NONLINEAR MAPPINGS IN METRIC DISCRETE LIMIT SPACES AND THEIR TOPOLOGICAL PROPERTIES

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This paper establishes some new topological characterizations of the basic concepts of the generalized perturbation theory developed in Stummel [9] — [11], Stummel-Reinhardt [12]. We can hereby extend results concerning continuity properties of discrete limits of mappings, and compactness properties of limits of sets and mappings as well as a generalization of the compactness criteria of Arzelá and Ascoli, which is of particular interest in view of applications.

In chapter 1 we study the topological structure of metric discrete limit spaces and establish topological properties of the sets B, $\mathscr{C}$, and D of the discretely bounded, discretely compact, and discretely convergent sequences. In the preliminary section 1.1 we present the definitions of a discrete limit space $X_0, \Pi_\iota X_\iota, lim$ and the discrete convergence according to [10], [12] as well as the definition of discrete boundedness according to Grigorieff [5]. Section 1.2 introduces a topology in the product space by means of a pseudo-metric. In the topological space of all sequences, the set B is always closed, whereas to prove that $\mathscr{C}$ and D are closed, we need a complete metric space X_0 and convergent metrics (cf. theorem 1.(4)). Moreover, in the case of convergent metrics, the mapping *lim* of the discrete convergence is an isometry (cf. 1.(2)). An extension property of continuous mappings then enables us to generalize the important result of the extension of the discrete convergence of Stummel [9]-*I* in the case of metric discrete limit spaces.

In chapter 2 we generalize classic compactness criteria by characterizing discrete compactness of a set of sequences in a metric discrete limit space by a finite covering property (cf. theorem 2.(4)). This needs the concept of asymptotic precompactness. As an immediate consequence of the equivalence of discrete compactness and asymptotic precompactness, we show that in a separable space the upper discrete limit of a discretely compact sequence of sets is always precompact (cf. 2.(6)). The general characterizations reduce to classic compactness conditions in the case of an approximation of subspaces studied in Anselone [1], [2], Petryshyn [7], Vainikko [13], [14].

In chapter 3, the basic concepts of the theory of discretely convergent sequences of mappings characterize topological properties of such sequences (A_ι). In this way, in section 3.1, the continuity is proven to be equivalent to the stability of $A = (A_\iota)$ (cf. (3.1)). An open mapping will be characterized by the generalized inverse stability of Grigorieff [5] (cf. 3.(2)), which is equivalent to the inverse stability of [10], [12] for injective mappings A_ι. In several cases even uniform stability conditions are valid. In 3.2 the discrete convergence of mappings is shown to be equivalent to the fundamental relation 3.(8). Thus we can prove the existence of a mapping which is the discrete limit of a given stable sequence of mappings. Moreover, relation 3.(8) yields complete information of the continuity properties of discrete limits. At the end of chapter 3, we shall prove an interesting characterization of the discrete convergence of mappings by means of the Lim-convergence of the associated graphs (cf. 3.(11)). This generalizes analogous theorems of classic functional analysis as well as a result of Stummel [9]-*III*,1.

In chapter 4 we apply the equivalence theorems of 2.1 to establish necessary and sufficient conditions for the discrete compactness of a sequence of mappings. Then as a corollary we obtain that a mapping, consistent with a discretely compact sequence of mappings, is precompact, which was first proved in Wolf [15] for linear operators. Our concepts also generalize compactness conditions of Anselone [1], [2], Vainikko [14] and yield new results and characterizations in their special cases. Finally, in section 4.2 the equivalence of discrete compactness and asymptotic precompactness enable us to prove a generalization of the classic compactness criteria of Arzelá and Ascoli in the case of discrete approximations of function spaces (cf. 4.(6)). These approximations are defined by the discretely uniform convergence 4.(5) containing the uniform convergence definitions of Aubin [3], Vainikko [13] as special cases. For linear and nonlinear initial value problems, as well as for integral equations, one can prove the existence of solutions by means of our equivalence theorem 4.(6), if classic compactness theorems are no longer applicable. For instance, this occurs if the approximating functions can not be embedded in the space containing the solution, or, if the possible embeddings do not have sufficiently good continuity properties.

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1. *Topology of metric discrete limit spaces*

For a metric discrete limit space $X_0, \Pi_\iota X_\iota, lim$ the topological properties of the important subsets $\mathfrak{B}$, $\mathscr{C}$, $\mathfrak{D}$ of the pseudo-metric space $\Pi_\iota X_\iota$ are studied. $\mathfrak{D}$ and X_0 are isometric with the discrete convergence *lim* as the isometry, where *lim* is viewed as a mapping of classes of equivalent sequences. The set $\mathfrak{B}$ is always closed, whereas to prove that $\mathscr{C}$ and $\mathfrak{D}$ are closed we need an additional assumption.

1.1. *Metric discrete limit spaces*

This section presents the basic difinitions of discrete convergence and discrete limit spaces in the sense of [10], [12]. We start from metric spaces X_ι, $\iota \in I$, with a denumerably infinite, linearly ordered index sequence I, a set X_0, and a mapping *lim* with values in X_0 and with domain of definition in the set of all sequences (u_ι) of elements $u_\iota \in X_\iota$, $\iota \in I$. The mapping *lim* is called a d i s c r e t e c o n v e r g e n c e. The tripel $X_0, \Pi_\iota X_\iota, lim$ is said to be a d i s c r e t e l i m i t s p a c e and, in the case of *lim* being surjective, it is called a d i s c r e t e a p p r o x i m a t i o n $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim)$. A sequence (u_ι) is called d i s c r e t e l y c o n v e r g e n t to $u_0 \in X_0$, if $lim(u_\iota) = u_0$. In the following we also write $lim\ u_\iota = u_0$ or $u_\iota \rightarrow u_0 (\iota \in I)$. We denote subsequences of I by I', I'' etc. and we call the cartesian product $\Pi_\iota Z_\iota$ of subsets $Z_\iota \subset X_\iota$, $\iota \in I$, a p r o d u c t s p a c e. The discrete convergence *lim* is said to be a m e t r i c d i s c r e t e c o n v e r g e n c e and a discrete limit space a m e t r i c d i s c r e t e l i m i t s p a c e, if, for every pair (u_ι), $(v_\iota) \in \Pi_\iota X_\iota$ such that (u_ι) or (v_ι) converges discretely, the following statements are equivalent:

$$(M) \qquad \lim |u_\iota, v_\iota|_{X_\iota} = 0 \iff lim\ u_\iota = lim\ v_\iota.$$

If X_0 is a metric space too, the metrics in a discrete limit space are called discretely convergent, $|.,.|_{X_\iota} \to |.,.|_{X_0}$ $(\iota \in I)$, if the following relation is valid:

$$lim\, u_\iota = u_0,\ lim\, v_\iota = v_0 \Rightarrow \lim |u_\iota, v_\iota|_{X_\iota} = |u_0, v_0|_{X_0}.$$

The set D of discretely convergent sequences is a subset of B, $D \subset B$, where B is the set of all discretely bounded sequences. An arbitrary subset Z of $\Pi_\iota X_\iota$ is called discretely bounded, if there exist a discretely convergent sequence $(u_\iota) \in D$ and a number $\alpha \geqslant 0$ such that $\overline{\lim}|v_\iota, u_\iota|_{X_\iota} \leqslant \alpha$ for every $(v_\iota) \in Z$.

If discretely convergent sequences in a discrete limit space have finite distances from each other, i. e.

$$\overline{\lim_{\iota \in I}}\, |u_\iota, v_\iota|_{X_\iota} \leqslant \gamma$$

for every pair discretely convergent of sequences (u_ι), (v_ι) and a number $\gamma \geqslant 0$, then this inequality is also valid for every pair of discretely bounded sequences. In a discrete limit space with discretely convergent metrics the discretely convergent sequences necessarily have finite distances.

Besides B and D, the set $\mathscr{C}$ of discretely compact sequences plays an important role. A subset $Z \subset \Pi_\iota X_\iota$ is called discretely compact, if for every sequence $(u_\iota) \in Z$ and every subsequence $I' \leqslant I$, there exist a subsequence $I'' \leqslant I'$ and an element $u_0 \in X_0$ such that $u_\iota \to u_0$ $(\iota \in I'')$. In the case of discretely convergent metrics we shall prove in 2.(3) the relation $D \subset \mathscr{C} \subset B$.

1.2. *The pseudo-metric space $\Pi_\iota X_\iota$*

The product space $\Pi_\iota X_\iota$ now becomes a topological space by means of the pseudo-metric (1). In it we study the topological properties of the sets B, $\mathscr{C}$ and D. By the continuity of the discrete convergence *lim* we further prove an important property of extension of the discrete convergence. For the properties of metric and pseudometric spaces we refer to Bourbaki [4], Kelley [6].

In the product space $\Pi_\iota X_\iota$, for metric spaces X_ι, $\iota \in I$, there is given a pseudo-metric $|.,.|: \Pi_\iota X_\iota \times \Pi_\iota X_\iota \longmapsto [0, \infty]$ by

$$(1) \qquad |(u_\iota), (v_\iota)| = \overline{\lim}\, |u_\iota, v_\iota|_{X_\iota},\ (u_\iota), (v_\iota) \in \Pi_\iota X_\iota,$$

i.e. the symmetry, the triangle inequality and $|(u_\iota), (u_\iota)| = 0$, $(u_\iota) \in \Pi_\iota X_\iota$, are valid. The set of open balls $B_\varrho((u_\iota)) = \{(v_\iota) \in \Pi_\iota X_\iota$:

$|(u_\iota), (v_\iota)| < \varrho\}, \varrho > 0$, is a basis for the pseudo-metric topology of $X = \Pi_\iota X_\iota$.

Before we continue the study of the pseudo-metric space X we introduce some notations. We also denote the sequences of X by $u = (u_\iota), v = (v_\iota)$, the distance between two non-void subsets U, $V \subset X$ is defined by $|U, V| = \inf \{|(u_\iota), (v_\iota)|:(u_\iota) \in U,(v_\iota) \in V\}$, and we define $B_\varrho(U) = \{v \in X: |v, U| < \varrho\}$ for $\varrho > 0$, $U \subset X$. Then the set of all non-void subsets of X constitutes again a pseudo-metric space with this definition of a distance. To simplify notation in the following we denote all occuring metrics and pseudo-metrics without any difference by $|.,.|$.

An important tool for our studies is the representation of the closure $\bar{Z}$ of a subset Z of a pseudo-metric space $X, \bar{Z} = \{u \in X: |u, Z| = 0\}$. This is proved in [6],4., for a finite pseudo-metric and is also true in our case of a not neccesarily finite one. Thus we can immediately see that the set $\mathfrak{B}$ is closed in X for a metric discrete limit space. Indeed, each $u \in X$ such that $|u, \mathfrak{B}| = 0$ has a finite distance from a discretely convergent sequence. As we shall see in theorem (4), this is generally not true for $\mathscr{C}$ and $\mathfrak{D}$.

If we consider the set of all closures $\overline{\{u\}} = \{v \in X: |v, u| = 0\}$ of sequences $u \in X$ for a metric discrete limit space $X_0, \Pi_\iota X_\iota, lim$, we again obtain a pseudo-metric space with the distance $|\overline{\{u\}}, \overline{\{v\}}|$, in which even the condition of definiteness is valid. Its topology is the quotient topology relative to the closure of sets consisting of one element and the topology in X. From the above representation of closures it follows that we just have the space of all classes of equivalent sequences, where two sequences $u = (u_\iota), v = (v_\iota)$ with $\lim |u_\iota, v_\iota| = 0$ are called *equivalent*. For every $u \in X$ we denote the class of all sequences $v \in X$ such that u and v are equivalent by $\hat{u}$, $\hat{u} = \overline{\{u\}}$, and we denote the space of all classes of sequences by $\hat{X}$. Every class $\hat{u} \in \hat{X}$ is already determined by one of its representatives. Obviously, $|\hat{u}, \hat{v}| = |u, v|$ for every representative u and v of the class $\hat{u}$ and $\hat{v} \in \hat{X}$ resp.. Moreover, for each $Z \subset X$, we denote by $\hat{Z}$ the set of classes determined by elements of Z, $\hat{Z} = \{\hat{v} \in \hat{X} | \exists\, z \in Z: \hat{v} = \hat{z}\}$. Then $\hat{\mathfrak{B}}$ is also closed in $\hat{X}$. If discretely convergent sequences have finite distances from each other, $\hat{\mathfrak{B}}$ is a metric space.

