DEFICIENT VALUES OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES

by

P. K. KAMTHAN (*)

1. INTRODUCTION: This paper is devoted to studying one aspect of Nevanlinna's theory of meromorphic functions, namely the deficient values of the derivatives of meromorphic functions. A part of this study has already been made in [7]. It is throughout assumed, unless the contrary is mentioned, that $f(z)$ is in general a transcendental meromorphic function. We assume the familiary of the symbols

$$\bar{n}(r, a), \ \bar{n}(r, f), \ \bar{N}(r, a), \ m(r, f), \ T(r, f), \ldots \text{ etc.}$$

occurring frequently in the Nevanlinna theory of meromorphic functions, for instance see [2]. Let l be an integer ≥ 1 and $f^{(l)}$ denote the l-th derivative of the meromorphic function f. We then also assume the familiarity of the symbols $\delta(\alpha, f^{(l)})$, $\theta(\alpha, f^{(l)})$ etc., where

$$\delta(\alpha, f^{(l)}) = 1 - \limsup_{r \to \infty} \frac{N(r, 1/(f^{(l)} - \alpha))}{T(r, f^{(l)})}, \ l \geq 1,$$

and so is defined $\theta(\alpha, f^{(l)})$ provided Ni s replaced by $\bar{N}$ in the definition of $\delta(\alpha, f^{(l)})$. Also, let $\lambda(\alpha, f^{(l)})$ denote the Valiron deficient value which is defined as $\delta(\alpha, f^{(l)})$ where lim sup is replaced by lim inf. An important role is played by the following result in our work.

LEMMA A: Let $f(z)$ be a meromorphic funtion of finite order and let $a_1, \ldots, a_q$ be a finite set of distinct complex numbers. Then for all $r \geq r_0 = r_0(f)$

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$$\sum_{\mu=1}^{q} m(r, a_\mu) + N(r, 1/f^{(l)}) + 0(\log r) \leq T(r, f^{(l)})$$
$$\leq 0(\log r) + m(r, f) + (l+1) N(r, f).$$

For proof see Lemma 1, p. 6 [7].

2. COMPARISON OF $T(r, f^{(l)})$ WITH $T(r, f)$: This section is basically concerned with comparing the growth of $T(r, f^{(l)})$ with respect to $T(r, f)$ and certain other auxiliary functions under various sufficient conditions. First of all we begin with.

THEOREM 1: Let $f(z)$ be a meromorphic function in the plane and be of finite order. Assume

(2.1) $\theta(\infty, f) = 1;$ (2.2) $\sum_{\alpha \neq \infty} \theta(\alpha, f) = 1.$

Then

$$T(r, f^{(l)}) \sim T(r, f), \quad \text{as} \quad r \to \infty.$$

PROOF: We have

$$T(r, f^{(l)}) = N(r, f^{(l)}) + m(r, f^{(l)})$$
$$\leq N(r, f^{(l)}) + m(r, f) + m(r, f^{(l)}/f).$$

But

$$N(r, f^{(l)}) = N(r, f) + \sum_{\nu=1}^{l-1} \overline{N}(r, f^{(\nu)});$$
$$\overline{N}(r, f^{(\nu)}) = \overline{N}(r, f).$$

Therefore

$$T(r, f^{(l)}) \leq l\,\overline{N}(r, f) + N(r, f) + m(r, f) + m(r, f^{(l)}/f).$$

But $f(z)$ is of finite order, therefore (see [7], Theorem 2.1)

$$T(r, f^{(l)} \leq l\,\overline{N}(r, f) + T(r, f) + 0(\log r).$$

Making use of (2.1) of the hypothesis and the fact that $f(z)$ is not rational (i.e., $\log r = o(T(r, f))$ for $r \to \infty$), one gets

$$(2.3) \qquad T(r, f^{(l)}) \leq (1 + o(1))\, T(r, f), \; r \to \infty$$

To prove the reverse inequality, we use lemma A to get

$$qT(r,f) - \sum_{\nu=1}^{q} N(r, a) + N(r, 1/f^{(l)}) + 0(\log r) \leq T(r, f^{(l)}).$$

If $N^*(r, 1/f^{(l)})$ corresponds to those zeros of $f^{(l)}$ which occur at points other than the zeros of $f(z) = a_\nu$ $(\nu = 1, \ldots, q)$, then

$$qT(r,f) + 0(\log r) \leq T(r, f^{(l)}) + \sum_{\nu=1}^{q} \bar{N}(r, a_\nu) - N^*(r, 1/f^{(l)})$$

$$\leq T(r, f^{(l)} + \sum_{\nu=1}^{q} \bar{N}(r, a_\nu)$$

Using (2.2) we get for all sufficiently large r and an arbitrary $\varepsilon > 0$ the following

$$qT(r,f) + 0(\log r) \leq T(r, f^{(l)}) + \sum_{\nu=1}^{q} \{1 - \theta(a_\nu, f) + \varepsilon\} T(r,f)$$

Hence

$$\liminf_{r\to\infty} \frac{T(R, f^{(l)})}{T(r,f)} \geq 1,$$

which when combined with (2.3) yields the required result.

