finite it is necessary and sufficient that f be of bounded variation». For surfaces we have an analogous result:

Theorem (*Tonelli,* 1926). *In order that a surface $z = F(x, y)$ have a finite area it is necessary and sufficient that the function F be of bounded variation in Tonelli's sense.*

Given the function $F(x, y)$ on Q: $0 \leq x, y \leq 1$, consider

$$V_1(\xi) = \text{total variation of } F(\xi, -)$$

$$V_2(\eta) = \text{total variation of } F(-, \eta).$$

Definition 3. The function F is said to be of bounded variation in Tonelli's sense if both functions V_1 and V_2 are integrable.

In order to prove Tonelli's theorem we are going to need two facts. The first one is an easy exercise of Calculus, that is: if the surface $z = F(x, y)$ has continuous partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and these functions are integrable, then

$$S(F) = \iint \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}\, dx\, dy.$$

The second fact is due to K. Krickeberg [10]. Suppose that F is a locally integrable function, then F is of bounded variation in Tonelli's sense if and only if $\frac{\partial F}{\partial x} = \mu, \frac{\partial F}{\partial y} = \nu$ (in the distribution sense), are measures of finite total variation.

To see this, assume first that F is of bounded variation in Tonelli's sense. Let Φ be a smooth function,

$$< \frac{\partial F}{\partial x}, \Phi > = - < F, \frac{\partial \Phi}{\partial x} > = - \iint F(x, y) \frac{\partial \Phi}{\partial x}(x, y)\, dx\, dy$$

and

$$| < \frac{\partial F}{\partial x}, \Phi > | \leq \int dy | \int F(x, y) \frac{\partial \Phi}{\partial y}(x, y)\, dx | \leq$$

$$\leq \int dy\, [V_1(y)\, ||\Phi(-, y)||_\infty] \leq ||\Phi||_\infty \int V_1(y)\, dy$$

Therefore $\frac{\partial F}{\partial x}$ is a measure of finite variation. Similarly for $\frac{\partial F}{\partial y}$.

To prove the result in the other direction we observe first that the conclusion is immediate if $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are integrable functions. Next we can show that if μ is a mesure of finite total variation, and Φ_ε is a good approximation of the identity, then $||\mu * \Phi_\varepsilon||_1 \leq$ total variation of μ. With these two observations the second implication is easy to deduce.

Proof of Tonelli's theorem.

(a) Suppose first that $S(F) < \infty$. Let $\{P_k\}$ be a sequence of Polyhedrons such that P_k converges uniformly to F and $A(P_k) \leq M < \infty$ for every k.

Given a smooth function Φ on Q we have

$$< \frac{\partial F}{\partial x}, \Phi > = - < F, \frac{\partial \Phi}{\partial x} > = - \iint F(x, y) \frac{\partial \Phi}{\partial x}(x, y)\, dx\, dy =$$

$$= - \lim_{k \to \infty} \iint_Q P_k(x,y) \frac{\partial \Phi}{\partial x} (x, y)\, dx\, dy$$

$$\text{Thus } \left| < \frac{\partial F}{\partial x}, \Phi > \right| \leq \operatorname*{Sup}_k \left| \iint_Q P(x, y) \frac{\partial \Phi}{\partial x} (x, y)\, dx\, dy \right| \leq$$

$$\leq \operatorname*{Sup}_k A(P_k)\, ||\Phi||_\infty \leq M\, ||\Phi||_\infty$$

Therefore $\frac{\partial F}{\partial x} = \mu$ is a measure of finite total variations. Similarly for $\frac{\partial F}{\partial y} = \nu$.

(b) Suppose now that F is of bounded variation in Tonelli's sense. It is clear that, if we consider the extension $\bar{F}$ of F defined by $\bar{F}/Q = F$ and $\bar{F} \equiv 0$ on $\mathbf{R}^2 - Q$, then $\bar{F}$ is also of bounded variation in Tonelli's sense with respect to the whole plane. In the following we will identify F and $\bar{F}$.