By means of condition (M) the metric discrete convergence lim: $\mathfrak{D} \mapsto X_0$ maps classes of discretely convergentse quences uniquely;

hence, *lim* is also a well-defined mapping from $\hat{D}$ into X_0. We use the notation *lim* for the mapping of classes as well. The inverse mapping lim^{-1} of X_0 onto $\hat{D}$ has the whole space X_0 as domain of definition, if *lim* is surjective, i.e. $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim)$ is a metric discrete approximation.

(2) *Let $X_0, \Pi_\iota X_\iota$, lim be a metric discrete limit space with discretely convergent metrics. Then lim is an isometry of $\hat{D}$ into X_0 and, if lim is surjective, $R = lim^{-1}$ is an isometry of X_0 onto $\hat{D}$, i, e.*

$|\hat{u}, \hat{v}| = |lim\, \hat{u}, lim\, \hat{v}|$, $\hat{u}, \hat{v} \in \hat{D}$, *and* $|R(u_0), R(v_0)| = |u_0, v_0|$, $u_0, v_0 \in X_0$.

Proof: Obviously $lim\colon \hat{D} \longmapsto X_0$ is injective. As a consequence of the discretely convergent metrics it follows that

$$|\hat{u}, \hat{v}| = \overline{\lim}|u_\iota, v_\iota| = \lim|u_\iota, v_\iota| = |lim\, u, lim\, v| = |lim\, \hat{u}, lim\, \hat{v}|$$

for every $\hat{u}, \hat{v} \in \hat{D}$ and arbitrary representatives $u = (u_\iota) \in \hat{u}$, $v = (v_\iota) \in \hat{v}$. Moreover $|R(u_0), R(v_0)| = |u_0, v_0|$, $u_0, v_0 \in X_0$, if *lim* is surjective. —

According to the above theorem, a surjective *lim* is, in particular, uniformly bicontinuous, thus it is a homeomorphism from $\hat{D}$ onto X_0.

The following theorem (4) concerning closure properties of the set of discretely convergent and compact sequences is very important for our studies in section 2.1 about discrete compactness. However, before we study the discrete compactness we shall prove the relation $D \subset \mathscr{C} \subset B$ which is already mentioned in section 1.1 and will be used in (4).

(3) *Let X_0 be a metric space and $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim)$ be a discrete approximation with discretely convergent metrics. Then every discretely compact product space $\Pi_\iota Z_\iota \subset \Pi_\iota X_\iota$ is discretely bounded.*

Proof: Suppose $Z = \Pi_\iota Z_\iota$ is not discretely bounded. By the discretely convergent metrics the discretely convergent sequences have finite distances. Hence, by assumption, there exist a convergent sequence (u_ι), $u_\iota \to u_0 (\iota \in I)$, a set of pairwise distinct indices $\iota_n \in I$ and elements $z_{\iota_n} \in Z_{\iota_n}$, $n \in \mathbf{N}$, such that $|z_{\iota_n}, u_{\iota_n}| > n$, $n \in \mathbf{N}$. By the discrete compactness of $\Pi_\iota Z_\iota$, there exist a subsequence $I'' \leqslant I' = (\iota_n)_{n \in \mathbf{N}}$ and an element $z_0 \in X_0$ such that $z_\iota \to z_0 (\iota \in I'')$. The surjectivity of *lim* allows the prolongation of the convergent subsequence $(z_\iota)_{\iota\in I''}$ to a convergent sequence $z_\iota \to z_0$ $(\iota \in I)$. By virtue of the discretely convergent metrics, this leads to the contradiction $|z_0, u_0| = \lim_{\iota\in I''} |z_\iota, u_\iota| = \infty$. —

(4) *Let X_0 be a complete metric space and $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim)$ be a metric discrete approximation with discretely convergent metrics. Then the sets* D *and* $\mathscr{C}$ *are closed in* X, *and* $\hat{\mathrm{D}}$ *and* $\hat{\mathscr{C}}$ *are closed in* $\hat{\mathrm{B}}$ *as well as in* $\hat{X}$.

Proof: In part (i) and (ii) we prove that D and $\mathscr{C}$, resp., are closed. Thus, we can show in (iii) that $\hat{\mathrm{D}}$ and $\hat{\mathscr{C}}$ are closed.

(i) Let $u \in X$ such that $|u, \mathrm{D}| = 0$. Then there exist discretely convergent sequences $u^{(n)} \in \mathrm{D}$, $lim\, u^{(n)} = u_0^{(n)}$, $n \in \mathbf{N}$, with the property $|u, u^{(n)}| \to 0$ $(n \to \infty)$. Since the metrics are discretely convergent, it follows from (2) that $\lim_{n,m\to\infty} |u_0^{(n)}, u_0^{(m)}| = \lim_{n,m\to\infty} |u^{(n)}, u^{(m)}| = 0$; hence $(u_0^{(n)})$ is a Cauchy sequence in X_0. By virtue of the completeness of X_0, there exists a $u_0 \in X_0$ such that $|u_0^{(n)}, u_0| \to 0$ $(n \to \infty)$. Hence, by (2), $|u, R(u_0)| \leqslant \lim_{n\to\infty} |u, u^{(n)}| + \lim_{n\to\infty} |u^{(n)}, R(u_0)| = \lim_{n\to\infty} |u, u^{(n)}| + \lim_{n\to\infty} |u_0^{(n)}, u_0| = 0$; this proves $u \in \mathrm{D}$.

(ii) Let $u \in X$ such that $|u, \mathscr{C}| = 0$. Then there exist discretely compact sequences $u^{(n)} = (u_\iota^{(n)}) \in \mathscr{C}$, $n \in \mathbf{N}$, with the property $|u, u^{(n)}| \to 0$ $(n \to \infty)$. Thus, for every subsequence $I' \leqslant I$, there exist subsequences $I^{(n)} \leqslant I'$, $I^{(n+1)} \leqslant I^{(n)}$, and elements $u_0^{(n)} \in X_0$ such that $u_\iota^{(n)} \to u_0^{(n)}$ $(\iota \in I^{(n)})$. Since the metrics are discretely convergent, $|u^{(n)}, u^{(m)}| = \overline{\lim}_{\iota\in I} |u_\iota^{(n)}, u_\iota^{(m)}| \geqslant \lim_{\iota\in I^{(k)}} |u_\iota^{(n)}, u_\iota^{(m)}| = |u_0^{(n)}, u_0^{(m)}|$, where $k = \max(m, n)$. Hence, $(u_0^{(n)})$ is a Cauchy sequence in X_0 and, since X_0 is complete, there exists a $u_0 \in X_0$ such that $|u_0^{(n)}, u_0| \to 0$ $(n \to \infty)$. Let $(v_\iota) \in R(u_0)$ and $\varepsilon_n = 2\,|u, u^{(n)}|$, $n \in \mathbf{N}$. Then, for every $n \in \mathbf{N}$, there is an index $\iota_n \in I^{(n)}$ such that $|u_{\iota_n}^{(n)}, v_{\iota_n}| \leqslant \varepsilon_n + |u_0^{(n)}, u_0|$ and $|u_{\iota_n}^{(n)}, u_{\iota_n}| \leqslant 2\,|u^{(n)}, u| = \varepsilon_n$. Since one can select the indices such that $\iota_{n+1} > \iota_n$, $I'' = (\iota_n)_{n\in\mathbf{N}}$ is a subsequence of I'. Therefore, we obtain $|u_{\iota_n}^{(n)}, v_{\iota_n}| \to 0$ and $|u_{\iota_n}, v_{\iota_n}| \to 0$ $(\iota_n \in I'')$; this proves u to be discretely compact.

(iii) As $\hat{B}$ is closed in $\hat{X}$, a set $\hat{Z} \subset \hat{B}$ is closed in $\hat{B}$ if and only if it is closed in $\hat{X}$. For $Z = \mathrm{D}$ or $Z = \mathscr{C}$ and every $u \in Z$, it follows, by condition (M), that each representative $v \in \hat{u}$ belongs to Z, $Z = \{v \in \hat{u} \mid \hat{u} \in \hat{Z}\}$. Thus $|u, Z| = |\hat{u}, \hat{Z}|$ for each representative u of

$\hat{u} \in \hat{X}$ and $Z = \mathrm{D}$ or $Z = \mathscr{C}$. Consequently, $\hat{Z} = \hat{\mathrm{D}}$ and $\hat{Z} = \hat{\mathscr{C}}$, resp., is closed in $\hat{X}$ if and only if Z is closed in X which is proved in (i) and (ii), resp. —

According to Stummel [9]-*I*,1.1.(10), one can characterize the discrete convergence of a sequence $u = (u_\iota) \in X$ in a metric discrete limit space with discretely convergent metrics by the existence of a $u_0 \in X_0$ such that, for every $\varepsilon > 0$, there exist an element $v_0 \in X_0$ and a discretely convergent sequence $v_\iota \to v_0 (\iota \in I)$ with the properties $|u_0, v_0| < \varepsilon$ and $\overline{\lim} |u_\iota, v_\iota| < \varepsilon$. With respect to the topology of X, this is equivalent to the existence of a $u_0 \in X_0$ such that in every neighbourhood U_0 of $u_0 \in X_0$ and U of $u \in X$ there exists a discretely convergent sequence $v \in U$ with limit in U_0. Moreover, for a complete X_0, the following characterization holds.

(5) *Under the assumptions of theorem* (4) *a sequence* $u \in X$ *is discretely convergent if and only if there exists a* $v \in B_\varepsilon(u) \cap \mathrm{D}$ *for all* $\varepsilon > 0$.

Since the statement of the theorem means $u \in \overline{\mathrm{D}}$, the proof is obvious, due to (4).

The last theorem of this section about the existence of a unique extension of the discrete convergence is fundamentally significant for the theory and generalizes a corresponding theorem for normed spaces in Stummel [9]-*I*. Here, the proof may be carried out completely analogously as in normed spaces. Following theorem (2), we shall now prove this theorem by means of an extension property of continuous functions.

(6) *Let* Φ_0 *be dense in the metric space* X_0 *and* $\mathcal{A}(\Phi_0, \Pi_\iota X_\iota, lim)$ *be a metric discrete approximation with discretely convergent metrics. Then there exists a unique extension of* $R = lim^{-1}: \Phi_0 \to \hat{X}$ *to* X_0 *that satisfies the conditions of a metric discrete convergence with discretely convergent metrics.*

Proof: First, we prove that, for every $u_0 \in X_0$ and every sequence $\varphi_0^{(n)} \in \Phi_0$, $n \in \mathbf{N}$, with the property $|u_0, \varphi_0^{(n)}| \to 0$ $(n \to \infty)$, there exists a $\hat{y} \in \hat{X}$ such that $|R(\varphi_0^{(n)}), \hat{y}| \to 0$ $(n \to \infty)$. Then, by continuity of R and denseness of Φ_0 in X_0, this statement will be shown to be sufficient for the existence of a unique extension with the desired properties.

