THE HARDY-LITTLEWOOD MAXIMAL FUNCTION AND DERIVATION OF INTEGRALS

BALDOMERO RUBIO
(University of Madrid)

Abstract. The Hardy-Littlewood maximal function is used to obtain a characterization of the derivation, with respect to a differentiation basis in R^n, for the integrals of locally integrable functions. New proofs of density properties relative to the bases of intervals and rectangles are given.

Introduction. We present in § 1 the necessary definitions. § 2 contains a version of theorems of Busemann-Feller using the maximal function. The theorem indicated above is in § 3, and § 4 is dedicated to density properties of special bases.

§ 1. A differentiation basis in R^n is a collection $\mathfrak{R}$ of bounded open sets in R^n such that for every $x \in R^n$ there is at least a sequence $\{R_k\} \subset \mathfrak{R}$ which verifies $R_k \to x$ (i. e. $x \in R_k$ for every k and, given a neighborhood U of x, $R_k \subset U$ for k greater than some k_0).

Let $f\colon R^n \to R$ be a locally (Lebesgue) integrable function. The upper derivative of $\int f$, with respect to $\mathfrak{R}$, at x is defined by

$$\bar{D}\left(\int f, x\right) = \sup \limsup \frac{1}{|R_k|} \int_{R_k} f(y)\, dy$$

being the sup taken over all the sequences $\{R_k\} \subset \mathfrak{R}$ such that $R_k \to x$. The lower derivative $\underline{D}\left(\int f, x\right)$ is defined setting inf lim inf. We say that $\mathfrak{R}$ differentiates $\int f$ if $\bar{D}\left(\int f, x\right) = \underline{D}\left(\int f, x\right) = f(x)$ a. e. in R^n. In this case we write $\bar{D} = \underline{D} = D$.

The Hardy-Littlewood maximal function associated with $\mathfrak{R}$ is defined by

$$Mf(x) = \sup_{x \in R \in \mathfrak{R}} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

Since $\{x\colon Mf(x) > \lambda\}$ is an open set, Mf is measurable.

There are connections between properties of M and derivation properties of $\mathfrak{R}$. Here we will consider properties of the type

$$|\{x: Mf_k(x) > \lambda\}| \to 0$$

where $\{f_k\}$ is some special sequence of locally integrable functions.

§ 2. A density basis is a differentiation basis which differentiates $\int f$ when f is the characteristic function of a measurable set. The Busemann-Feller theorems [1] relative to density bases are presented now in connection with the maximal function.

2.1. THEOREM. *A differentiation basis $\mathfrak{R}$ in R^n is a density basis if and only if for every λ in the interval* $(0,1)$, *every contractive sequence $\{E_k\}$ of bounded measurable sets such that $|E_k| \to 0$, and every non increasing sequence $\{\alpha_k\}$ of positive numbers such that $\alpha_k \to 0$ one has*

$$\lim |\{x: M_k \chi_{E_k}(x) > \lambda\}| = 0$$

where M_k is the maximal function associated with the basis

$$\mathfrak{R}_k = \{R \in \mathfrak{R}: \text{diameter } R < \alpha_k\}.$$

The density property for a differentiation basis which is homothecy invariant (i. e. if $R \in \mathfrak{R}$ and R' is homothetic to R, then $R' \in \mathfrak{R}$) admits the following characterization.

2.2. THEOREM. *A homothecy invariant basis in R^n is a density basis if and only if for every λ in the interval* $(0,1)$ *there exists a number $c(\lambda)$ such that*

$$|\{x: M\chi_E(x) > \lambda\}| \leq c(\lambda)\,|E|$$

for every bounded measurable set E.

§ 3. We present now a new theorem which is suggested by the Busemann-Feller theorems.

3.1. THEOREM. *The two following conditions for a basis $\mathfrak{R}$ in R^n are equivalent:*

(a) *R differentiates $\int f$ for every locally integrable function f.*

(b) *For every* $\lambda > 0$, *every non increasing sequence* $\{\alpha_k\}$ *of positive numbers such that* $\alpha_k \to 0$, *and every non increasing sequence* $\{f_k\}$ *of non negative functions in* $L^1(R^n)$ *with compact support such that* $\|f_k\|_1 \to 0$, *one has*

$$\lim |\{x: M_k f_k(x) > \lambda\}| = 0$$

being M_k *defined as before.*

Proof. Suppose (*b*) is not true. There exist λ, $\{\alpha_k\}$, and $\{f_k\}$ as stated above, and there exists $\eta > 0$ such that for $k = 1, 2, \ldots$ $|\{x: M_k f_k(x) > \lambda\}| > \eta$. We can suppose, considering a subsequence of $\{f_k\}$ if necessary, that $\|f_k\|_1 < 2^{-k}$.