Consider an approximation of the identity $\Phi_\delta(x) = \delta^{-2}\Phi(\delta^{-1}x)$, where Φ is a smooth, positive function such that

$$\int \Phi(x)\, dx = 1$$

Then $F_\delta = F * \Phi_\delta$ in a smooth function with partial derivatives

$$\frac{\partial F_\delta}{\partial x} = \frac{\partial F}{\partial x} * \Phi = \mu * \Phi_\delta$$

$$\frac{\partial F_\delta}{\partial y} = \frac{\partial F}{\partial y} * \Phi_\delta = \nu * \Phi_\delta$$

$$\text{Therefore } S(F_\delta) = \iint_Q \sqrt{1 + \left(\frac{\partial F_\delta}{\partial x}\right)^2 + \left(\frac{\partial F_\delta}{\partial y}\right)^2}\, dx\, dy \leq$$

$$\leq \iint_Q \left\{1 + \left|\frac{\partial F_\delta}{\partial x}\right| + \left|\frac{\partial F_\delta}{\partial y}\right|\right\} dx\, dy \leq$$

$$\leq 1 + \iint_Q \left| \int \Phi_\delta(u - t)\, d\mu\,(t) \right| du + \iint_Q \left| \int \Phi_\delta(u - t)\, d\nu(t) \right| du \leq$$

$\leq 1 +$ total variation of $\mu +$ total variation of $\nu < \infty$ (independently of δ).

Let Q' be a square contained in Q and such that $\operatorname{dist}(\partial Q', \partial Q) > 0$. Given $\varepsilon > 0$ we can find $\delta > 0$ such that $|F(x, y) - F_\delta(x, y)| \leq \leq \varepsilon/2$ for every (x, y) in Q'. With this F_δ we can now find a polyhedron P such that $|F_\delta(x, y) - P(x, y)| \leq \varepsilon/2$ if $(x, y) \in Q'$ and

$$A(P) \leq 2(1 + \text{tot. variation } \mu + \text{tot. variation } \nu).$$

Since this is true for every $\varepsilon > 0$ and $Q' \subset Q$, we have that $S(F) < \infty$.

Remark. It is possible to repeat the proof of Tonelli's theorem more carefully to obtain an analytical expression for the surface area. That expression is

$$S(F) = \iint \sqrt{d\lambda^2 + d\mu_1^2 + d\mu_2^2}$$

where $\mu_1 = \dfrac{\partial F}{\partial x}$, $\mu_2 = \dfrac{\partial F}{\partial y}$, $\lambda =$ Lebesgue measure on $\mathbf{R}^2$ and the integral means the total variation of the set function $\sqrt{d\lambda^2 + d\mu_1^2 + d\mu_2^2}$.

II

The analogy between curves and surfaces breaks down when we look at the following result of Lebesgue: if $x = f(t)$ is a function of bounded variation then it has a tangent almost everywhere. The corresponding result for a surface is false. R. Cacciopoli [1] showed the existence of a surface of finite variation in Tonelli's sense without having a tangent plane on a set of positive measure. Therefore in order to continue the analogy we only have two choices: either we generalize what we mean by a tangent plane or we introduce some additional conditions in the definition of bounded variation. The first approach was carried out by Calderon-Zygmund [2], [3].

Definition 4 (Calderon-Zygmund). Let $f : \Omega \to \mathbf{R}$, Ω open subset of $\mathbf{R}^2$, be a locally integrable function. We say that f has a derivative of order k in the L^q-sense at x_0, if there exists a polynomial $P_{x_0}(y)$ of degree $\leq k$, such that:

$$\left(h^{-2}\int_{|y|\leq h}|f(x_0+y) - P_{x_0}(y)|^q\,dy\right)^{1/q} = o(h^k), \text{ as } h \to 0.$$

* If this polynomial $P_{x_0}(y)$ exists it is uniquely determined and if $P_{x_0}(y) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} a_\alpha(x_0) y^\alpha$, the coefficient $a_\alpha(x_0)$ is called the L^q-derivative of order α at x_0. This is a generalization of the ordinary and Peano derivatives (see [2], [3]). Furthermore if f has a derivative of order p at x_0, then it has a derivative of order p_0 for every $1 \leq p_0 \leq p \leq \infty$.