(i) Let $u_0 \in X_0$, $\varphi_0^{(n)} \in \Phi_0$, $n \in \mathbf{N}$, such that $|u_0, \varphi_0^{(n)}| \to 0$ $(n \to \infty)$. Without loss of generality let $|u_0, \varphi_0^{(n)}| \leqslant \frac{1}{2^{n+3}}$, $n \in \mathbf{N}$, thus, by theorem (2), $|R(\varphi_0^{(n+1)}), R(\varphi_0^{(n)})| = |\varphi_0^{(n+1)}, \varphi_0^{(n)}| \leqslant \frac{1}{2^{n+2}}$. For each $n \in \mathbf{N}$ and each representative $(\varphi_\iota^{(n)})$ of the class $R(\varphi_0^{(n)})$, there exists, by definition (1), an index $\nu_n \in I$, $\nu_{n+1} > \nu_n$, such that $|\varphi_\iota^{(n)}, \varphi_\iota^{(n+1)}| \leqslant \leqslant \frac{1}{2^{n+1}}$, $\iota \geqslant \nu_n$. Denote $y_\iota = \varphi_\iota^{(n+1)}$, $\nu_{n+1} > \iota \geqslant \nu_n$, $y_\iota = \varphi_\iota^{(1)}$, $\iota < \nu_1$, then the class $\hat{y}$, which is defined by (y_ι), has the desired properties. Indeed, for every $\varepsilon > 0$ there is an integer $n < 0$ such that $\varepsilon \geqslant \frac{1}{2^n}$, and for each $m \geqslant n$, $\iota \geqslant \nu_m$, there exists, by the definition of (y_ι), a $s \geqslant \geqslant m$ such that $\nu_{s+1} > \iota \geqslant \nu_s \geqslant \nu_m$; hence $|y_\iota, \varphi_\iota^{(m)}| = |\varphi_\iota^{(s)}, \varphi_\iota^{(m)}| \leqslant \frac{1}{2^s} + + \ldots + \frac{1}{2^{m+1}} \leqslant \frac{1}{2^m} \leqslant \varepsilon$. Therefore, $\overline{\lim}\, |y_\iota, \varphi_\iota^{(m)}| \leqslant \varepsilon$ and $|\hat{y}, R(\varphi_0^{(m)})| \leqslant \varepsilon$ for each $m \geqslant n$, which proves our first assertion.

(ii) For every $u_0 \in X_0$, there exists a sequence $\varphi_0^{(n)} \in \Phi_0$, $n \in \mathbf{N}$, such that $|\varphi_0^{(n)}, u_0| \to 0$ $(n \to \infty)$. We define the extension of $R\colon \Phi_0 \mapsto \hat{X}$ by $R(u_0) = \lim_{n\to\infty} R(\varphi_0^{(n)})$. Obviously, $R\colon X_0 \mapsto \hat{X}$ is well defined and continuous, by virtue of the continuity of $R\colon \Phi_0 \mapsto \hat{X}$. Moreover, the extension is unique, since any other extension $\tilde{R}$, any $u_0 \in X_0$, and any $\varphi_0^{(n)} \in \Phi_0$, $n \in \mathbf{N}$, such that $|\varphi_0^{(n)}, u_0| \to 0$ $(n \to \infty)$ satisfy the relation $\tilde{R}(u_0) = \lim_{n\to\infty} \tilde{R}(\varphi_0^{(n)}) = \lim_{n\to\infty} R(\varphi_0^{(n)}) = R(u_0)$. By (2), for arbitrary $u_0, v_0 \in \in X_0$, $\varphi_0^{(n)}, \psi_0^{(n)} \in \Phi_0$, $n \in \mathbf{N}$, $|u_0, \varphi_0^{(n)}| \to 0$, $|v_0, \psi_0^{(n)}| \to 0$ $(n \to \infty)$, it follows that $|R(u_0), R(v_0)| = \lim_{n\to\infty} |R(\varphi_0^{(n)}), R(\psi_0^{(n)})| = \lim_{n\to\infty} |\varphi_0^{(n)}, \psi_0^{(n)}| = |u_0, v_0|$, and for $u \in R(u_0)$, $\varphi^{(n)} \in R(\varphi_0^{(n)})$, $v \in X$ such that $|u, v| = 0$, we have $\lim_{n\to\infty} |v, \varphi^{(n)}| = |v, u| + \lim_{n\to\infty} |u, \varphi^{(n)}| = 0$, i.e. $v \in R(u_0)$. Hence, $R\colon X_0 \mapsto \hat{X}$ is injective and its inverse satisfies the conditions of a metric discrete convergence with discretely convergent metrics. —

2. *Compactness*

In this chapter we establish a series of necessary and sufficient conditions for the discrete compactness of sets of sequences, where the concept of asymptotic precompactness is of essential significance. In the special case of an approximation by subspaces our general conditions permit the analysis and characterization of collectively compact sequences of sets. The notion «collectively compact» has been introduced in Anselone [1], [2] for linear operators in Banach spaces and will be generalized to sequences of subsets of metric spaces.

2.1. *Asymptotic precompactness and discrete compactness*

For the whole chapter we shall assume that $X_0, \Pi_\iota X_\iota, lim$ is a metric discrete limit space with discretely convergent metrics. In order to define asymptotic precompactness, we need the notion of a finite product space. For non-void subsets $F_\iota \subset X_\iota$, $\iota \in I$, we call $F = \Pi_\iota F_\iota$ finite, if there exist a natural number $N \in \mathbf{N}$ and sequences $v^{(n)} = (v_\iota^{(n)}) \in X$ such that $F_\iota \subset \bigcup_{n=1}^{N} \{v_\iota^{(n)}\}$ for each $\iota \in I$. For two subsets S, Z of X, we now call the set Z asymptotically S-precompact, if, for every $\varepsilon > 0$, there is a finite product space F such that the finite number of sequences $v^{(n)}$ in the finiteness condition of F belong to S, $v^{(n)} \in S$, and $|u, F| < \varepsilon$ for every $u \in Z$. Obviously, F is asymptotically S-precompact if and only if, for every $\varepsilon > 0$, there are finitely many sequences $v^{(n)} = (v_\iota^{(n)}) \in S$, $n = 1, \ldots, N$ such that, for every $u = (u_\iota) \in Z$, the inequality $\overline{\lim_{\iota \in I}} \min_{1 \leq n \leq N} |u_\iota, v_\iota^{(n)}| < \varepsilon$ is valid. If we define, according to Stummel [10], 4., the upper and lower discrete limit of a subset $Z \subset X$ by

$$\text{Lim sup } Z = \{u_0 \in X_0 | \exists\, u \in Z\ \exists\, I' \leqslant I\colon \lim_{\iota \in I'} u = u_0\}$$

and

$$\text{Lim inf } Z = \{u_0 \in X_0 | \exists\, u \in Z\colon \lim_{\iota \in I} u = u_0\}, \text{ resp.,}$$

then we obtain the following necessary conditions for the asymptotic precompactness.

(1) *Let lim be surjective and $Z \subset X$ be asymptotically $\mathcal{D}$ (resp. $\mathscr{C}$)-precompact. Then* Lim sup Z *(resp.* Lim inf Z*) is precompact and (in both cases) we have $Z \subset \overline{\mathscr{C}}$.*

Proof: (i) Let Z be asymptotically $\mathcal{D}$-precompact. For every $\varepsilon > 0$, there exist N discretely convergent sequences $v^{(n)} = (v_\iota^{(n)})$, $lim\, v^{(n)} = v_0^{(n)}$, $n = 1, \ldots, N$, such that $\overline{\lim_{\iota \in I}} \min_{1 \leq n \leq N} |u_\iota, v_\iota^{(n)}| < \varepsilon$ for every $u = (u_\iota) \in Z$. We prove now that $\{v_0^{(n)}\}_{n=1}^{N}$ constitutes an ε-net for

Lim sup Z. Let $u_0 \in$ Lim sup Z, then there are a $u = (u_\iota) \in Z$ and a subsequence $I' \leqslant I$ such that $\lim_{\iota \in I'} u_\iota = u_0$. By assumption, we have $\overline{\lim}_{\iota \in I} \min_{1 \leq n \leq N} |u_\iota, v_\iota^{(n)}| < \varepsilon$ and therefore, there exist a subsequence $I'' \leqslant I'$ and a $n' \in \{1, ..., N\}$ such that $|u_\iota, v_\iota^{(n')}| < \varepsilon$, $\iota \in I''$. In view of the discretely convergent metrics and the surjectivity of *lim*, we obtain $|u_\iota, v_\iota^{(n')}| \to |u_0, v_0^{(n')}|$ $(\iota \in I'')$, hence $|u_0, v_0^{(n')}| \leqslant \varepsilon$.

(ii) Let Z be asymptotically $\mathscr{C}$-precompact. For every $\varepsilon < 0$, there exist N discretely compact sequences $v^{(n)} = (v_\iota^{(n)})$ out of this condition which we assume to be convergent to $v_0^{(n)} \in X_0$ for a common subsequence $I' \leqslant I$, $\lim_{\iota \in I'} v_\iota^{(n)} = v_0^{(n)}$, $n = 1, ..., N$. The elements $v_0^{(n)}$, $n = 1, ..., N$, constitute an ε-net for Lim inf Z. Indeed, for every $u_0 \in$ Lim inf Z, there exists a $u = (u_\iota) \in Z$, $\lim_{\iota \in I} u_\iota = u_0$, such that $\overline{\lim}_{\iota} \min_{n} |u_\iota, v_\iota^{(n)}| < \varepsilon$ and, as in part (i), there are a subsequence $I'' \leqslant I'$ and a $n' \in \{1, ..., N\}$ such that $|u_\iota, v_\iota^{(n')}| < \varepsilon$, $\iota \in I''$; hence $|u_0, v_0^{(n')}| \leqslant \varepsilon$.

(iii) Since asymptotic D-precompactness implies asymptotic $\mathscr{C}$-precompactness, it is sufficient to deduce the relation $Z \subset \overline{\mathscr{C}}$ from the latter. Given $\varepsilon > 0$ then, for $\varepsilon/2$, there are N discretely compact sequences $v^{(n)} = (v_\iota^{(n)}) \in \mathscr{C}$ such that, for every $u = (u_\iota) \in Z$, there is an index $\nu \in I$ and, for every $\iota \geqslant \nu$, there exists a number $n_\iota \in \{1, ..., N\}$ such that $|u_\iota, v_\iota^{(n_\iota)}| < \varepsilon/2$. If we denote $v_\iota = v_\iota^{(n_\iota)}$, $\iota \geqslant \nu$, then, for every subsequence $I' \leqslant I$, there exist a number $n' \in \{1, ..., N\}$ and a subsequence $I'' \leqslant I'$ such that $v_\iota = v_\iota^{(n')}$, $\iota \in I''$. By the discrete compactness of $v^{(n')}$, there exist a subsequence $I''' \leqslant I''$ and a $v_0 \in X_0$ such that $\lim_{\iota \in I'''} v_\iota = \lim_{\iota \in I'''} v_\iota^{(n')} = v_0$. Thus, $v = (v_\iota)$ is discretely compact and $\overline{\lim} |u_\iota, v_\iota| = |u, v| \leqslant \varepsilon/2 < \varepsilon$, which proves the relation $Z \subset \overline{\mathscr{C}}$. —

For the proof of the precompactness of Lim sup Z in the last theorem it is sufficient to assume $lim(\Pi_\iota X_\iota$ Lim sup $Z)$ instead of $lim = lim(\Pi_\iota X_\iota, X_0)$ being surjective. Moreover, we need no assumption of surjectivity for the relation $Z \subset \overline{\mathscr{C}}$. By means of an idea in Wolf [15], 2.(4), we shall now prove the following converse of theorem (1). Here, for abbreviation, we call a set $Z \subset X$ a s y m p t o t i c a l l y p r e c o m p a c t, if Z is asymptotically D-precompact.