Remark: If $l = 1$, then the above theorem gives rise to a result of Shah and Singh [9] which they stated without proof.

By weakening the hypothesis in one way more information could be sought in the preceding result and therefore in this direction we may now state and prove:

Theorem 2: Let $f(z)$ a meromorphic function of finite order, such that $\delta(0,f) = 1$; $\theta(\infty, f) = 1$. Tehn

$$T(r, f^{(l)}) \sim T(r,f) \sim \bar{N}\left(r, \frac{1}{f^{(l)} - x}\right),$$

for all x, except possibly 0 and ∞.

PROOF: Let $\psi = f^{(l)}$ in Theorem 3.2 of Hayman [2] and writing x for 1 there, we have

$$T(r, f) < \bar{N}(r, f) + N(r, 1/f) + \bar{N}(r, 1/(f^{(l)} - x)), \; x \neq 0, \infty$$

$$(2.4) \qquad < 0(T(r, f)) + \bar{N}(r, 1/(f^{(l)} x)), \; r \geq r_0$$

Since $\delta(0, f) = \theta(\infty, f) = 1$; also using the first fundamental theorem of Nevannlinna, we have:

$$T(r, f^{(l)}) \geq \bar{N}\left(r, \frac{1}{f^{(l)} - x}\right).$$

Hence

$$(2.5) \qquad \liminf_{r\to\infty} \frac{T(r, f^{(l)})}{T(r, f)} \geq \liminf_{r\to\infty} \frac{\bar{N}(r, 1/(f^{(l)} - x))}{T(r, f)} \geq 1$$

from (2.4). Further

$$\frac{\bar{N}(r, 1/(f^{(l)} - x))}{T(r, f)} \leq 0(1) + \frac{T(r, f^{(l)})}{T(r, f)} \leq 1 + o(1), \; r \geq r_0$$

from (2.3) where due care is to be taken that we have made use of the fact that $\Theta(\infty, f) = 1$. Consequently

$$(2.6) \qquad \limsup_{r\to\infty} \frac{N(r, 1/(f^{(l)} - x))}{T(r, f)} \leq \limsup_{r\to\infty} \frac{T(r, f^{(l)})}{T(r, f)} \leq 1$$

The inequalities (2.5) and (2.6) when combined together lead to Theorem 2.

3. ESTIMATION OF $\sum_{a\neq\infty} \delta(a, f)$: In this section we estimate $\sum_{a\neq\infty} \delta(a, f)$ in terms of $\delta(0, f^{(l)})$ and $\delta(\infty, f^{(l)})$ and certain other deficient values which arise in a natural course of our discussion that follows. It may be mentioned that Theorem 3 of this section answers partially a problem raised by Hayman in his book ([2], p. 105, last paragraph). To begin with, let us therefore start from

THEOREM 3: With the notations mentioned above, we have

$$(3.1) \qquad \lambda(0, f^{(l)}) \geq \frac{1}{l+1} \sum_{a\neq\infty} \delta(a, f),$$

if $f(z)$ is meromorphic and of finite order; and

$$\lambda(0, f^{(l)}) \geq \sum_{a \neq \infty} \delta(a, f) \tag{3.2}$$

if $f(z)$ is entire and of finite order.

Also we have

$$2 - K(\varrho) - \delta(\infty, f^{(l)}) \geq \frac{1}{l+1} \sum_{a \neq \infty} \delta(a, f), \tag{3.3}$$

if $f(z)$ is meromorphic and of non-integral order ϱ; and

$$1 - K(\varrho) \geq \sum_{a \neq \infty} \delta(a, f) \tag{3.4}$$

if $f(z)$ is entire and of non-integral order ϱ, where $K(\varrho)$ is the quantity defined below:

$$K(\varrho) \geq 1 - \varrho, \quad \text{if} \quad 0 < \varrho < 1;$$

$K(\varrho) \geq (q + 1 - \varrho)(\varrho - q)/2\varrho(q+1)\{2 + \log(q+1)\}$, if $\varrho < 1$ and $q = [\varrho]$.