Because f_k has compact support and diameter $R < \alpha_k$ for every $R \in \mathfrak{R}_k$, $\{x: M_k f_k(x) > \lambda\}$ is a bounded set. The sequence

$$\{x: M_k f_k(x) > \lambda\}$$

is contractive and so its limit E verifies $|E| > \eta$. We denote also $\{f_k\}$ a subsequence of $\{f_k\}$ which converges to 0 a. e. in E. By Egorov theorem, there exists a measurable set $F \subset E$ such that $|F| > \eta$ and $\{f_k\}$ converges uniformly in F, and so for every k there exists $j(k)$ which verifies

$$f_{j(k)}(x) \leq \frac{\lambda}{2} 2^{-k}$$

for every $x \in F$. Now we consider f defined by

$$f = \sum_{k=1}^{\infty} f_{j(k)}.$$

It is clear that f is in $L^1(R^n)$ and $f(x) \leq \frac{\lambda}{2}$ for $x \in F$. Given $x \in F$, there exists a sequence $\{R_k\}$ such that $R_k \to x$ and

$$\int_{R_k} f_{j(k)}(y)\, dy > \lambda\, |R_k|.$$

Hence, we have also

$$\int_{R_k} f(y)\, dy \geq \int_{R_k} f_{j(k)}(y)\, dy > \lambda\, |R_k|$$

and so $\overline{D}(\int f, x) \geq \lambda$ in F. This shows that $\mathfrak{R}$ no differentiates $\int f$.

Conversely, now we suppose (b) is true. It is sufficient to prove (*a*) for non negative functions in $L^1(R^n)$ with compact support. Let f be such a function. There is a non decreasing sequence $\{g_k\}$ of non negative simple functions such that $\|f - g_k\|_1 \to 0$ and $f - g_k$ is non negative for every k. Now we have for every non increasing sequence $\{\alpha_k\}$ of positive numbers such that $\alpha_k \to 0$,

$$\lim |\{x: M_k(f - g_k)(x) > \lambda\}| = 0$$

where λ is an arbitrary positive number and M_k is the maximal function associated with the basis $\mathfrak{R}_k = \{R \in \mathfrak{R}: \text{diameter } R < \alpha_k\}$. Only remains to prove that $\{x: |\bar{D}(\int f, x) - f(x)| > 0\}$ and $\{x: |\underline{D}(\int f, x) - f(x)| > 0\}$ have measure zero.

Condition (b) implies that $\mathfrak{R}$ is a density basis, by theorem 2.1. This means that $\mathfrak{R}$ differentiates $\int g_k$ for every k, and so

$$\bar{D}(\int f, x) = \bar{D}(\int (f - g_k), x) + g_k(x)$$

a. e. in R^n. Now we have, for every k and $\lambda > 0$,

$$\left|\{x: |\bar{D}(\int f, x) - f(x)| > 2\lambda\}\right|_e \leq$$
$$|\{x: M_k(f - g_k)(x) > \lambda\}| + |\{x: (f - g_k)(x) > \lambda\}|$$

and this converges to zero as $k \to \infty$. The remainder of the proof is easy.

§ 4. Now we consider the density property for two important bases in R^2. First we consider the basis of intervals. The following theorem admits an easy generalization.

4.1. THEOREM. *The basis of bounded intervals in* R^2 *has the density property.*

Proof. It is sufficient to prove that for λ in the interval (0,1) there exists a number $c(\lambda)$ such that

$$|\{(x, y) \in R^2: M\chi_G(x, y) > \lambda\}| \leq c(\lambda)|G|$$

for every bounded open set G in R^2.

We denote $\mathfrak{R}'$ the basis of intervals in R, and M' the maximal function associated. It is known that $\mathfrak{R}'$ differentiates $\int f$ for every $f \in L^1(R)$ and so M' is of weak type (1,1) (see [2] or [5]). This leads to the inequality

$$|\{t \in R: M' \chi_S(t) > \lambda\}| \leq \frac{c}{\lambda} |S|$$

for every $\lambda > 0$ and S measurable in R.