(2) *For subsets* $Z_\iota \subset X_\iota$, $\iota \in I$, *let* $Z = \Pi_\iota Z_\iota$ *be discretely compact, let* $Z_0 = \text{Lim sup } Z$ *be separable, and* $lim\,(\Pi_\iota X_\iota, Z_0)$ *be surjective. Then* Z *is asymptotically precompact.*

Proof: Let $\{u_0^{(k)}\}_{k\in\mathbf{N}}$ be a denumerably dense subset in $Z_0 = \text{Lim sup } Z$, and let $u^{(k)} = (u_\iota^{(k)}) \in X$, $k \in \mathbf{N}$, be discretely convergent sequences $u_\iota^{(k)} \to u_0^{(k)}$ $(\iota \in I)$, which exist by the surjectivity of $lim\,(\Pi_\iota X_\iota, Z_0)$. Suppose Z is not asymptotically precompact. Then there exists *a* $\varepsilon_0 > 0$ and, for every $n \in \mathbf{N}$ and for the sequences $u^{(k)}$, $k = 1, \ldots, n$, there exist a sequence $w^{(n)} = (w_\iota^{(n)}) \in Z$ and a subsequence $I^{(n)} \leqslant I$ such that $|w_\iota^{(n)}, u_\iota^{(k)}| \geqslant \varepsilon_0$, $\iota \in I^{(n)}$, $k = 1, \ldots, n$. In particular, for every $n \in \mathbf{N}$, there exists an index $\iota_n \in I^{(n)}$, $\iota_{n+1} > \iota_n$, such that $|w_{\iota_n}^{(n)}, u_{\iota_n}^{(k)}| \geqslant \varepsilon_0$, $k = 1, \ldots, n$. By the discrete compactness of Z, there is a subsequence $I'' \leqslant I' = (\iota_n)_{n\in\mathbf{N}}$ and an element $z_0 \in Z_0$ such that $z_\iota \to z_0(\iota \in I'')$, where $(z_\iota)_{\iota\in I'} = (w_{\iota_n}^{(n)})_{n\in\mathbf{N}}$. To every $u_0^{(k)}$ of the dense set in Z_0 by the convergence $u_\iota^{(k)} \to u_0^{(k)}$, $z_\iota \to z_0$ $(\iota \in I'')$ there now corresponds an index $\iota_0 \in I$ such that $||u_\iota^{(k)}, z_\iota| - |u_0^{(k)}, z_0|| \leqslant \leqslant \varepsilon_0/2$, $\iota \geqslant \iota_0$, $\iota \in I''$. For every $\iota = \iota_n \in I'$ such that $\iota_n \geqslant \max\,(\iota_k, \iota_0)$, by assumption we have $|u_\iota^{(k)}, z_\iota| \geqslant \varepsilon_0$, thus $|u_0^{(k)}, z_0| \geqslant |u_\iota^{(k)}, z_\iota| - - \varepsilon_0/2 \geqslant \varepsilon_0/2$ contrary to the deneseness of $\{u_0^{(k)}\}$ in Z_0. —

As an abvious generalization of a well-known definition the relation $Z \subset \overline{\mathscr{C}}$ in theorem (1) can be characterized by the denseness of $\mathscr{C}$ with respect to Z. For two subsets S, Z of a topological space T the set S is said to be d e n s e w i t h r e s p e c t to Z, if $Z \subset \bar{S}$. Evidently, in a pseudo-metric space T a set S is dense with respect to Z if and only if, for every $\varepsilon > 0$ and any $u \in Z$, there exists a $v \in S$ such that $|u, v| < \varepsilon$. By means of the notion «dense», theorem 1.(4) immediately yields a necessary and sufficient condition for the discrete compactness.

(3) *Let* X_0 *be a complete metric space and* lim *be surjective. Then a subset* $Z \subset X$ *is discretely compact if and only if the set* $\mathscr{C}$ *of discretely compact sequences is dense with respect to* Z.

The property of the denseness of $\mathscr{C}$ with respect to a product space $Z = \Pi_\iota Z_\iota$ also characterizes the asymptotic precompactness of Z, as it is seen in (1) and (2).

(4) *Let* X_0 *be a complete metric space, let* Z_ι *be subsets of* X_ι, $\iota \in I$, *and let lim be surjective. Then the asymptotic precompactness of* $Z =$

$= \Pi_\iota Z_\iota$ *is necessary and sufficient for the compactness of* Lim sup Z *together with the denseness of* $\mathscr{C}$ *with respect to* Z.

Proof: The sufficiency immediately follows from theorem (1), if one notes that the precompact set Lim sup Z is, by the completeness of X_0, relatively compact, and, moreover, the upper and lower discrete limits are always closed in a metric discrete approximation with discretely convergent metrics. Conversely, under the above assumptions the set $\mathscr{C}$ of all discretely compact sequences is closed in X, hence $Z \subset \overline{\mathscr{C}} = \mathscr{C}$. Therefore, Z is discretely compact and, in particular, the compact set Lim sup Z is separable. Thus theorem (2) implies the asymptotic precompactness.—

It is now readily seen that the results of this section yield the characterization of the discrete compactness of a product space by its asymptotic precompactness.

(5) *Let* X_0 *be a complete metric space, let* $Z_\iota \supset X_\iota$, $\iota \in I$, *let lim be surjective, and let* $Z_0 =$ Lim sup $\Pi_\iota Z_\iota$ *be separable. Then* $Z = \Pi_\iota Z_\iota$ *is discretely compact if and only if it is asymptotically precompact.*

Proof: By (2), the discrete compactness is sufficient for the asymptotic precompactness of Z. Conversely, the asymptotic precompactness implies $\mathscr{C}$ to be dense with respect to Z, which, by (3) and the completeness of X_0, is equivalent to the discrete compactness of Z.—

As a corollary of the above theorems, we obtain the important result that the set of all limits of subsequences of a discretely compact product space is necessarily precompact, if the approximated space X_0 is separable. Even the following more general statement is true, which yields the just mentioned result.

(6) *Let* $Z_\iota \subset X_\iota$, $\iota \in I$, *let* $Z = \Pi_\iota Z_\iota$ *be discretely compact, and let lim* $(\Pi_\iota X_\iota,$ Lim sup $Z)$ *be surjective. Then, if* Lim sup Z *is separable, it is precompact.*

Proof: By (2), the asymptotic precompactness follows from the assumptions, hence Lim sup Z is precompact, due to (1).—

2.2. *Collectively compact sequences of sets*

In the special case of a metric discrete approximation $A(X_0,\Pi_\iota X_\iota, lim)$ by subsets X_0, X_ι, $\iota \in I$, of a metric space M, the above conditions can be reduced to classic compactness conditions in M. According to Stummel [9]$-I$,5.(4), the convergence $|u_0,X_\iota| \to 0$ $(\iota \in I)$ for each $u_0 \in X_0$ is necessary and sufficient for the existence of the metric discrete approximation by subsets, where the discrete convergence is the convergence in M. In this situation, we necessarily have discretely convergent metrics. We now introduce the collective compactness of sequences of subsets $Z_\iota \subset X_\iota$, $\iota \in I$, generalizing Anselone [1], [2] who has defined the notion of collective compactness for sequences of linear operators in Banach spaces. We call a sequence $(Z_\iota)_{\iota \in I}$ collectively compact, if the set $\bigcup_{\iota \in I} Z_\iota$ is precompact. For a complete space M the precompactness is equivalent to the relative compactness as it is well known. By means of the asymptotic precompactness, we obtain the following characterization of a collectively compact sequence of subsets.

(7) *A sequence of subsets $Z_\iota \subset X_\iota$, $\iota \in I$, is collectively compact if and only if each Z_ι, $\iota \in I$, is precompact and $\Pi_\iota Z_\iota$ is asymptotically precompact.*

Proof: Suppose $\bigcup Z_\iota$ is not precompact, then there exist a $\varepsilon_0 > 0$ and elements $z_0^{(k)} \in \bigcup Z_\iota$, such that $|z_0^{(j)}, z_0^{(k)}| \geqslant \varepsilon_0$, $j < k$, $k \in \mathbf{N}$. Either there exists a subsequence $I' \leqslant I$ and, for every $\iota \in I'$, there is a $m \in \mathbf{N}$ such that $z_0^{(m)} \in Z_\iota$, or all $z_0^{(k)}$, $k \in \mathbf{N}$, belong to only finitely many Z_ι. But the latter case contradicts the precompactness of each single Z_ι. In the first case, it follows from the asymptotic precompactness that, for $\varepsilon_0/4$, we have discretely convergent sequences $v_\iota^{(i)} \to v_0^{(i)}$ $(\iota \in I)$, $(v_\iota^{(i)}) \in \Pi_\iota Z_\iota, v_0^{(i)} \in X_0$, $i = 1, \ldots, N_0$, and, for $z_\iota = z_0^{(m)} \in Z_\iota$, $\iota \in I'$, there is an index $\nu \in I'$ such that $\min_{1 \leq i \leq N_0} |z_\iota, v_\iota^{(i)}| < \varepsilon_0/4$, $\iota \geqslant \nu$, $\iota \in I'$, Hence, there is a subsequence $I'' \leqslant I'$ and a $n_0 \in \{1, \ldots, N_0\}$ such that $|z_\iota, v_\iota^{(n_0)}| < \varepsilon_0/4$, $\iota \in I''$, and therefore the relation $|z_\iota, v_0^{(n_0)}| < \varepsilon_0/4$, $\iota \geqslant \mu$, $\iota \in I''$. holds for a $\mu \in I''$. Then for arbitrary $\iota, \varkappa \in I_1'' = \{\iota \in I'' \mid \iota \geqslant \mu\}$ we have $|z_\iota, z_\varkappa| \leqslant |z_\iota, v_0^{(n_0)}| + |v_0^{(n_0)}, z_\varkappa| < \varepsilon_0$, in contradiction to $|z_\iota, z_\varkappa| \geqslant \varepsilon_0$, $\iota \neq \varkappa \in I'$, which proves the precompact ness of $\bigcup Z_\iota$. The converse is trivial.—

In a complete metric space we now characterize a collectively compact sequence by its discrete compactness together with the relative compactness of each set.

(8) *Let M be a complete metric space and X_0 be a closed subset of M. Then a sequence $(Z_\iota)_{\iota\in I}$ is collectively compact if and only if each Z_ι, $\iota \in I$, is relatively compact and $\Pi_\iota Z_\iota$ is discretely compact.*

Proof: (i) Let (Z_ι) be collectively compact. By (7) and (1), $\mathscr{C}$ is dense with respect to $Z = \Pi_\iota Z_\iota$. Since M is complete and X_0 is closed, X_0 is a complete metric space. Thus, by (3), Z is discretely compact. The relative compactness of each Z_ι, $\iota \in I$, follows trivially from the relative compactness of $\bigcup Z_\iota$.

(ii) Proving $Z = \Pi_\iota Z_\iota$ to be collectively compact, in view of (7) and (5) it is sufficient to prove that Lim sup Z is separable. According to a well-known theorem of the set theory that the union of a denumerable number of denumerable sets is at most denumerable, this follows from the precompactness of each Z_ι, $\iota \in I$.—

Theorem (7) and (8) can be further simplified, if the assumption of Vainikko [14], 1., example 2, is valid. If M is complete and the relation Lim sup $Z = \bigcup_{\iota\in I} Z_\iota$ with $Z = \Pi_\iota Z_\iota$ is satisfied, then the collective compactness of $(Z_\iota)_{\iota\in I}$ is, by definition, equivalent to the compactness of Lim sup Z as well as to the asymptotic precompactness of Z. Moreover, it is equivalent to the discrete compactness of Z, if M is separable and X_0 is closed.

3. *Mappings*

In this chapter we study the topological properties of sequences of mappings in the underlying pseudo-metric spaces. The basic concepts of the theory of discretely convergent sequences of mappings yield characterizations of topological properties. For example, stability is equivalent to continuity and inverse stability is equivalent to openness. The main result of section 3.2 is the equivalence of discrete convergence and the fundamental relation (8). Thus we establish important statements concerning existence as well as continuity of limits of discretely convergent sequences of mappings. Finally, theorem (11) proves an interesting characterization of the discrete convergence by means of the Lim-convergence of the associated graphs. Hereby, we generalize and extend results of classic functional analysis as well as a result of Stummel [9]$-III$,1. We assume that the definitions of stability and inverse stability and their various characterizations in [10], [12] are known.

Let $X_0, \Pi_\iota X_\iota, lim^X$ and $Y_0, \Pi_\iota Y_\iota, lim^Y$ be two metric discrete limit spaces with the same denumerably infinite, linearly ordered index sequence I, and let (A_ι) be a sequence of mappings $A_\iota: X_\iota \longmapsto Y_\iota, \iota \in I$. The sequence (A_ι) is viewed as a mapping of $X = \Pi_\iota X_\iota$ into $Y = \Pi_\iota Y_\iota$ and we write, for abbreviation, $A = (A_\iota)$.