Proof of Theorem 3: Let $f(z)$ be meromorphic. Then Lemma A yields

$$\sum_{\mu=1}^{q} \frac{m(r, a_\mu)}{T(r, f^{(l)})} \leq 1 - \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})}$$
$$< \lambda(0, f^{(l)}) + \varepsilon,$$

for sufficiently large r. Hence

$$\sum_{\mu=1}^{q} \liminf_{r \to \infty} \frac{m(r, a_\mu)}{T(r, f^{(l)})} \leq \lambda(0, f^{(l)}) \tag{3.5}$$

But from a result of Milloux, See [2], Theorem 3.1,

$$T(r, f^{(l)}) \leq (l+1)\, T(r, f),$$

provided $f(z)$ is meromorphic. Hence from (3.5)

$$\lambda(0, f^{(l)}) \geq \frac{1}{l+1} \sum_{1}^{q} \liminf_{r \to \infty} \frac{m(r, a_\mu)}{T(r, f)}, \tag{3.6}$$

and from which (3.1) follows.

Let $f(z)$ be entire, then

$$T(r, f^{(l)}) = m(r, f^{(l)}) \leq m(r, f) + m(r, f^{(l)}/f)$$
$$= T(r, f) + 0(\log r)$$
$$= (1 + o(1) T(r, f),$$

since $f(z)$ is not a polynomial and that $f(z)$ is of finite order. Therefore from (3.5)

$$\lambda(0, f^{(l)}) \geq \sum_{\mu=1}^{q} \liminf_{r\to\infty} \frac{m(r, a_\mu)}{T(r, f)}, \tag{3.7}$$

and from (3.7) we get (3.2).

To prove the other part of the theorem, let now $f(z)$ be meromorphic and of non-integral order. By a well-known fact, $f^{(l)}(z)$ is also of order ϱ. Therefore using a result of Hayman ([2], p. 101; see also [5] Theorem A)

$$\limsup_{r\to\infty} \frac{N(r, f^{(l)}) + N(r, 1/f^{(l)})}{T(r, f^{(l)})} \geq K(\varrho),$$

and so for arbitrarily large r

$$\frac{N(r, 1/f^{(l)})/}{T(r, f^{(l)})} \geq K(\varrho) - \frac{N(r, f^{(l)})}{T(r, f^{(l)})} - \varepsilon \tag{3.8}$$

But for all $r \geq r_0$ and $\varepsilon > 0$

$$\frac{N(r, f^{(l)})}{T(r, f^{(l)})} \leq 1 - \delta(\infty, f^{(l)}) + \varepsilon$$

Thus for arbitrarily large r, from (3.8)

$$\frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \geq K(\varrho) - 1 + \delta(\infty, f^{(l)}) - 2\varepsilon,$$

or,

$$\limsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \geq K(\varrho) - 1 + \delta(\infty, f^{(l)})$$

Therefore

$$(3.9) \qquad \delta(0, f^{(l)}) = 1 - \limsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \leq 2 - K(\varrho) - \delta(\infty, f^{(l)})$$

If $f(z)$ is entire, then $\delta(\infty, f^{(l)}) = 1$, and so (3.9) redeces to

$$(3.10) \qquad \delta(0, f^{(l)}) \leq 1 - K(\varrho)$$

Therefore (3.9) and (3.1) give (3.3), and (3.10) and (3.2) give (3.4). The proof of Theorem 3 is now complete.

The next two results are also devoted to finding certain other types of estimations of $\delta(0, f^{(l)})$, $\sum_{a\neq\infty} \delta(a, f)$ and $\lambda(0, f^{(l)})$ where $\lambda(0, f^{(l)}))$ is defined below. First I state and prove

LEMMA 1: Let $f(z)$ be meromorphic in the plane and let its order be finite. Let A_i, $|A_i| < \infty$ $i = 1, 2, \ldots, p$ be a set of p $(p < 2)$ distinct complex numbers. Let also B_i, $0 < |B_i| < \infty$, $i = 1, 2, \ldots, q$ be another set of q $(q > 2)$ distinct numbers in the complex plane. Then

$$pq\,T(r, f) \leq q \sum_{i=1}^{p} N(r, A_i) + \sum_{i=1}^{q} N\left(r, \frac{1}{f^{(l)} - B_i}\right) + \bar{N}(r, f^{(l)})$$
$$- [(q-1) N(r, 1/f^{(l)}) + N(r, 1/f^{(l+1)})] + 0(\log r)$$

PROOF: Rewriting lemma A, we have

$$(3.11) \quad pq\,T(r, f) \leq q\,T(r, f^{(l)}) + q \sum_{i=1}^{p} N(r, A_i) - q\,N(r, 1/f^{(l)}) + 0(\log r)$$