THEOREM (CALDERON-ZYGMUND). *If f is of bounded variation on $\mathbf{R}^2$ in the sense of Tonelli, then for almost every point, f has a derivative in the L^2-sense.*

In the following we will be interested in the second approach and we will show some negative results.

(A) Suppose that $z = F(x, y)$ is a surface defined on the unit square Q and such that

$$V_1(\xi) = \text{total variation of } F(\xi, -)$$
$$V_2(\eta) = \text{total variation of } F(-, \eta)$$

are bounded functions. Question: does the surface have an ordinary tangent plane at almost every point?

The following example due to J. P. Kahane shows that the answer to this question is no.

Consider a sequence $\varepsilon_n \searrow 0$. Start with a cross of width ε_1 contained on the unit square (*Fig.* 1). That cross contains a square Q_1 of

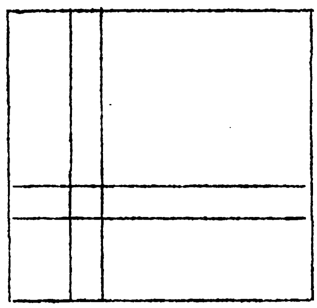

Fig. 1

side ε_1 in the intersection of its two branches. Over Q_1 we consider a pyramid of height $\varepsilon_1^{1/2}$; let Φ_1 be the function defined by $\Phi_1 \equiv 0$ on $Q - Q_1$ and let Φ_1/Q_1 be given by the pyramid.
Now the complement of the cross consists of four rectangles; we pick the rectangle with biggest diameter and we consider a cross of width ε_2 with square Q_2 inside that rectangle (Fig. 2)

Over Q_2 we consider a pyramid of height $\varepsilon_2^{1/2}$ and a function Φ_2 defined by:

$$\Phi_2 = \Phi_1 \text{ on } Q - Q_2$$

$$\Phi_2/Q_2 \text{ given by the pyramid.}$$

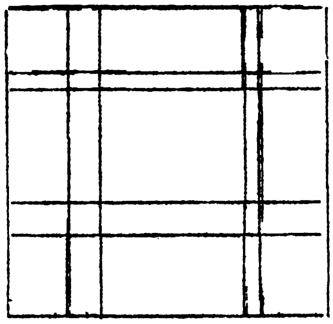

Fig. 2

We can repeat the process to get a sequence of squares $\{Q_k\}$ and a sequence of functions $\{\Phi_k\}$ such that:

i) $F(x, y) = \lim_{k \to \infty} \Phi_k(x, y)$ is continuous

ii) $\cup Q_k$ is dense in Q.

Furthermore $V_1(\xi) =$ total variation $F(\xi, -) \leq \varepsilon_1^{1/2}$. $V_2(\eta) =$ total variation $F(-, \eta) \leq \varepsilon_1^{1/2}$. And $F \equiv 0$ on the complement of $\cup Q_k$.

We can now use the fact that $\cup Q_k$ is dense in Q to see that, for each point in the complement of $\cup Q_k$, the function F does not have a tangent plane. Finally it is easy to check that if the sequence ε_n converges very quickly to zero, then $Q - \cup Q_k$ has a positive measure.

(B) Suppose now that F has bounded variations not only with respect to vertical and horizontal lines, but with respect to any straight line and we ask the same question as in part (A). The answer is still negative.

