A NOTE ON THE GROWTH OF ENTIRE FUNCTIONS

CHUNG-CHUN YANG[1]

1. INTRODUCTION AND RESULTS

Le $f(z)$ be a nonconstant entire function in the finite plane $|z| < \infty$, then the growth of f can be measured in terms of $M(r,f)$ $(= \operatorname*{Max}_{|z|=r} |f(z)|)$ or $T(r,f)$ $\left(= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta\right.$, where $\log^+ x = \operatorname{Max}\{\log x, 0\}$), the Nevanlinna characteristic function. We note that the order ϱ_f of f is defined to be $\varrho_f = \varlimsup_{r\to\infty} \log\log M(r,f)/\log r$, $(= \varlimsup_{r\to\infty} \log T(r,f) \log r)$, and that both $T(r,f)$ and $\log M(r,f)$ are monotone increasing functions of r. There are many results done on comparing the growth of $T(r,f)$ with that of $M(r,f)$. Among them the following one is fundamental:

THEOREM 1 ([3, p. 8]). If $f(z)$ is regular for $|z| \leq R$, then

$$(1) \qquad T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r}\, T(R,f) \; (0 \leq r < R).$$

It is shown by examples that $T(r,f) \sim \log M(r,f)$ as $r \to \infty$ need not be necessarily true (see e. g., [3, p. 19]). However, in [4, p. 48] Kamthan proved the following result.

THEOREM 2. Let $f(z)$ be an entire function of finite order. Then there exists a sequence $\{r_n\}$, $r_n \to \infty$ with n, such that

$$(2) \qquad \lim_{r\to\infty} \log\log M(r,f)/\log T(r,f) = 1.$$

We remark here that the above result can be extended to entire functions of all orders by an application of a result of T. SHIMIZU (see [3, p. 20]).

---

[1] Mathematics Research Center, Naval Research Laboratory, Washington, D C, 20390

In the conclusion of the paper [4] Kamthan remarked that it was still an open question if

$$\lim_{r\to\infty} \log \log M(r, f) / \log T(r, f) = 1 \tag{3}$$

for entire functions of all orders.

In this note, although we do not not give any positive answer to this question we have succeeded to prove that given any $\epsilon > 0$ the quotient $\log \log M(r, f) / \log T(r, f)$ is always less than $1 + \epsilon$ outside a set E of r values of finite measure which depends on ϵ. We shall also show that for certain entire functions of infinite order one can replace ϵ by 0 in the above assertion. More precisely we have.

THEOREM 3. Let $f(z)$ be a transcendental entire function and let ϵ be any given positive number. Then possibly outside a set $E(\epsilon)$ of r values of finite measure, we have for $r \notin E(\epsilon)$

$$\overline{\lim_{r\to\infty}} \log \log M(r, f) / \log T(r, f) \leq 1 + \epsilon. \tag{4}$$

THEOREM 4. Let $f(z)$ be an entire function. Suppose that

$$\underline{\lim_{r\to\infty}} \log T(r, f) / r^{12} \geq c < 0. \tag{5}$$

Then

$$\lim_{r\to\infty} \log \log M(r, f) / \log T(r, f) = 1$$

outside a set of r values of finite measure.

2. Lemmas Needed for the Proofs.

In the proofs of Theorems 3 and 4 we shall need the following two lemmas of Borel's type argument on monotone functions.

Lemma 1 ([3, p. 38]). Suppose that $T(r)$ is continuous, increasing and $T(r) \geq 1$ for $r_0 \leq r < \infty$. Then we have

$$T\{r + 1 / T(r)\} < 2\, T(r) \tag{6}$$

outside a set E of r which has linear measure at most 2.

Remark. In view of the proof of Lemma 1, one can see easily that there is no sacredness about the constant multiple 2 in the right hand side of inequality (6). It can be replaced by any other fixed constant $\in$ with $\in > 1$, and, of course, the measure of the exceptional set will no longer be bounded by 2 but $\in/(\in - 1)$.