But from the second fundamental theorem, when applied to $f^{(l)}(z)$, one gets

$$q\,T(r, f^{(l)}) \leq \sum_{i=1}^{q+1} N\left(r, \frac{1}{f^{(l)} - B_i}\right) + \bar{N}(r, f^{(l)}) - N(r, 1/f^{(l+1)}) + 0(\log r),$$

where $B_i'S$ are distinct $q+1$ complex numbers confined to finite values. We can write

$$\sum_{i=1}^{q+1} N\left(r, \frac{1}{f^{(l)} - B_i}\right) = \sum_{i=0}^{q} N\left(r, \frac{1}{f^{(l)} - B_i}\right),$$

where we can choose $B_0 = 0$. Thus

$$(3.12) \qquad qT(r, f^{(l)}) \leq \sum_{i=1}^{q} N\left(r, \frac{1}{f^{(l)} - B_i}\right) + \bar{N}(r, f^{(l)}) - N(r, 1/f^{(l+1)}) + N(r, 1/f^{(l)}) + 0(\log r),$$

where $B_i \neq 0$ for $i = 1, 2, \ldots, q$ and they are distinct. Substituting the value of $qT(r, f^{(l)})$ from (3.12) in (3.11) and arranging the various terms, we get the proof of the lemma.

We can now pass on to prove

THEOREM 4: Let $f(z)$ be meromorphic in the plane and of finite order. Suppose $\{a_i\}$ $(i = 1, \ldots, p)$, $|a_i| < \infty$ and $\{b_i\}$ $(i = 1, \ldots, q)$' $0 < |b_i| < \infty$ be two sets of complex numbers. Let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{T(r, f^{(l)})}{T(r, f)} = \frac{A}{B};$$

$$\lambda(0, f^{(l)}) = 1 - \liminf_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})}.$$

Then we have:

$$(3.13) \qquad q\sum_1^p \delta(a_i, f) + \theta(\infty, f) - 1 \leq B(q-1)\,\delta(0, f^{(l)}) - A\sum_1^q \delta(b_i, f^{(l)}) + Aq - B(q-1);$$

$$(3.14) \qquad q\sum_1^p \delta(a_i, f) + \theta(\infty, f) - 1 \leq B\{1 + (q-1)\}\,\lambda(0, f^{(l)}) - \sum_1^q \delta(b_i, f^{(l)}).$$

COROLLARIES From (3.13) and (3.14) we derive respectively

$$(3.15) \qquad \sum_1^p \delta(a_i, f) \leq B\,\delta(0, f^{(l)}) + A - B;$$

$$(3.16) \qquad \sum_1^p \delta(a_i, f) \leq B\,\lambda(0, f^{(l)}).$$

As far as (3.15) and (3.16) are considered, we use (3.13) and (3.14) and there we let q tend to infinity and we get (3.15) and (3.16) where it is to be noted that $\sum_1^{\infty} \delta(b_i, f^{(l)})$ is equal to a finite number.

PROOF OF THEOREM 4: We note that $\bar{N}(r, f^{(l)}) = \bar{N}(r, f)$; also $T(r, f^{(l)})/T(r, f) < A + \varepsilon$, $T(r, f^{(l)})/T(r, f) > B - \varepsilon$ for all $r \geq r_0 = r_0(\varepsilon)$ where $\varepsilon > 0$. Therefore from the preceding lemma, one has for all $r \geq r_0$ and $\varepsilon > 0$, the following inequality:

$$pq < 1 - \theta(\infty, f) + \varepsilon + q \sum_1^p (1 - \delta(a_i, f) + \varepsilon) + (A + \varepsilon \sum_1^q \frac{N\left(r, \dfrac{1}{f^{(l)} - b_i}\right)}{T(r, f^{(l)})}$$
$$- (q - 1)(B - \varepsilon) \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} + o(1),$$

or we have:

$$0 \leq 1 - \theta(\infty, f) - q \sum_1^p \delta(a_i, f) + Aq - A \sum_1^q \delta(b_i, f^{(l)})$$
$$- (q - 1) B \limsup_{r \to \infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})};$$

and the rearrangement of various terms of this inequality leads to (3.13). Similarly, we have for arbitrarily large r, the following inequality:

$$0 < 1 - \theta(\infty, f) - q \sum_1^p \delta(a_i, f) + Bq - B \sum_1^q \delta(b_i, f^{(l)})$$
$$- (q - 1) B \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} + 0(\varepsilon),$$

and this results in (3.14).

Towards the close of this article, *I* state and prove the following result which is concerned with the estimation of $\sum_{a \neq \infty} \delta(a, f)$ in terms of A and B defined above.