3.1. *Continuous and open mappings*

We shall now prove the following characterization concerning the continuity of $A = (A_\iota)$.

(1) *The following statements are equivalent:*

(i) $A = (A_\iota)$ *is continuous at the point* $u = (u_\iota) \in X$.

(ii) *For every* $v \in \overline{\{u\}}$, *the relation* $A\,v \in \overline{\{Au\}}$ *holds.*

(iii) A *is stable at* u.

Proof: According to the definition of stability in [12],2., and the representation $\overline{\{u\}} = \{v \in X : |u, v| = 0\}$, the statements (ii) and (iii) are obviously equivalent.

(i) $\Rightarrow$ (ii). For every $\varepsilon > 0$ and every $v \in \overline{\{u\}}$, by $|u, v| = 0$ and the continuity of A at the point u, the relation $|Au, Av| < \varepsilon$ holds. Hence $|Au, Av| = 0$, i.e. $Av \in \overline{\{Au\}}$.

(iii) $\Rightarrow$ (i). According to the stability at u, which is equivalent to the asymptotic equicontinuity at u (cf. [12],2.(1)), for every $\varepsilon > 0$ resp. $\varepsilon/2$, there exist a $\delta > 0$ and a $\nu \in I$ such that the relation $|u_\iota, v_\iota| < \delta$ implies $|A_\iota u_\iota, A_\iota v_\iota| < \varepsilon/2$ for every $\iota \geqslant \nu$ and every $v_\iota \in X_\iota$. Let $v \in X$ with $|u, v| < \delta$, then there exists a $\mu \geqslant \nu$ such that $|u_\iota, v_\iota| < \delta$, $\iota \geqslant \mu$, and $|A_\iota u_\iota, A_\iota v_\iota| < \varepsilon/2$, $\iota \geqslant \mu$. Hence $|Au, Av| \leqslant \varepsilon/2 < \varepsilon$.$-$

The concept of inverse stability is suitable for the characterization of an open mapping. A sequence (A_ι) is said to be inversely stable at $u = (u_\iota) \in X$ in the generalized sense, if, for every sequence $v = (v_\iota) \in X$, the condition $\lim |A_\iota u_\iota, A_\iota v_\iota| = 0$ implies $\lim |u_\iota, A_\iota^{-1} A_\iota v_\iota| = 0$. Further, a mapping A between topological spaces T_1 and T_2 is called open at the point $u \in T_1$, if, for every neighbourhood U of u, the set AU is a neighbourhood of Au, and A is said to be open, if it is open at every point $u \in T_1$. The following theorem shows that an open mapping $A: X \longmapsto Y$, may be characterized not only by the generalized inverse stability but also by the continuity of the associated inverse A^{-1}, which is a mapping of $AX \subset Y$ into the pseudo-metric space of all non-void subsets of X. Moreover, we characterize the above generalized inverse stability condition of Grigorieff [5],3., which originates in a definition of Wolf [15],3., for linear operators, and is equivalent to the inverse stability of [10], [12] for injective mappings A_ι, $\iota \in I$.

(2) *The following condition* (i) *implies* (ii), *and condition* (ii) *implies* (iii). *Conversely, if A is surjective,* (ii) *implies* (i), *and, if* $\{u\} = A^{-1} Au$, (iii) *implies* (ii).

(i) *A is open at the point $u \in X$.*

(ii) *A is inversely stable at u in the generalized sense.*

(iii) *A^{-1} is continuous at the point Au.*

Proof: (i) $\Rightarrow$ (ii). For every $\varepsilon > 0$ and $v \in X$ such that $|Av, Au| = 0$, Av belongs to the neigbourhood $V = AB_\varepsilon(u)$ of $Au, Av \in V$. Hence, there exists a $v' \in B_\varepsilon(u)$ such that $v' \in A^{-1} Av$. Therefore, for every $\varepsilon > 0$, we have $|u, A^{-1} Av| \leqslant |u, v'| < \varepsilon$; thus $|u, A^{-1} Av| = 0$, which proves the generalized inverse stability.

(ii) $\Rightarrow$ (iii). From the generalized inverse stability at u it follows that, for every $\varepsilon > 0$ resp. $\varepsilon/2$, there exist a $\nu \in I$ and a $\delta > 0$ such that the relation $|A_\iota u_\iota, A_\iota v_\iota| < \delta$ implies $|u_\iota, A_\iota^{-1} A_\iota v_\iota| < \varepsilon/2$ for every $\iota \geqslant \nu$, $v_\iota \in X_\iota$ (cf. [5],3.(16)). Let $w \in B_\delta(Au) \cap AX$, then there exist a $v \in A^{-1}w$ and a $\mu \geqslant \nu$ with $|A_\iota v_\iota, A_\iota u_\iota| < \delta$, $\iota \geqslant \mu$, and therefore we have $|u_\iota, A_\iota^{-1} w_\iota| < \varepsilon/2$, $\iota \geqslant \mu$, and $|u, A^{-1} w| \leqslant \varepsilon/2 < \varepsilon$. Hence, in particular, $|A^{-1} Au, A^{-1} w| < \varepsilon$, which proves the continuity of A^{-1} at Au.

(iii) $\Rightarrow$ (ii). Let $\{u\} = A^{-1}Au$. For each $v \in X$ such that $|Av, Au| = 0$, we have $w = Av \in B_\delta(Au)$ for every $\delta > 0$. Thus, by (iii), the relation $|A^{-1}Av, A^{-1}Au| < \varepsilon$ is true for every $\varepsilon > 0$. Hence $|A^{-1}Av, A^{-1}Au| = 0$ and, by the assumption $A^{-1}Au = \{u\}$, $|u, A^{-1}Av| = 0$.

(ii) $\Rightarrow$ (i). Let A be surjective. For an arbitrary neighbourhood U of u there exists a $\varrho > 0$ such that $B_\varrho(u) \subset U$. As we have proved in part '(ii) $\Rightarrow$ (iii)', for $\varrho/2$, there exists a $\delta > 0$ such that for all $w \in B_\delta(Au) \cap AX$ the relation $|u, A^{-1} w| \leqslant \varrho/2 < \varrho$ is valid. Thus, by the surjectivity of A, for every $w \in B_\delta(Au)$, there exists a $v \in A^{-1}w$ such that the relation $|u, v| < \varrho$ holds. Hence $w \in AB_\varrho(u)$ and $B_\delta(Au) \subset AB_\varrho(u) \subset AU$, which ends the proof. —

Usually one calls a bijective mapping A **bicontinuous at the point** u, if A is continuous and open at the point u. According to Stummel [11], a sequence $A = (A_\iota)$ is said to be **bistable at** u, if A is stable und inversely stable at u. By (1) and (2), the bistability of a bijective mapping A at u is equivalent to its bicontinuity at u.

If we consider the images of all representatives v of a class $\hat{u} \in \hat{X}$, then, in general, the set of classes defined by Av, $v \in \hat{u}$, consists of different classes. However, if there is a subset $\hat{Z} \subset \hat{X}$ and, for every class $\hat{u} \in \hat{Z}$, there is a class $\hat{w} \in \hat{Y}$ such that all representatives $v \in \hat{u}$ are mapped into $\hat{w}$, then A defines a mapping of $\hat{Z}$ into $\hat{Y}$, which we denote by $\hat{A}$. Condition (1ii) of theorem (1) now affirms that this mapping $\hat{A}: \hat{Z} \mapsto \hat{Y}$ exists for $A = (A_\iota)$, if A is stable at each sequence u of a subset $Z \subset X$. Moreover, $\hat{A}$ is continuous, which yields a new equivalent stability condition.

(3) *A sequence $A = (A_\iota)$ is stable at each $u \in Z \subset X$ if and only if $\hat{A}$ is a continuous mapping of $\hat{Z}$ into $\hat{Y}$. Moreover, the stability of A at $u = (u_\iota) \in X$ is equivalent to the following condition:*

(4) *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $z = (z_\iota) \in \overline{\{u\}}$, there exists an index $\nu \in I$ such that the relation $|z_\iota, v_\iota| < \delta$ implies $|A_\iota z_\iota, A_\iota v_\iota| < \varepsilon$ for every $\iota \geqslant \nu$, $v_\iota \in X_\iota$.*

Proof: (i) From the functional property of $\hat{A}$, for every $u \in Z$ and $v \in \overline{\{u\}} = \hat{u}$, it follows that $Av \in \hat{A}\hat{u} = (\widehat{Au})$. Thus, by (1), A is stable at each $u \in Z$. Conversely, by means of the stability of A at each $u \in Z$, there exists the well-defined mapping $\hat{A}: \hat{Z} \mapsto \hat{Y}$, as we already mentioned above. Moreover, A is continuous, as for every $\hat{z} \in \hat{Z}$ and an arbitrary representative $z \in \hat{z}$ and, for every $\varepsilon > 0$, there is, by (1iii), a $\delta > 0$ such that $Av \in B_\varepsilon(Az)$ for all $v \in B_\delta(z)$. Thus, let $\hat{v} \in \hat{Z}$ such that $|\hat{v}, \hat{z}| < \delta$, then for an arbitrary representative $v \in \hat{v}$ we have $|v, z| = |\hat{v}, \hat{z}| < \delta$; hence $|\hat{A}\hat{z}, \hat{A}\hat{v}| = \inf\{|Ax, Ay| : x \in \hat{z}, y \in \hat{v}\} \leqslant |Az, Av| < \varepsilon$.

(ii) The above proof also establishes that, setting $Z = \{u\}$ and given $\varepsilon > 0$, there exists a $\delta > 0$ such that the relation $|v, \hat{u}| < \delta$ implies $|Av, \hat{A}\hat{u}| < \varepsilon$ for every $v \in X$. Further, let $\nu_1 \in I$ be the index in the condition of the asymptotic equicontinuity at u, then, for any $z \in \hat{u}$, there exists an index $\nu_2 \geqslant \nu_1$ such that $|u_\iota, z_\iota| < \delta/2$, $\iota \geqslant \nu_2$. For every $v = (v_\iota) \in X$ such that $|v_\iota, z_\iota| < \delta/2$, $\iota \geqslant \nu_2$, we have $|v, z| = |v, \hat{u}| < \delta$ and therefore $|Av, Az| = |Av, \widehat{Az}| = |Av, \hat{A}\hat{u}| < \varepsilon$. Conversely, (4) immediately implies the asymptotic equicontinuity at u and, thus, the stability at u.—

The stability condition (4) shows that in the condition of asymptotic equicontinuity of [12],2.(1), which is equivalent to the stability, we can choose the number δ uniformly for all $z \in \overline{\{u\}}$. For the inverse stability and the generalized inverse stability analogous characterizations are true, where the number $\delta > 0$ can be chosen uniformly for all z such that $Az \in \overline{\{Au\}}$. We omit the proof. Concerning the bistability, we can state the following theorem.

(5) *A bijective mapping* $A{:}X \longmapsto Y$ *is bistable at all* $u \in Z \subset X$ *if and only if the mapping* $\hat{A}{:}\hat{Z} \longmapsto \hat{Y}$ *and the mapping* $(\widehat{A^{-1}}){:}\widehat{AZ} \longmapsto \hat{Z}$ *associated with the sequence of inverses* $A^{-1} = (A_\iota^{-1})$ *exist, are continuous and satisfy the relation*

$$\hat{A}^{-1} = (\widehat{A^{-1}}).$$

Proof: Since a bijective mapping A is inversely stable in Z if and only if A^{-1} is stable in AZ, by means of (3), it is enough to prove the relation $\hat{A}^{-1} = (\widehat{A^{-1}})$. The inverse $\hat{A}^{-1}{:}\hat{A}\hat{Z} \to \hat{Z}$ exists, as, for arbitrary $\hat{u}, \hat{v} \in \hat{Z}$ such that $\hat{A}\hat{u} = \hat{A}\hat{v}$, by the inverse stability the relation $|\hat{u}, \hat{v}| = 0$ holds, i.e. $\hat{u} = \hat{v}$. Moreover $\hat{A}\hat{Z} = \widehat{AZ}$ and, for every $\hat{w} = \hat{A}\hat{z} \in \hat{A}\hat{Z}$ and every representative $w \in \hat{w}$, the equality $(\widehat{A^{-1}})\,\hat{A}\hat{z} = (\widehat{A^{-1}})\,\hat{w} = (\widehat{A^{-1}w})$ is true. Choose $w = Az$, $z \in \hat{z} \in \hat{Z}$, then $(\widehat{A^{-1}})\,\hat{A}\hat{z} = \hat{z}$; hence the assertion is proved.—

At the end of our studies concerning continuity and stability we characterize the uniform continuity of A in a product space by a uniform stability condition, which is even necessary for the stability at each discretely convergent sequence of a discretely compact product space. Besides this, we state the important result that stability at every sequence of a product space is equivalent to the uniform stability and uniform continuity.