Given $P \subset R^2$ and $(x, y) \in R^2$, we define the sections

$$P(x) = \{t: (x, t) \in P\} \qquad P(y) = \{t: (t, y) \in P\}$$

and, for every bounded open set G we consider the following open sets

$$A = \{(x, y): M' \chi_{G(x)}(y) > \lambda\}$$
$$B = \{(x, y): M' \chi_{A(y)}(x) > \lambda\}.$$

Using the Fubini theorem and the inequalities above, we obtain

$$|A| \leq \frac{c}{\lambda} |G| \qquad |B| \leq \frac{c}{\lambda} |A|$$

and so

$$|B| \leq \frac{c^2}{\lambda^2} |G|.$$

Let $I \times J$ be an interval no contained in B, and choose (a, b) in $I \times J$ such that $(a, b) \notin B$. Being $S = \{x \in I: (x, b) \in A\}$, it is clear that $|S| \leq \lambda |I|$. For every $x \in I - S$ the set $T(x) = \{y \in J: (x, y) \in G\}$ verifies $|T(x)| \leq \lambda |J|$. It is sufficient to use again the Fubini theorem to obtain $|(I \times J) \cap G| \leq 2\lambda |I \times J|$. This means that

$$\{(x, y) \in R^2: M \chi_G(x, y) > 2\lambda\} \subset B$$

being M the maximal operator associated with the basis of intervals in R^2. Finally, we have

$$|\{(x, y) \in R^2: M \chi_G(x, y) > 2\lambda\}| \leq \frac{c^2}{\lambda^2} |G|.$$

This completes the proof.

The basis of rectangles in R^2 does not verify the density property. One proof of this is based on the famous set of Nikodym [3]. Another one can be found in Busemann and Feller [1]. Here we present a new proof based on the construction of Rademacher [4] of the Perron tree. This construction permits to observe that, given $\varepsilon > 0$, there exists a Perron tree P of a triangle T such that $|P| < \varepsilon |T|$ and the set

$$\left\{x \in R^2 : M\chi_P(x) > \frac{1}{2}\right\}$$

contains a triangle with the same size as T, being M the corresponding maximal operator. Then it is sufficient to apply theorem 2.2. We begin with a simple lemma.

4.2. Lemma. *Let M be the Hardy-Littlewood maximal function associated with the basis of rectangles in R^2. Given an arbitrary open triangle T, the set $\left\{x : M\chi_T(x) > \frac{1}{2}\right\}$ contains the image of T by the homothecy of ratio 2 with a vertex of T as center.*

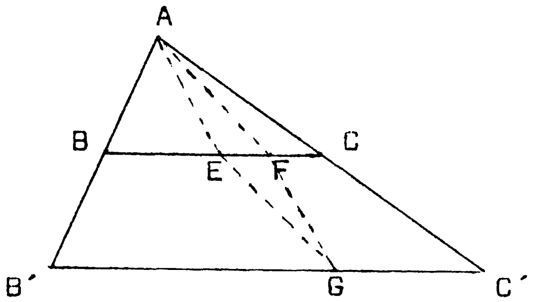

Fig. 1

Proof. Let T be the triangle ABC (see fig. 1). It is sufficient to observe that for every interior point x of the parallelogram $AEGF$ there exists a rectangle R such that $x \in R$ and $|R \cap T| > \frac{1}{2}|R|$, and that all the parallelograms $AEGF$, with E and F in the segment BC, cover $AB'C'$.

The idea of the Perron tree is the following: We cut a triangle ABC in partial triangles with A as common vertex and their bases in the segment BC. Shifting them on the line BC it is possible to obtain a figure P (Perron tree) with small area.

4.3. THEOREM. *Let T be a triangle. Given $\varepsilon > 0$, there exists a Perron tree P obtained from T such that $|P| < \varepsilon |T|$ and*

$$\left\{ x \in R^2 : M\chi_P(x) > \frac{1}{2} \right\}$$

contains a triangle with the same size as T, being M the maximal function for rectangles.