THEOREM 5: Let $f(z)$ be meromorphic and of non-integral order ϱ $(0 < \varrho < 1)$ and further let $\delta(\infty, f) = 1$, then

$$\sum_{a \neq \infty} \delta(a, f) \begin{cases} \leq A - B, \left(0 < \varrho < \dfrac{1}{2}\right) \\ \leq A - B \sin \pi \varrho, \left(\dfrac{1}{2} \leq \varrho < 1\right) \end{cases}$$

REMARK: This result generalises a result of Edrei and Fuchs ([1], Corollary 1.3), since, when $f(z)$ is entire, then $A = B = 1$.

PROOF OF THE THEOREM: It is a well-known fact that if $f(z)$ has order ϱ, then $f^{(l)}(z)$ has also the order ϱ. Also as $\delta(\infty, f) = 1$ and hence $\delta(\infty, f^{(l)}) = 1$, (See lemma 3, (vii) [7], where we put $r = 0$). Now let

$$u = 1 - \delta(0, f^{(l)})\,; \quad v = 1 - \delta(\infty, f^{(l)}).$$

Then applying Theorem 1 of Edrei and Fuchs [1], we get

$$u \geq \sin \pi \varrho$$

or,

$$\delta(0, f^{(l)}) \leq 1 - \sin \pi\varrho$$

and this together with (3.15) yields the result for $\frac{1}{2} \leq \varrho < 1$. Let $0 < \varrho < \frac{1}{2}$, then $v < \cos \pi \varrho$ and so (Theorem 1, [1]) $u = 1$, that is $\delta(0, f^{(l)}) = 0$, and once again we get the result from (3.15). The theorem is, therefore, proved completely.

4. Next I verify under what conditions the derivative of a meromorphic function does not assume a particular value. In this direction I prove two results. First I have

THEOREM 6: Let $f(z)$ be a meromorphic function in the plane and let its order be finite. Further, let $\delta(0, f) = 1$; $\delta(1, f) = 1$. Then $\delta(x, f^{(l)}) = 1$ possibly for $x = 0$ only and for no other value of x, finite or infinite.

REMARK: In the above result, we may replace 0 and 1 in the function δ by any other distinct numbers x_1 and x_2, $|x_1| < \infty$, $|x_2| < \infty$.

PROOF OF THEOREM 6: We have (lemma 4, [7])

$$T(r, f) \sim N(r, f)\,; \; \bar{N}(r, f) \sim T(r, f),$$
$$T(r, f^{(l)}) \sim (l + 1)\, T(r, f).$$

Therefore

$$\frac{N(r, f^{(l)})}{T(r, f^{(l)})} \sim \frac{N(r, f) + l\overline{N}(r, f)}{(l+1)\,T(r, f)}$$
$$\to 1, \quad (r \to \infty)$$

and so $\delta(\infty, f^{(l)}) = 0$. Now

$$N(r, 1/f^{(l)}) \leq T(r, f^{(l)}) - \sum_{1}^{1} m(r, a_{\mu}),$$

and as

$$\liminf_{r\to\infty} \frac{m(r, a_{\mu})}{T(r, f)} = \delta(a_{\mu}, f)\,; \qquad m(r, a_{\mu}) > (\delta(a_{\mu}, f) - \varepsilon)\,T(r, f)$$

for all $r \geq r_0 = r_0(\varepsilon)$, $\varepsilon < 0$, one finds that

$$N(r, 1/f^{(l)}) < T(r, f^{(l)}) - T(r, f) \sum_{1}^{q} (\delta(a_{\mu}, f) - \varepsilon),$$

or,
$$\frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} < 1 - \frac{T(r, f)}{T(r, f^{(l)})} \sum_{1}^{q} \delta(a_{\mu}, f) + 0(\varepsilon),$$

for all $r \geq r_0$, $\varepsilon > 0$. Therefore

$$\limsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \leq 1 - \frac{1}{l+1} \sum_{1}^{q} \delta(a_{\mu}, f)$$

$$= 1 - \frac{2}{l+1}. \tag{4.1}$$

Now (4.1) shows that there is a possibility of $\delta(0, f^{(l)})$ being equal to 1. Next, we shall show that $\delta(x, f^{(l)}) \neq 1$ for any other $x \neq 0$. Now

$$T(r, f^{(l)}) = T(r, 1/f^{(l)}) + 0(1)$$
$$= N(r, 1/f^{(l)}) + m(r, 1/f^{(l)}) + 0(1)$$

But from a result of Milloux ([2], p. 57, equation (3.8))

$$m(r, 1/f^{(l)}) < \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^{(l)} - x}\right) - N_0(r, 1/f^{(l+1)}) + S(r, f) \tag{4.2}$$

and hence

$$T(r, f^{(l)}) < N(r, 1/f^{(l)}) + \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f^{(l)} - x}\right) + 0(\log r),$$

since $f(z)$ is of finite order. Let $\delta(x, f^{(l)}) = 1$, $(x, \neq 0)$. Then for all $r \geq r_0$

$$T(r, f^{(l)}) < \left(1 - \frac{2}{l+1} + \varepsilon\right) T(r, f^{(l)}) + (1 + \varepsilon) T(r, f) + \varepsilon$$

and so

$$1 \leq 1 - \frac{2}{l+1} + \frac{1}{l+1},$$

and this gives a contradiction. Hence $\delta(x, f^{(l)}) \neq 1$ except possibly for $x = 0$.