(6) *Let* Z_ι *be non-void subsets of* X_ι, $\iota \in I$, *and* $Z = \Pi_\iota Z_\iota$. *Then the following statements* (i), (ii), (iii) *are equivalent to each other:*

(i) $A = (A_\iota)$ *is uniformly continuous in* Z.

(ii) *For every* $\varepsilon < 0$, *there exist a* $\nu \in I$ *and a* $\delta < 0$ *such that, for every* $\iota \geqslant \nu$ *and all* $u_\iota, v_\iota \in Z_\iota$, *the relation* $|u_\iota, v_\iota| < \delta$ *implies* $|A_\iota u_\iota, A_\iota v_\iota| < \varepsilon$.

(iii) A *is stable at all* $u \in Z$.

If lim^X *is surjective and* $Z = \Pi_\iota Z_\iota = \Pi_\iota X_\iota = X$ *is discretely compact, the following statement* (iv) *is equivalent to* (i), (ii), *and* (iii):

(iv) A *is stable at all discretelly convergent* $u \in X$.

Proof: Refering to theorem (1), condition (iii) follows from (i).

(iii) $\Rightarrow$ (ii). Suppose the contrary, then there exists a $\varepsilon_0 > 0$ such that, for every $\nu \in I$ and every $\delta = 1/t$, $t = 1, 2, \ldots$, there exist an index $\iota_t \geqslant \nu$ and elements $u_\iota^{(t)}, v_\iota^{(t)} \in Z_\iota$ with $|u_\iota^{(t)}, v_\iota^{(t)}| < 1/t$, $|A_\iota u_\iota^{(t)}, A_\iota v_\iota^{(t)}| \geqslant \varepsilon_0, \iota = \iota_t$. We can select the indices such that $\iota_1 < \iota_2 < \iota_3 \ldots$ and thus $I' = (\iota_t)_{t \in \mathbf{N}}$ is a subsequence of I. Given a fived sequence $u = (u_\iota) \in Z$ and denote $u_\iota' = u_\iota$, $v_\iota' = u_\iota$, $\iota \in I - I'$, $u_\iota' = u_\iota^{(t)}$, $v_\iota' = v_\iota^{(t)}$, $\iota = \iota_t \in I'$, then the convergence $|u_\iota', v_\iota'| \to 0$ $(\iota \in I)$ holds. On the other hand, $|A_\iota u_\iota', A_\iota v_\iota'| \geqslant \varepsilon_0$, $\iota \in I'$, contrary to the stability of (A_ι) at (u_ι').

(ii) $\Rightarrow$ (i). For an arbitrary $\varepsilon > 0$, let $\delta > 0$, $\nu \in I$ be derived from condition (ii) and $u = (u_\iota)$, $v = (v_\iota) \in Z$ such that $|u, v| < \delta$. Then there exists a $\nu_1 \geqslant \nu$ such that $|u_\iota, v_\iota| < \delta$, $\iota \geqslant \nu_1$, and by (ii), it follows that $|A_\iota u_\iota, A_\iota v_\iota| < \varepsilon$, $\iota \geqslant \nu_1$. Hence $|Au, Av| < \varepsilon$.

(ii) $\Leftrightarrow$ (iv). Let $\Pi_\iota X_\iota$ be discretely compact, let lim^X be surjective, and let (iv) be valid. Suppose (ii) is not true, then there exist a $\varepsilon_0 > 0$, a subsequence $I' \leqslant I$, and elements $u_\iota, v_\iota \in X_\iota$, $\iota \in I'$, such that $|u_\iota, v_\iota| \to 0$ $(\iota \in I')$ and $|A_\iota u_\iota, A_\iota v_\iota| \geqslant \varepsilon_0$, $\iota \in I'$. By assumption, there exist a subsequence $I'' \leqslant I'$ and a $u_0 \in X_0$ such that $u_\iota \to u_0$ $(\iota \in I'')$. For lim^X is surjective, we can prolong the sequence $(u_\iota)_{\iota \in I''}$ to a convergent sequence $(u_\iota)_{\iota \in I}$, $u_\iota \to u_0$ $(\iota \in I)$. Set $z_\iota = v_\iota$, $\iota \in I'$,

$z_\iota = u_\iota$, $\iota \in I - I'$ then $\lim\limits_{\iota \in I} |u_\iota, z_\iota| = 0$, and, by (iv), it follows that $\lim\limits_{\iota \in I} |A_\iota u_\iota, A_\iota z_\iota| = 0$, contrary to $|A_\iota u_\iota, A_\iota z_\iota| \geqslant \varepsilon_0$, $\iota \in I'$. The converse is trivial.—

3.2. *Discrete convergence*

The mapping $\hat{A}$ defined by the sequence $A = (A_\iota)$ enables us now to establish a new characterization of the discrete convergence of (A_ι) by means of an equality of mappings. For the sake of clearness we denote the set of all discretely convergent sequences in X and Y by $\mathfrak{D}^X$ and $\mathfrak{D}^Y$, resp.. A sequence of mapping $A_\iota : X_\iota \longmapsto Y_\iota$, $\iota \in I$, is called discretely convergent to $A_0 : X_0 \longmapsto Y_0$, $lim\, A_\iota = A_0$, if lim^X is surjective and if, for every $u \in \mathfrak{D}^X$, the relations $Au \in \mathfrak{D}^Y$ and $lim^Y Au = A_0 lim^X u$ hold. Furthermore, a sequence $u \in X$ is said to be discretely (A_ι)-convergent, if $u \in \mathfrak{D}^X$ and $Au \in \mathfrak{D}^Y$. We denote the set of all discretely (A_ι)-convergent sequences by $\mathfrak{D}_A$. Then we obtain the following equivalence theorem.

(7) *Let lim^X be surjective. Then (A_ι) converges discretely to A_0, $lim\, A_\iota = A_0$, if and only if $\hat{A}$ is a well-defined, continuous mapping of $\hat{\mathfrak{D}}^X$ into $\hat{\mathfrak{D}}^Y$ and the following relation holds:*

$$A_0 = lim^Y \hat{A} (lim^X)^{-1}. \tag{8}$$

Proof: By definition, the convergence $lim\, A_\iota = A_0$ follows from relation (8), since the mapping of the discrete convergence also maps classes of discretely convergent sequences. Conversely, $lim\ A_\iota = A_0$ yields condition (1ii), and thus A is stable at each $u \in \mathfrak{D}^X$. Then by (3), $\hat{A}$ is a well-defined, continuous mapping of $\hat{\mathfrak{D}}^X$ into $\hat{\mathfrak{D}}^Y$, and $lim^Y \hat{A}(lim^X)^{-1}$ is also defined. Hence, the definition of $lim\ A_\iota = A_0$ implies $A_0 = lim^Y \hat{A}(lim^X)^{-1}$.—

A simple corollary of (7) yields the existence of a mapping $A_0 : X_0 \longmapsto Y_0$ which is a discrete limit of a given stable sequence $A = (A_\iota)$.

(9) *Let lim^X be surjective, let $A = (A_\iota)$ be stable at each $u \in \mathfrak{D}_A$, and for every $v_0 \in X_0$, let there be a $v \in (lim^X)^{-1} v_0$ such that $Av \in \mathfrak{D}^Y$. Then there exists a mapping $A_0 : X_0 \longmapsto Y_0$ such that $lim\ A_\iota = A_0$.*

Proof: Since $A = (A_\iota)$ is stable at each $u \in D_A$, it follows that $\hat{A}$ is a well-defined mapping from $\hat{D}_A$ in $\hat{D}^Y$ (cf. (3)). Moreover, by assumption, we have $\hat{D}_A = \hat{D}^X$, since, for every $u \in D^X$, there exists a $v \in D_A$ such that $u \in \overline{\{v\}}$, and the stability at v implies $Au \in \overline{\{Av\}} = \widehat{(Av)} \in \hat{D}^Y$ (cf. (1ii)). Thus, theorem (7) yields the assertion for $A_0 = lim^Y \hat{A}(lim^X)^{-1}$.

As another important result of the equivalence theorem (7) we obtain complete information of the continuity properties of discrete limits of sequences of mappings, which relation (8) makes immediately obvious. The continuity of a discrete limit has been already proved in [12],6.(2).

(10) *Let $X_0, \Pi_\iota X_\iota, lim^X$ and $Y_0, \Pi_\iota Y_\iota, lim^Y$ be metric discrete limit spaces with discretely convergent metrics, and let $A_0{:}X_0 \mapsto Y_0$ be the limit of a discretely convergent sequence of mappings $A_\iota{:}X_\iota \mapsto Y_\iota$, $\iota \in I$. Then A_0 is continuous in X_0. Moreover, A_0 is uniformly continuous in X_0 or Lipschitz-continuous or an isometry, if $\hat{A}$: $\check{D}^X \mapsto \hat{D}^Y$ has the corresponding property.*

The proof is an immediate consequence of the continuity of $\hat{A}$, the isometry property of lim^Y and $(lim^X)^{-1}$ (cf. 1. (2)) and the representation of A_0 in (8). The Lipschitz-continuity of $\hat{A}$ holds, for instance, if the mappings A_ι, $\iota \in I$, are uniformly Lipschitz-continuous.

The discrete convergence of mappings may be described by convergence properties of the associated graphs as in classic functional analysis. In Stummel [9]—*III*,1., a generalized strong convergence permits a simple characterization by means of the graphs. The following characterization theorem generalizes and extends this result.

For two metric discrete limit spaces

$$X_0, \Pi_\iota X_\iota, lim^X \quad \text{and} \quad Y_0, \Pi_\iota Y_\iota, lim^Y$$

there obviously exists a metric discrete limit space of the products, $X_0 \times Y_0, \Pi_\iota(X_\iota \times Y_\iota), lim$. If we denote the graphs of the mappings A_0, A_ι by $G(A_0)$, $G(A_\iota)$, then the following theorem can be stated.