PROOF. Divide the square Q into four dyadic squares $Q_1^1, Q_2^1, Q_3^1, Q_4^1$ of side 2^{-1}. Pick one of these squares, for exemple Q_1^1, and construct a pyramid of height $2^{-1/2}$. Now we can take three points $p_i \varepsilon Q_i^1$, $i = 2, 3, 4$, in such a way that no straight line contains more than two of them. Next, we can find a positive number $\varepsilon > 0$ such that no straight line intersects more than two of the balls $\{B(p_i, \varepsilon)\}_{i=,2,3\,4}$. Divide every one of the three squares $Q_2^1 Q_3^1, Q_4^1$ into a grid of dyadic squares such that:

i) The dyadic squares Q_i^2 that contains the point p_i is contained in $B(p_i, \varepsilon)$

ii) $|Q_i^2| \leq \frac{1}{3} 2^{-3}$

Now we construct over each Q_i^2 $i = 2, 3, 4$, a pyramid of height 2^{-1}. Observe that every point in Q is at a distance less than $1/3$ of $F_2 = U_i Q_i^2 \cup Q_1^1$.

By induction we can get for every k a family of dyadic squares $\{Q_i^k\}$ such that:

i) $\underset{i}{\Sigma} |Q_i^k| \leq 2^{-(k+2))}$

ii) $d(x, \overset{k}{\underset{j=1}{\cup}} \underset{i}{\cup} Q_i^j) \leq 2^{-k}$ for every point x in Q.

iii) No straight line intersects more than two squares of the family $\{Q_i^k\}$.

Furthermore we construct a pyramid of height $2^{-k/2}$ on each square of the family $\{Q_i^k\}$. Then we get a continuous function F in the square Q, because it is a uniform limit of continuous functions and such that:

1. $\underset{k=1}{\Sigma} \underset{i}{\Sigma} |Q_i^k| \leq 1/2$. Therefore $F = Q - \underset{k}{\cup} \underset{i}{\cup} Q_i^k$ has positive measure.

2. $\underset{k}{\cup} \underset{i}{\cup} Q_i^k$ is dense in Q by construction.

3. F does not have a tangent plane at any point of density of F.

4. F has total variation uniformly bounded on each straight line.

Q.E.D.

(C) The same basic example (Cacciopoli's pyramid), can be modified to show that the Calderon-Zygmund result [3] is the best possible.

Take numbers $p \geq 3$ and $\varepsilon_1 > 0$ and consider the sequences $\{\varepsilon_n\}$ $\{h_n\}$ given by:

$$\varepsilon_n = \frac{1}{2}\varepsilon_{n-1}^{p} = 2^{-(n-1)} A_n^{p-1} \varepsilon_1, \text{ where } A_n = \varepsilon_{n-1} \dots \varepsilon_1$$

$$h_n = A_n^{\frac{3-p}{2}}$$

Divide the square Q into a grid of squares of area ε_1. Pick one of the squares of the grid, Q_1^1, and construct a pyramid of height h_1. Subdivide each one of the remainder squares into a grid of squares of area ε_2; select one square of area ε_2 on each one of the squares of the first grid (except for Q_1^1) and construct a pyramid of height h_2 and so on (Fig. 3). Call F the surface obtained by this process.

Then:

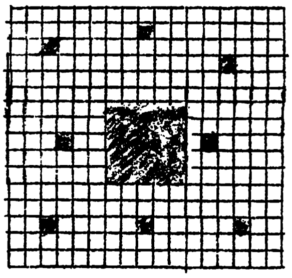

Fig. 3

a) $\Sigma |Q_j^k| \leq \varepsilon_1 + (\varepsilon_1^{-1} - 1)\, \varepsilon_2 + \dots + (\varepsilon_1^{-1} - 1) \dots (\varepsilon_1^{-1} - 1)\varepsilon_n + \dots \leq$

$$\leq \Sigma A_n^{-1} \varepsilon_n \leq 2\varepsilon_1 < 1 \text{ if } \varepsilon_1 < \frac{1}{2}$$

Thus the bad set $F = Q - \cup Q_j^k$ has positive measure.