REMARKS (i) Let $l = 1$. Then from (4.1),

$$\limsup_{r\to\infty} \frac{N(r, 1/f^{(1)})}{T(r, f^{(1)})} = 0$$

Therefore $\delta(0, f^{(1)}) = 1$. Following as above we find that $\delta(x, f^{(1)}) = 1$ for $x = 0$ only. Thus in the above theorem if $l = 1$, we rule out the statement «possibly».

(ii) The result (4.1) is also a consrquence of corollary (3.15), for here $A = B = l + 1$ and therefore $\sum_{a\neq\infty} \delta(a, f) \leq (l + 1)\,\delta(0, f^{(l)})$ and so

$$\frac{2}{l+1} \leq 1 - \limsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})},$$

and which is (4.1).

(iii) We may also have an alternative proof of remark (i) above without a recourse to Milloux's result (4.2). For, let $l = 1$, then (3.15) we find since $T(r, f^{(1)}) \sim 2\,T(r, f)$, that

$$\sum_{a\neq\infty} \delta(a, f) \leq 2\,\delta(0, f^{(1)}),$$

that is, in this case we have $1 \leq \delta(0, f^{(1)})$ and as $\delta(0, f^{(1)}) \leq 1$ always, one gets $\delta(0, f^{(1)}) = 1$ and hence from (3.13) of Theorem 4 of this paper we get ($A = B = 2$ in this case)

$$2q - 1 \leq 2(q-1) - 2\sum_1^q \delta(b_i, f^{(1)}) + 2,$$

or

$$\sum_1^q \delta(b_i, f^{(1)}) \leq \frac{1}{2};$$

and if $\delta(b_i, f^{(1)}) = 1$, for some i ($b_i \neq 0$), the preceding inequality will give

$$\sum_{\substack{j=1 \\ j \neq i}}^q \delta(b_j, f^{(1)}) \leq -\frac{1}{2},$$

and this is certainly a contradiction and our assertion follows.

Next, we wish to know if we could get a sharper form of the result of Theorem 6 by weaking the hypothesis of this Theorem. To formulate problem, let us recall ([5], p. 13).

Definition: Let S be the family of all increasing functions $\phi(x)$, such that $\log x = 0(\phi(x))$ and that $x^\varepsilon/\phi(x)$ ($\varepsilon > 0$) is non-decreasing $\to \infty$ with x. We will call a complex number α as e. v. S. of a meromorphic function f, if for $\phi \varepsilon S$

$$\liminf_{r \to \infty} \frac{T(r, f)}{n(r, \alpha)\phi(r)} < 0.$$

We are then concerned with

Theorem 7: Let $f(z)$ be meromorphic in the plane and let its order be finite. Let 0 and λ ($\lambda \neq 0, \neq \infty$) be e. v. S. of $f(z)$. Then $f^{(l)}(z)$ has possibly 0 as its only e. v. S.

Remark: It has not been possible for me to prove the above assertion for $l \geq 2$ and I prove this result for $l = 1$, and in this particular case 0 is necessarily an e. v. S. for $f^{(1)}(z)$.

Proof of the theorem for $l = 1$: Now as noted earlier we see that $\delta(\infty, f^{(l)}) = 0$ and this implies that ∞ is not an e. v. S. for $f^{(l)}(z)$, for if it were then

$$T(r, f^{(l)}) > A\, n(r, f^{(l)})\, \phi(r)$$

for all $r \geq r_0$, where A is some positive constant and $\phi(r)$ is suitably chosen an increasing function such that $\log r = 0(\phi(r))$. Therefore

$$N(r, f^{(l)} \leq n(r, f^{(l)}) \log r < T(r, f^{(l)}) \log r / A\,\phi(r), \; r \geq r_0$$

and

$$\delta(\infty, f^{(l)}) = 1$$

and we arrive at a contradiction. Hence is not an e. v. S. for $f^{(1)}(z)$ (put $l = 1$ in the above discussion). Let

$$g(z) = \frac{f(z)}{f(z) - \lambda}, \; (\lambda \neq 0, \; \neq \infty).$$

Then

$$T(r, g) = T(r, f) + 0(1);$$

$$n(r, 1/g) = n(r, 1/f), \; n(r, g) = n\left(r, \frac{1}{f - \lambda}\right).$$

Therefore from the hypothesis, it follows that 0 and ∞ are two e. v. S. for $g(z)$. Hence from Theorem 2 ([7]) 0 and ∞ are two e. v. S. for $g^{(l)}(z)$ and therefore for $g^{(1)}(z)$