Lemma 2 ([2, p. 118]). Let $T(r)$ be a nonnegative, nondecreasing unbounded function defined in $r > r_0$. Then there is a set E with

$$m\{E \cap [\varrho, 2\varrho]\} \leq \frac{8\,\varrho}{[\log T\,(\varrho)]^{\frac{1}{2}}} \qquad (\varrho > \varrho_0) \tag{7}$$

such that outside E, we have

$$T\{r + r/\log^2 T\,(r)\} < e\,T\,(r). \tag{8}$$

Remark. This is a special case of a Lemma 10.2 [|] ($\alpha = 0, \in = 1$).

3. Proof of Theorem 3.

From (1) by setting $R = r + \dfrac{1}{\log T\,(r,\,f)}$ and taking logarithm, we have

$$\log\log M\,(r,f) \leq \log\left|\frac{2r + 1/\log T\,(r,f)}{1/\log T\,(r,f)}\right| + \tag{9}$$
$$+ \log\{T\,(r + 1/\log T\,(r,f),\,f)\}$$

By applying the Borel's lemma to the monotone function $T(r) = \log T(r,f)$ and noting the remark mentioned earlier, we obtain

$$\log T\,(r + 1/\log T\,(r,f),\,f) < (1 + \in)\log T\,(r,\,f), \tag{10}$$

provided $r \notin E\,(\in)$. From this and (9), we have for $r \notin E\,(\in)$

$$\log\log^+ M\,(r,f) \leq \log\{2\,r\log T\,(r,f) + 1\} + (1 + \in)\log T\,(r,f) \tag{11}$$
$$\leq (1 + \in + {}_0(1))\log T\,(r,f) + \log 2\,r.$$

Since for any transcendental entire f, $\overline{\lim\limits_{r\to\infty}} \log r/\log T\,(r,\,f) = 0$, we have from (11) that

$$\text{(12)} \qquad \log\log^{+} M(r,f) \leq (1 + \in + o(1)) \log T(r,f),$$

and Theorem 3 is thus proved.

4. Proof of Theorem 4.

Put $T(r) = T(r,f)$, and without loss of generality, we may assume that $\varrho_0 = r_0 = 1$. Let E be the set of r values greater than 1 where inequality (8) fails to hold.

By Lemma 2 and the assumption (5)

$$\text{(13)} \qquad m\{E\} = \bigcup_{n=1}^{\infty} m\,\{E \cap [n, n+1]\} \leq \sum_{n=1}^{\infty} \frac{8\,n}{\{\log\, T\,(n)\}^{\frac{1}{4}}}$$

$$\leq \sum_{n=1}^{\infty} \frac{8\,(c - \in)^{-\frac{1}{4}}}{n^2} < + \infty$$

where $\in$ is a positive number less than c.

Then, for $r \notin E$ with $r > 1$,

$$\text{(14)} \qquad T\,(r + \frac{r}{\log^2 T\,(r,f)}, f) = T\,(r + \frac{r}{\log^2\, T\,(r)}) < e\,T\,(r).$$

Now applying Theorem 1 by setting $R = r + \dfrac{r}{\log^2\, T\,(r)}$ with $r \notin E$ and taking logarithm, we deduce

$$\text{(15)} \quad \log\,\log^{+} M\,(r,f) \leq \log \frac{2\,r + r/\log^2 T\,(r)}{r/\log^2 T\,(r)} + \log T\,(r + \frac{r}{\log^2\, T\,(r)})$$

$$\leq \log\,\{2 \log^2 T\,(r) + 1\} + \log T\,(r + r/\log^2 T\,(r)).$$

Hence, by (14), for $r \notin E$

$$\text{(16)} \qquad \log\log^{+} M\,(r,\, f) \leq (1 + o\,(1)) \log T\,(r,\, f).$$

Our assertion follows from this and the fact that $\log\log^{+} M(r,f) \geq$ $\geq \log T(r,f)$ for $r > r_0$.

REFERENCES

1 A EDREI & W H J FUCHS: *Bounds for the number of certain classes of functions*, Proc Lond Math Soc 3 , Ser 12 (1962), 315-344

2 A EDREI & W H J FUCHS: *On meromorphic functions with regions free of poles and zeros*, Acta Math Vol 108, 1962

3 W K HAYMAN: *Meromorphic Functions*, Oxford (1964)

4 P K KAMTHAN: *Growth of a meromorphic function*, Collectana Mathematica, Vol XIX - Fases 1 º y 2 º - Año 1968