b) F has finite area, because

$$\varepsilon_1^{1/2} h_1 + \dots + (\varepsilon_1^{-1} - 1) \dots (\varepsilon_{n-1}^{-1} - 1)\, \varepsilon_n^{1/2} h_n + \dots \leq$$

$$\leq \varepsilon_1^{1/2}\, \Sigma\, 2^{-\frac{n-1}{2}} < \infty.$$

We want to show that given $q = 2 + \delta$, $\delta > 0$ we can choose p such that the corresponding surface has no tangent plane, in the

L^q-sense, at any density point of the bad set F. In order to do that we can observe, first of all, that if F has a differential in such a point, then it must be zero. Therefore the function F must satisfy

$$\Big(h^{-2}\int_{|y|\leq h}|F(x_0+y)|^q dy\Big)^{1/q} = o(h), \text{ as } h \to 0.$$

With $h = \varepsilon_n^{1/2}$ we have:

$$\Big(\varepsilon_n^{-1}\int_{|y|\leq \varepsilon_n^{1/2}}|F(x_0+y)|^q dy\Big)^{1/q} \geq (\varepsilon_n^{-1}\,\varepsilon_{n+1}h_{n+1}^q)^{1/q} =$$

$$= h_{n+1}\Big(\frac{\varepsilon_{n+1}}{\varepsilon}\Big)^{1/q} = 2^{-1/q}\,h_{n+1}\,\varepsilon_n^{\frac{p-1}{q}} =$$

$$= 2^{(n-1)/2}\,\varepsilon_1^{\frac{p-3}{p-1}}\,\varepsilon_n^{\frac{p(3-p)}{2(p-1)}+\frac{p-1}{q}}.$$

With $q = 2 + \delta$, $\delta > 0$ we have:

$$\frac{p}{2}\frac{3-p}{p-1}+\frac{p-1}{2+\delta} = -\frac{p}{2}\frac{\delta}{2(2+\delta)}+\frac{p}{p-1}-\frac{1}{2+\delta}\leq -\frac{\delta}{8}p$$

if p is big enough. Thus

$$\Big(\varepsilon_n^{-1}\int_{|y|\leq \varepsilon_n^{1/2}}|F(x_0+y)|^q dy\Big)^{1/q} \geq C\,\varepsilon_n^{-\frac{\delta p}{8}}$$

which is a contradiction with the fact that F has derivative zero in the L^q-sense at x_0.

Q.E.D.

REMARK. It is possible to modify our basic example to show that there exists a function F of bounded variation in Tonelli's sense and such that $V_1(\xi)$, $V_2(\eta)$ are in $L^p(0, 1)$, but fails to have a derivative in the L^q-sense for $q > 2p$.

REFERENCES

1. R. CACCIOPOLI, *Sul Carattere infinitesimale delle Superficie quadrabili*, Atti. della Lincei, 7 (1928), 901-905.
2. A. CALDERON and A. ZYGMUND, *Local Properties of solutions of elliptic partial differential equations*, Studia Mat. 20 (1961), 179-225.
3. —, *On the differentiability of functions which are of bounded variation in Tonelli's sense*, Revista de la Union Matem. Argent., 20 (1960), 102-121.
4. L. CESARI, *Surface Area*, 1956.
5. G. PEANO, *Sulla formula di Taylor*, Atti della Ac. Sci. Torino, 27 (1891) 40-46.
6. —, *Sur la definition de la derive'e*, Mathesis (2) 2 (1892) 12-14.
7. T. RADO, *Length and Area*, 1948.
8. S. SAKS, *Theory of the integral*, 1937.
9. A. ZYGMUND, *Algunos resultados y problemas sobre diferenciabilidad de funciones*, Lecture notes University of Madrid, 1971.
10. K. KRICKEBERG, *Distributionen, Funktionen beschränter Variation und Lebesguescher Inhalt Flächen*, Annali di Mat. (IV) 44 (1957), 105-134.