Now we have

$$f^{(1)}(z) = -\frac{g^{(1)}(z)}{(g(z) - 1)^2}$$

and so

$$n(r, 1/f^{(1)}) \leq n(r, 1/g^{(1)}) + 2n(r, g)$$
$$< \frac{T(r, g^{(1)})}{A\,\phi(r)} + 2n(r, g), \quad (r \leq r_0),$$

or, we have, on choosing $\phi(r)$ to be arbitrary which we can do here

$$\frac{T(r, f^{(1)})}{n(r, 1/f^{(1)})\,\phi(r)} \geq \frac{A\,T(r, g^{(1)})(1 + o(1))}{T(r, g^{(1)}) + 2A\,\phi(r)\,n(r, g)}$$
$$\geq \frac{A\,T(r, g^{(1)})(1 + o(1))}{T(r, g^{(1)}) + 2AB\phi(r)},$$

where A and B are positive constants; and as $T(r, g) \sim T(r, g')$, since $\delta(0, g) = \delta(\infty, g) = 1$, one finds that 0 is an e. v. S. for $f^{(1)}(z)$. Now if $x \neq 0$, then x is not an e. v. S. for $f^{(1)}(z)$, for if it were then

$\delta(x, f^{(1)}) = 1$ and following the last part of the proof of the preceding theorem, we arrive at a contradiction and so the result is proved.

5. This article is supplementary. First we have the following

LEMMA 2: Let $f(z)$ be meromorphic in $|z| \leq R$, $f(0) = 0$. Then for $0 < r < R$

$$m(r, f) \leq (1 + \psi(r/R))\, T(R, f^{(l)}) + l \log^+ r,$$

where

$$\psi(t) = \frac{(1-t)\log\left(1 + \dfrac{2\pi\sqrt{t}}{1-t}\right)}{\pi\sqrt{t}\,\log(1/t)}, \quad 0 < t < 1.$$

PROOF: Substantially this is theorem 2 of Hayman [4] (see also [3]) and we need merely note that

$$\sup_{0<t<r} |f^{(l)}(te^{i\theta})| \geq r^l |f(z)|, \quad |z| = r < 1.$$

We are now ready to prove the following

THEOREM 8: Let $f(z)$ be meromorphic in the plane, $f(0) = 0$, and have finite non-zero order ϱ, then

$$\lim_{r\to\infty} \frac{T(r, f^{(l)})}{T(r, f)} \geq \frac{1}{(1 + \psi(1/k))(1 + k^{\varrho})}$$

where $k > 1$ and $r \to \infty$ through a set E:

$$E = \{x : T(x, f^{(l)}) = x^{\varrho(x)}\}.$$

PROOF: Since $f(z)$ is of order ϱ and therefore

$$\limsup_{r\to\infty} \frac{\log T(r, f^{(l)})}{\log r} = \varrho$$

Then there exists a proximate order $\varrho(r)$, suchthat ([8], p. 35 see also [6])

$$\varrho(r) \to \varrho, \ r \to \infty;$$
$$r\varrho'(r)\log r \to 0, \ r \to \infty;$$
$$T(r, f^{(l)}) \leq r^{\varrho(r)} \qquad \text{all } r \geq r_0;$$
$$T(r, f^{(l)}) = r^{\varrho(r)},$$

for a sequence of r, $r = r_n \to \infty$ with n. Then for $R > r$

$$T(r, f^{(l)}) \leq R^{\varrho(R)-\varrho(r)} (R/r)^{\varrho(r)} r^{\varrho(r)},$$

or, $$T(R, f^{(l)}) \leq \exp\left((\varrho(R) - \varrho(r)) \log R\right) (R/r)^{\varrho(r)} r^{\varrho(r)}.$$

But

$$\varrho(R) - \varrho(r) = \int_r^R \varrho'(x)\, dx \leq \frac{\varepsilon}{\log r} \log (R/r).$$

Hence

$$T(R, f^{(l)}) \leq \exp\left\{\frac{\varepsilon \log (R/r) \log R}{\log r}\right\} (R/r)^{\varrho(r)} T(r, f^{(l)})$$

where $r \varepsilon E$. Let $R = kr$, $k > 1$. Then

$$T(k, f^{(l)}) \leq (1 + o(1)) k^{\varrho(r)} T(r, f^{(l)})$$

Therefore from lemma 2

$$m(r, f) \leq (1 + \psi(1/k)) (1 + o(1)) k^{\varrho(r)} T(r, f^{(l)})$$

and as $N(r, f) \leq T(r, f^{(l)})$, we get the result.