(11) *The* Lim-*convergence of the graphs, that means*

(12) $$\text{Lim sup}\ \Pi_\iota G(A_\iota) = \text{Lim inf}\ \Pi_\iota G(A_\iota) = G(A_0),$$

is necessary for the discrete convergence $A_\iota \to A_0(\iota \in I)$. *Conversely*, (12) *is sufficient for* $A_\iota \to A_0(\iota \in I)$, *if, for every discretely convergent sequence* $(u_\iota) \in D^X$, *the sequence of images* $(A_\iota u_\iota)$ *is discretely compact.*

Proof: (i) Necessity. Obviously, $G(A_0) \subset \text{Lim inf } \Pi_\iota G(A_\iota)$ holds, if $A_\iota \to A_0(\iota \in I)$. Suppose $G(A_0) \not\supset \text{Lim sup } \Pi_\iota G(A_\iota)$, then there exists a $(u_0, w_0) \in \text{Lim sup } \Pi_\iota G(A_\iota)$ such that $(u_0, w_0) \notin G(A_0)$, i.e. $A_0 u_0 \neq w_0$. Furthermore, there are a subsequence $I' \leqslant I$ and elements $u_\iota' \in X_\iota$, $\iota \in I'$, such that $u_\iota' \to u_0$, $A_\iota u_\iota' \to w_0(\iota \in I')$. By assumption $A_\iota \to A_0$ $(\iota \in I)$, we have the convergence $A_\iota u_\iota' \to A_0 u_0(\iota \in I')$ and thus $A_0 u_0 = w_0$, contrary to $A_0 u_0 \neq w_0$. Therefore, we have proven that $G(A_0) \subset \text{Lim inf } \Pi_\iota G(A_\iota) \subset \text{Lim sup } \Pi_\iota G(A_\iota) \subset G(A_0)$ and all sets are equal.
(ii) Sufficiency. Let (12) be valid. Let $u_\iota \to u_0(\iota \in I)$ and let $I' \leqslant I$ be an arbitrary subsequence. Then, by the discrete compactness of $(A_\iota u_\iota)$ there exist a subsequence $I'' \leqslant I'$ and a $w_0 \in Y_0$ such that $A_\iota u_\iota \to w_0(\iota \in I'')$. Thus the definition of the upper discrete limit entails the relation $(u_0,w_0) \in \text{Lim sup } \Pi_\iota G(A_\iota)$. Assumption (12) yields $(u_0, w_0) \in G(A_0)$ and therefore $w_0 = A_0 u_0$ and $A_\iota u_\iota \to A_0 u_0(\iota \in I'')$. As it is well-known, this proves $A_\iota u_\iota \to A_0 u_0(\iota \in I)$.—

4. *Compactness of mappings*

The equivalence theorems in chapter 2 now immediately yield the equivalence of discrete compactness and asymptotic precompactness of a sequence of mappings. As a corollary, we obtain an interesting compactness property of a mapping by means of discrete compactness and asymptotic precompactness. Thus we also obtain a result of Wolf [15],2.(4), concerning the precompactness of a mapping which is consistent with a discretely compact sequence of mappings. In the case of an approximation by subspaces, the concepts of discrete compactness and asymptotic precompactness characterize a collectively compact sequence of mappings. The definitions of collective compactness in Anselone [1], [2] and in this chapter are equivalent in the special case of [1], [2]. As an application of our general characterizations, we finally prove a generalization of the classic compactness criteria of Arzelá and Ascoli.

4.1. *Discrete compactness*

A sequence of mappings $A_\iota : X_\iota \to Y_\iota$, $\iota \in I$, is called d i s c r e t e l y c o m p a c t, if, for every discretely bounded $\Pi_\iota Z_\iota \subset \Pi_\iota X_\iota$, the product space $\Pi_\iota A_\iota Z_\iota$ is discretely compact. Obviously, (A_ι) is discretely compact if and only if, for every discretely bounded sequence $(u_\iota) \in \Pi_\iota X_\iota$, the sequence of images $(A_\iota u_\iota)$ is discretely compact. Furthermore, we call (A_ι) asymptotically precompact, if, for every discretely bounded $\Pi_\iota Z_\iota \subset \Pi_\iota X_\iota$, the product space $\Pi_\iota A_\iota Z_\iota$ is asymptotically precompact. Then, according to theorem 2. (5), the following theorem is true.

(1) *Let Y_0 be a complete metric space, let $A(Y_0, \Pi_\iota Y_\iota, lim^Y)$ be a metric discrete approximation with discretely convergent metrics, and let* Lim sup $\Pi_\iota A_\iota X_\iota$ *be separable. Then (A_ι) is discretely compact if and only if it is asymptotically precompact.*

As a further result of the theorems of section 2.1, we obtain sufficient conditions for the precompactness of a mapping $A_0 : X_0 \mapsto Y_0$ by means of discrete compactness and asymptotic precompactness. Assumption (3) of the following theorem especially holds, if A_0, A_ι, $\iota \in I$, are consistent (cf. [10], 2.).

(2) *Let $X_0, \Pi_\iota X_\iota, lim^X$ and $Y_0, \Pi_\iota Y_\iota, lim^Y$ be metric discrete limit spaces with discretely convergent metrics, let $lim^X(\Pi_\iota X_\iota, X_0)$ be surjective, and for every bounded set $Z_0 \subset X_0$ let the relation*

$$(3) \qquad A_0 Z_0 \subset \text{Lim sup } A(lim^X)^{-1} Z_0$$

be satisfied. Then each of the following conditions (i), (ii) *is sufficient for the precompactness of $A_0 : X_0 \mapsto Y_0$:*

(i) *(A_ι) is asymptotically precompact.*

(ii) *$lim^Y(\Pi_\iota Y_\iota,$* Lim sup *$\Pi_\iota A_\iota X_\iota)$ is surjective,* Lim sup *$\Pi_\iota A_\iota X_\iota$ is separable, and (A_ι) is discretely compact.*

Proof: (i) Given a bounded set $Z_0 \subset X_0$, then $\delta \equiv \sup\{|u_0, v_0| : u_0, v_0 \in Z_0\} < \infty$. According to 1.(2), the relation $|R^X(v_0), R^X(z_0)| = |v_0, z_0|$ holds for every $v_0, z_0 \in Z_0$, where $R^X = (lim^X)^{-1}$. Hence $\sup\{|v, z| : v, z \in R^X(z_0)\} \leqslant \delta$ and $R^X(Z_0) \subset B_\delta(u) \subset \Pi_\iota B_\delta(u_\iota)$ for each $u = (u_\iota) \in R^X(Z_0)$. Let $u = (u_\iota)$ be an arbitrary sequence of $R^X(Z_0)$. Since the product space $\Pi_\iota B_\delta(u_\iota)$ is discretely bounded, by assumption (i), it follows that $\Pi_\iota A_\iota B_\delta(u_\iota)$ is asymptotically precompact. Thus, according to 2.(1), Lim sup $\Pi_\iota A_\iota B_\delta(u_\iota)$ is precompact. By assumption (3), we have $A_0 Z_0 \subset$ Lim sup $A R^X(Z_0) \subset$ Lim sup $\Pi_\iota A_\iota B_\delta(u_\iota)$ and hence the precompactness of $A_0 Z_0$.

(ii) Under the assumptions of (ii) the sequence (A_ι) is asymptotically precompact, due to 2.(2). Thus, part (i) yields the precompactness of A_0. —

Theorem 2.(7) and 2.(8) now enable us to characterize the notion of collective compactness of a sequence of mappings in Anselone [1], [2]. In [1], [2] linear operators are being studied which map the same Banach space into itself for all indices, whereas here we shall consider a more general situation.

Let (A_ι) be a sequence of mappings $A_\iota: X_\iota \longmapsto Y_\iota$, $\iota \in I$, where X_ι and Y_ι, $\iota \in I$, are subsets of a metric space M_1 and M_2, resp., which constitute discrete approximations $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim^{M_1})$ and $\mathcal{A}(Y_0, \Pi_\iota Y_\iota\, lim^{M_2})$ with subsets $X_0 \subset M_1$, $Y_0 \subset M_2$. The sequence (A_ι) is called c o l l e c t i v e l y c o m p a c t, if, for every discretely bounded sequence of subsets $Z_\iota \subset X_\iota$, $\iota \in I$, the product space $\Pi_\iota A_\iota Z_\iota$ is collectively compact, i.e. if $\bigcup_{\iota \in I} A_\iota Z_\iota$ is precompact in M_2. The discrete boundedness of $\Pi_\iota Z_\iota$ in an approximation of subspaces is equivalent to its uniform boundedness, if each set Z_ι, $\iota \in I$, is bounded. Hence in the special case of [1], [2], the definition of collective compactness in [1], [2] and the above definition are equivalent. From theorem 2.(7) it now follows that a sequence (A_ι) is collectively compact if and only if every operator A_ι, $\iota \in I$, is precompact and (A_ι) is asymptotically precompact. For a complete metric space M_2 and a closed subset Y_0, theorem 2.(8) yields equivalence of collective compactness and the compactness of each A_ι together with the discrete compactness of (A_ι).

Vainikko [14] has studied a similar situation as in [1], [2], considering mappings $A_\iota: X_\iota \longmapsto X_\iota$ and subspaces X_ι of a complete space X_0. There the relation $A_0 X_0 \subset \text{Lim inf}\, \Pi_\iota X_\iota$ is assumed instead of surjectivity of $lim\,(\Pi_\iota X_\iota, X_0)$, where A_0 is a mapping of X_0 into itself. This constitutes a metric discrete limit space $X_0, \Pi_\iota X_\iota$, lim of the domains of definition and an approximation $\mathcal{A}(A_0 X_0, \Pi_\iota X_\iota, lim)$ of the images. For a discretely compact sequence (A_ι) which is con sistent with the restriction of A_0 to $Z_0 = \text{Lim inf}\, \Pi_\iota X_\iota$, theorem (11) implies the compactness of the restriction $A_0|Z_0$, since the domains of definition of $A_0|Z_0$ and A_ι, $\iota \in I$, also constitute an approximation $\mathcal{A}(\text{Lim inf}\, \Pi_\iota X_\iota, \Pi_\iota X_\iota, lim)$.

4.2. *Compactness criteria in spaces of functions*

Equicontinuity together with compactness at each point of a product space is necessary and sufficient for the discrete compactness of a sequence of functions in a discrete approximation of spaces of functions. In view of applications this is an important generalization of the compactness criteria of Arzelá and Ascoli.

Let Y_0 be a complete normed space and, for a denumerably infinite, linearly ordered index sequence I, let Y_ι, $\iota \in I$, be linear subspaces of Y_0, which constitute a metric discrete approximation $\mathcal{A}(Y_0, \Pi_\iota Y_\iota, lim^Y)$ with the convergence in Y_0 as the discrete convergence, i.e. $|y_0, Y_\iota| \to 0$ $(\iota \in I)$, $y_0 \in Y_0$, or $Y_0 \subset \mathrm{Lim\,inf}\, \Pi_\iota Y_\iota$. In such a case there always exist restriction operators $r_\iota^Y : Y_0 \longmapsto Y_\iota$, $\iota \in I$, this means that $lim^Y y_\iota = y_0$ is equivalent to the convergence relation $\lim \|r_\iota^Y y_0 - y_\iota\| = 0$. Then, the metrics are discretely convergent. Here we still assume the existence of a sequence of restriction operators r_ι^Y, $\iota \in I$, being equicontinuous at each point $y_0 \in Y_0$.

Further, let X_0 be a compact metric space and X_ι, $\iota \in I$, be closed subspaces of X_0. Then the Banach spaces $E_0 = C_{Y_0}(X_0)$, $E_\iota = C_{Y_\iota}(X_\iota)$, $\iota \in I$, of bounded continuous functions of X_0 and X_ι with values in Y_0 and Y_ι, resp., constitute a metric discrete approximation $\mathcal{A}(E_0, \Pi_\iota E_\iota, u{-}lim)$ with discretely convergent metrics, if the relation

$$\text{(4)} \qquad \mathrm{Lim\,inf}\ \Pi_\iota X_\iota = \mathrm{Lim\,sup}\ \Pi_\iota X_\iota = X_0$$

holds. The discretely uniform convergence *u-lim* is defined by

$$\text{(5)} \qquad u\text{-}lim\ u_\iota = u_0 \iff \sup_{y \in X_\iota} \|u_\iota(x) - r_\iota^Y u_0(x)\| \to 0 \quad (\iota \in I)$$

for every sequence $u_0 \in C_{Y_0}(X_0)$, $u_\iota \in C_{Y_\iota}(X_\iota)$, $\iota \in I$. Here we do not prove the existence of the metric discrete approximation $\mathcal{A}(E_0, \Pi_\iota E_\iota, u\text{-}lim)$ and refer to corresponding results for spaces of $\mathbf{K}$-valued, bounded continuous functions in Stummel [11], § 8, where $\mathbf{K} = \mathbf{R}$ resp. $\mathbf{K} = \mathbf{C}$. By $X_\iota \subset X_0$, relation (4) is equivalent to $X_0 \subset \mathrm{Lim\,inf}\ \Pi_\iota X_\iota$ and moreover, by the compactness of X_0, it is equivalent to the convergence $d_0(X_0, X_\iota) = \sup_{x \in X_0} |x, X_\iota| \to 0$ $(\iota \in I)$. Thus, $\mathcal{A}(X_0, \Pi_\iota X_\iota, lim^X)$ is a metric discrete approximation with the convergence in X_0 as the discrete convergence lim^X. Therefore, it is again possible to define restriction operators $r_\iota^X : X_0 \longmapsto X_\iota$, $\iota \in I$, which still satisfy (cf. Stummel [10], 5.(5)) $|x_0, r_\iota^X x_0| \leqslant 2\, d_0(X_0, X_\iota)$, $x_0 \in X_0$. We do not need r_ι^X, $\iota \in I$, being equicontinuous but we assume that each r_ι^X is continuous.