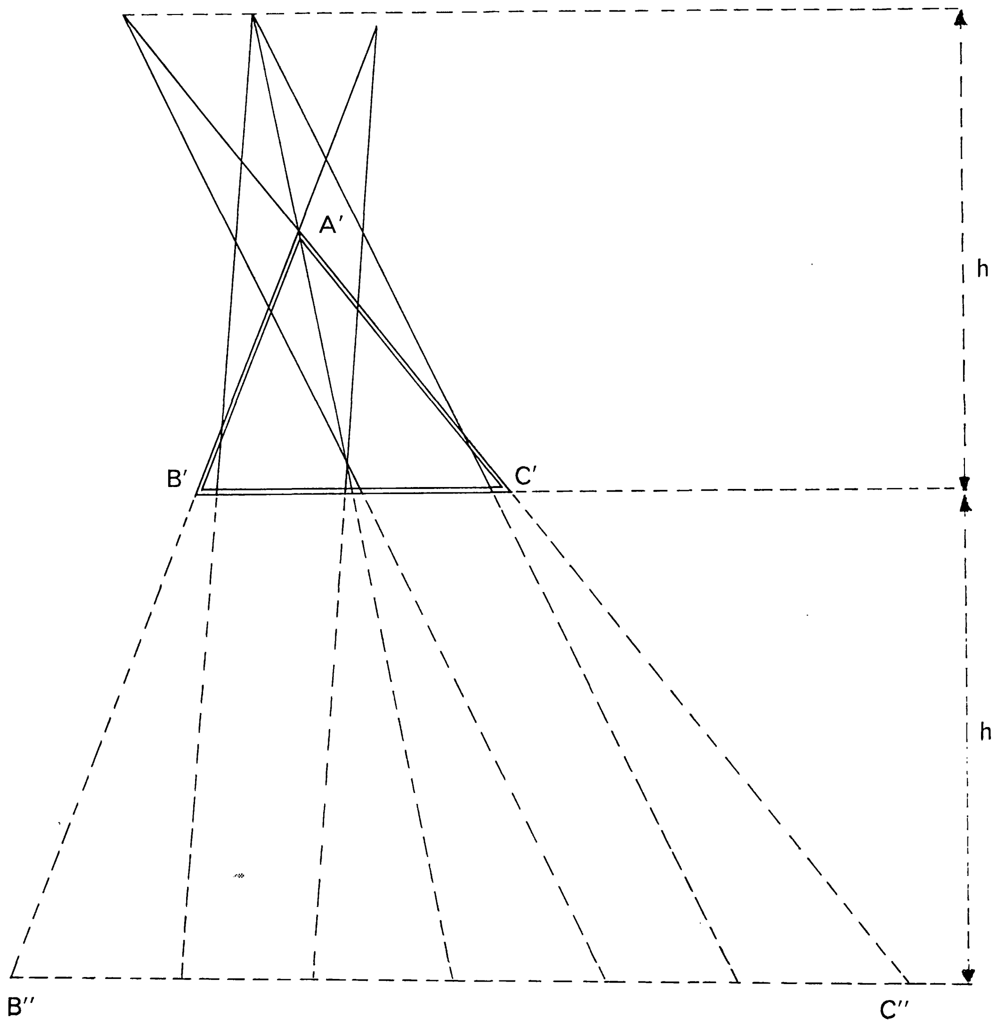

Fig. 2

Proof. The construction of Rademacher [4] shows how it is possible to obtain a Perron tree from a triangle $T = ABC$ by dividing the base BC into 2^n equal parts and considering the partial triangles T_k so obtained. After they have been shifted, the union of their bases is the base $B'C'$ (see fig. 2) of a triangle $A'B'C'$ similar to ABC. We denote P'_k the last position of P_k, and P''_k the image of P'_k by the homothetic transformation of ratio 2 with its upper vertex as center. The union of the P'_k is a Perron tree P, and the union of the P''_k covers the triangle $A'B''C''$ and so, by the lemma above, the set $\left\{x : M\chi_P(x) > \frac{1}{2}\right\}$ contains a triangle with the same size as T. For n large this construction can be made so that P verifies $|P| < \varepsilon |T|$.

Now can be used theorem 2.2. and easily is obtained that the basis of rectangles in R^2 is not a density basis.

Acknowledgment. These results are contained in my doctoral thesis at the University of Madrid, being M. de Guzmán my thesis adviser, to whom *I* wish to thank for his help.

REFERENCES

[1] M. BUSEMANN und W. FELLER, *Zur Differentiation der Lebesgueschen Integrale,* Fund. Math. 22 (1934) 226-256.

[2] M. DE GUZMAN and G. V. WELLAND, *On the Differentiation of Integrals,* Revista de la Unión Matemática Argentina, Vol. 25 (1971) 253-276.

[3] O. NIKODYM, *Sur la mesure des ensembles plans dont tous les points sont rectilineairement accesibles,* Fund. Math. 10 (1927) 116-168.

[4] H. RADEMACHER, *A. new construction of the Perron tree, In the book «Studies in Mathematical Analysis and related topics»,* edited by Gilbarg, Solomon and others, Stanfor (1962).

[5] B. RUBIO, *The property of weak type (p, p) for the Hardy-Littlewood maximal operator and derivation of integrals.* Studios Math. Vol. 57.3.