To proceed further in this direction *I* give one more result. For this purpose *I* follow Hayman [4] and define the following the term

DEFINITION: Let C_1 and C_2 be constants, $C_1 > 1$; $C_2 > 1$. We shall say that a positive number r is *l-normal* (C_1, C_2) for $f(z)$ if

$$T(C_1 r, f^{(l)}) < C_2 T(r, f^{(l)}),$$

where $f(z)$ is meromorphic in the plane.

LEMMA 3: Let $f(z)$ be transcendental in the plane, $f(0) = 0$ and r is l-normal (C_1, C_2), then

$$m(r, f) < C_3 T(r, f^{(l)}) + l \log^+ r$$

where $C_3 = C_2 (1 + \psi(1/C_1))$. Further

$$T(r, f) < (1 + C_3) T(r, f^{(l)}) + l \log^+ r;$$

und

$$T(r, f) < \left(\frac{p}{p+l} + C_3\right) T(r, f^{(l)}) + \log^+ r,$$

if $f(z)$ has no poles of multiplicity greater than p.

PROOF: This is in fact lemma 2 of Hayman [4] where we note that

$$N(r, f) \leq N(r, f^{(l)}) \leq T(r, f^{(l)}),$$

and if q the of order of multiplicity of a pole of $f(z)$ then $f^{(l)}(z)$ has a pole of order $q + l$, thus

$$q + l = \left(1 + \frac{l}{q}\right) q$$

$$\geq \frac{p+l}{p} q$$

and so

$$n(r, f) \leq \frac{p}{p+l} n(r, f^{(l)}),$$

and therefore

$$N(r, f) \leq \frac{p}{p+l} N(r, f^{(l)}).$$

6. The following result is of independent interest

THEOREM 9: Let $f(z)$ be meromorphic in the plane and of finite order ϱ, such that $T(r, F(z)) = o(T(r, f(z)))$ where $F(z)$ is another meromorphic function. Then

$$\liminf_{r \to \infty} \frac{T(r, f)}{\bar{n}(r, f - F)} \leq \frac{3}{\varrho} \tag{6.1}$$

except possibly two meromorphic functions F_1, F_2 $(T(r, f) = o(T(r, F_1))$; $T(r, f) = o(T(r, F_2)))$.

PROOF: Let F_i $(i = 1, 2, 3,)$ be three meromorphic functions where (6.1) does not hold good. Let $0 < \varrho < \infty$. Then for all $r > r_0$

$$T(r, f) > A_i n(r, f - F_i)\ ;\ i = 1, 2, 3,$$

where $A_i > 3/\varrho$. Then

$$3\,T(r, f) > \sum_{i=1}^{3} A_i \bar{n}(r, f - F_i),$$

or,

$$3 \int_{r_0}^{r} \frac{T(x, f)}{x}\, dx > \left(\frac{3}{\varrho} + \alpha\right) \sum_{i=1}^{3} \bar{N}(r, f - F_i) + O(\log r), \quad \alpha > 0$$

$$> \left(\frac{3}{\varrho} + \alpha\right) T(r, f) + 0(\log r)$$

Therefore

$$\text{(6.2)} \qquad \liminf_{r \to \infty} \frac{1}{T(r, f)} \int_{r_0}^{r} \frac{T(x, f)}{x}\, dx \geq \frac{1}{\varrho} + \frac{\alpha}{3} > \frac{1}{\varrho}.$$

Let

$$\phi(r) = \log \int_{r_0}^{r} (T(x, f)/x)\, dx\,; \; \psi(r) = \log r$$

Then

$$\phi(r) < (\varrho + \varepsilon)(1 + o(1)) \log r, \; (r \geq r_0)$$

and therefore

$$\limsup_{r \to \infty} \{\phi(r)/\psi(r)\} \leq \varrho.$$

Hence

$$\limsup_{r \to \infty} \{\phi'(r)/\psi'(r)\} \geq \varrho,$$

that is

$$\text{(6.3)} \qquad \limsup_{r \to \infty} \frac{T(r, f)}{\int_{r_0}^{r} (T(r, f)/x)\, dx} \geq \varrho.$$

But (6.3) is inconsistant with (6.2) and therefore the result is proved.

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Department of Mathematics,
Indian Institute of Technology,
Kanpur 208016, India.