Let us mention that, by the discrete compactness of $\Pi_\iota X_\iota$, the discretely uniform convergence of $u = (u_\iota) \in \Pi_\iota E_\iota$ is equivalent to the discrete convergence of mappings in the sense of section 3.2, i.e. $lim^Y \hat{u}(lim^X)^{-1} = u\text{-}lim\ u$.

We shall now state the following characterization of the discrete compactness of a sequence in $\mathcal{A}(E_0, \Pi_\iota E_\iota, u\text{-}lim)$.

(6) *Let condition* (4) *and the above assumptions with continuous* r_ι^X, $\iota \in I$, *and equicontinuous* r_ι^Y, $\iota \in I$, *be satisfied. Then the discrete compactness of a sequence* $u_\iota \in C_{Y_\iota}(X_\iota)$, $\iota \in I$, *is necessary and sufficient* for *the following two conditions:*

(i) (u_ι) *is equicontinuous.*

(ii) *For every sequence* $(x_\iota) \in \Pi_\iota X_\iota$, *the sequence of images* $(u_\iota(x_\iota))$ *is compact in* Y_0.

Proof: (i) Necessity. Assume that, for every subsequence $I' \leqslant I$, there exist a subsequence $I'' \leqslant I'$ and a function $u_0 \in C_{Y_0}(X_0)$ such that $u\text{-}lim_{\iota \in I''}\, u_\iota = u_0$. Suppose (i) is not true, then there exists a $\varepsilon_0 > 0$ and, for every $n \in \mathbf{N}$, there are an index ι_n and elements $x_{\iota_n}^{(n)}, z_{\iota_n}^{(n)} \in X_{\iota_n}$ with $|x_{\iota_n}^{(n)}, z_{\iota_n}^{(n)}| < 1/n$ and $\|u_{\iota_n}(x_{\iota_n}^{(n)}) - u_{\iota_n}(z_{\iota_n}^{(n)})\| \geqslant \varepsilon_0$. If the index set $\{\iota_n\}_{n \in \mathbf{N}}$ consists of finitely many pairwise distinct indices, it contradicts the assumption of the uniform continuity of each u_ι, $\iota \in I$. If $I' = (\iota_n)$ is a subsequence, the compactness of X_0 implies the existence of a subsequence $I'' \leqslant I'$ and of an element $x_0 \in X_0$ such that $|x_{\iota_n}^{(n)}, x_0| \to 0$, $|z_{\iota_n}^{(n)}, x_0| \to 0$ $(\iota_n \in I'',\ n \to \infty)$. By assumption, there is another subsequence $I' \leqslant I''$ and a function $u_0 \in C_{Y_0}(X_0)$ such that $u\text{-}lim_{\iota \in I'''}\, u_\iota = u_0$. Since the restriction operators r_ι^Y, $\iota \in I$, are assumed to be equicontinuous, it follows that $\|u_{\iota_n}(x_{\iota_n}^{(n)}) - u_0(x_0)\| \leqslant$ $\leqslant \sup \|u_{\iota_n}(x) - r_{\iota_n}^Y u_0(x)\| + \|r_{\iota_n}^Y u_0(x_{\iota_n}^{(n)}) - u_0(x_0)\| \to 0$ $(\iota_n \in I''',\ n \to \infty)$. Analogously, $\|u_{\iota_n}(z_{\iota_n}^{(n)}) - u_0(x_0)\| \to 0$ $(\iota_n \in I''',\ n \to \infty)$, contrary to $\|u_{\iota_n}(x_{\iota_n}^{(n)}) - u_{\iota_n}(z_{\iota_n}^{(n)})\| \geqslant \varepsilon_0$, $\iota_n \in I'$, which proves condition (i).

Moreover, we have proven that, for every sequence $x_0 \in X_0$, $x_\iota \in X_\iota$, $\iota \in I$, and every subsequence $K \leqslant I$, the relations $u\text{-}lim_{\iota \in K}\, u_\iota = u_0$ and $|x_\iota, x_0| \to 0$ $(\iota \in K)$ imply the convergence $\|u_\iota(x_\iota) - u_0(x_0)\| \to 0$ $(\iota \in K)$. Hence condition (ii) is valid, since, for every sequence $(x_\iota) \in$ $\in \Pi_\iota X_\iota$ and every subsequence $I' \leqslant I$, there exist a subsequence

$I'' \leqslant I'$ and a $x_0 \in X_0$ with $|x_\iota, x_0| \to 0$ $(\iota \in I'')$ and, moreover, there exist a subsequence $I''' \leqslant I''$ and a function $u_0 \in C_{Y_0}(X_0)$ such that $u\text{-}\lim_{\iota \in I'''} u_\iota = u_0$.

(ii) Sufficiency. For $E_0 = C_{Y_0}(X_0)$ is complete, by theorem 1.(4) and 2.(1), it is sufficient to prove the asymptotic precompactness of $\Pi_\iota\{u_\iota\} \subset \Pi_\iota E_\iota$. Given $\varepsilon > 0$ and, for $\varepsilon/6$, let $\delta > 0$ be the number in the condition of uniform equicontinuity of (u_ι) following from (6*i*) and the assumption that X_0 is compact and X_ι is closed for every $\iota \in I$. Moreover, there exist points $x_0^{(1)}, \ldots, x_0^{(N_1)}$ in X_0 such that the balls $B_{\delta/4}(x_0^{(n)})$, $n = 1, \ldots, N_1$, cover X_0. By assumption, there is an index $\nu_1 \in I$ such that $|r_\iota^X x_0^{(n)}, x_0^{(n)}| \leqslant \delta/4$, $\iota \geqslant \nu_1$, $n = 1, \ldots, N_1$, and there is another index $\nu_2 \geqslant \nu_1$ such that $d_0(X_0, X_\iota) < \delta/4$, $\iota \geqslant \nu_2$. From condition (ii), it follows that $(u_\iota(r_\iota^X x_0^{(n)}))_{\iota \in I}$ is a compact sequence in Y_0 for every $n = 1, \ldots, N_1$; thus, the set $Z = \bigcup_{n,\iota} \{u_\iota(r_\iota^X x_0^{(n)})\}$ is relatively compact in Y_0. Therefore, there exist points $z^{(1)}, \ldots, z^{(N_2)}$ in Z such that the balls $B_{\varepsilon/6}(z^{(m)})$, $m = 1, \ldots, N_2$, cover Z. By means of the set $\Phi = \{\varphi\colon \{1, \ldots, N_1\} \longmapsto \{1, \ldots, N_2\}\}$ of finitely many functions, we define $I_\varphi = \{\iota \in I : ||u_\iota(r_\iota^X x_0^{(n)}) - z^{(\varphi(n))}|| < \varepsilon/6,\ n = 1, \ldots, N_1\}$ and $H_\varphi = \{v_\iota \in C_{Y_\iota}(X_0) : v_\iota(x) = u_\iota(r_\iota^X x),\ x \in X_0,\ \iota \in I_\varphi,\ \iota \geqslant \nu_2\}$ for each $\varphi \in \Phi$. Then $I_1 \equiv \{\iota \in I : \iota \geqslant \nu_2\} = \bigcup_{\varphi \in \Phi} I_\varphi$ and the diameter of H_φ, $\varphi \in \Phi$, is bounded by $2\varepsilon/3$. Indeed, for arbitrary $\varphi \in \Phi$, $\iota_1, \iota_2 \in I_\varphi$ and $x_0 \in X_0$ there exists a $n \in \{1, \ldots, N_1\}$ such that $|x_0, x_0^{(n)}| \leqslant \delta/4$ and $|r_{\iota_1}^X x_0, r_{\iota_1}^X x_0^{(n)}| \leqslant |r_{\iota_1}^X x_0, x_0| + |x_0, x_0^{(n)}| + |x_0^{(n)}, r_{\iota_1}^X x_0^{(n)}| \leqslant 2d_0(X_0, X_\iota) + \delta/2 < \delta$. Thus $||u_{\iota_1}(r_{\iota_1}^X x_0) - u_{\iota_1}(r_{\iota_1}^X x_0^{(n)})|| = ||v_{\iota_1}(x_0) - v_{\iota_1}(x_0^{(n)})|| < \varepsilon/6$ and, analogously, $||v_{\iota_2}(x_0) - v_{\iota_2}(x_0^{(n)})|| < \varepsilon/6$, and, by definition of H_φ, $||v_{\iota_1}(x_0) - v_{\iota_2}(x_0)|| \leqslant 4\varepsilon/6$ for every $x_0 \in X_0$. If we denote an index of I_φ by ι_φ for each $\varphi \in \Phi$, then the asymptotic preompactness is satisfied with the sequences $u_\iota^{(\varphi)} = v_{\iota\varphi}|X_\iota \in C_{Y_\iota}(X_\iota)$, $\iota \in I$, $\varphi \in \Phi$. Indeed, for every $\iota \geqslant \nu_2$, there is a φ such that $\iota \in I_\varphi$ and for an arbitrary $x_\iota \in X_\iota$, there exists a $n \in \{1, \ldots, N_1\}$ such that $|x_\iota, x_0^{(n)}|| \leqslant \delta/4$. Thus, as $|x_\iota, r_\iota^X x_0^{(n)}| \leqslant \delta/2$, we have $||u_\iota(x_\iota) - v_\iota^{(\varphi)}(x_\iota)|| \leqslant ||u_\iota(x_\iota) - u_\iota(r_\iota^X x_0^{(n)})|| + ||u_\iota(r_\iota^X x_0^{(n)}) - u_{\iota_\varphi}(r_{\iota_\varphi}^X x_0^{(n)})|| + ||v_{\iota_\varphi}(r_{\iota_\varphi}^X x_0^{(n)}) - v_\iota^{(\varphi)}(x_\iota)|| \leqslant \frac{\varepsilon}{6} + \frac{2}{3}\varepsilon + \frac{\varepsilon}{6} = \varepsilon$. Hence $\overline{\lim_{\iota \in I}} \min_{\varphi \in \Phi} \sup_{x \in X_\iota} ||u_\iota(x) - v_\iota^{(\varphi)}(x)|| < \varepsilon$, and we need only prove that $u\text{-}\lim v_\iota^{(\varphi)} = v_0^{(\varphi)}$, $\varphi \in \Phi$, where $v_0^{(\varphi)}(x_0) = u_{\iota_\varphi}(r_{\iota_\varphi}^X x_0)$, $x_0 \in X_0$. But this is obviously true, since the continuity of $v_0^{(\varphi)}$ implies the discrete convergence of $(v_\iota^{(\varphi)})$ to $v_0^{(\varphi)}$ (in the sense of 3.2).—

An anlogous equivalence theorem can also be stated, if X_ι, $\iota \in I$, are not closed in X_0. Then, in theorem (6), one must use the uniform stability condition 3.(6ii) instead of (i). In the case of **K**-valued functions or functions with values in $\mathbf{K}^n$, $n \in \mathbf{N}$, i.e. $Y_0 = Y_\iota = \mathbf{K}^n$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, theorem (6) remains valid, if X_0, X_ι, $\iota \in I$, are subsets of a compact metric space M but the relation $X_\iota \subset X_0$ is not assumed. In such a case, by means of the extension theorem of Tietze, it is proved in [11), § 8, that condition (4) is even necessary and sufficient for the existence of the metric discrete approximation $A(E_0, \Pi_\iota E_\iota, u\text{-}lim)$ with discretely convergent metrics.

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