FINITE DIMENSIONALITY AND DUALITY OF B*-ALGEBRAS

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The purpose of this note is to give alternate proofs of the main theorems (Theorem 3.1, p. 58 and Theorem 4.1, p. 59) obtained by Tomiuk [9]. The proof of these theorems depends heavily on Lemma 3.1 [9, p. 58] whose construction is very technical in nature. Our approach avoids the use of Lemma 3.1 cited above and uses only the well-known techniques of modular annihilator algebras developed by Barnes [1] and Yood [10]. The proof of Theorem 4.1 [9, p. 59] is also considerably simplified. We shall follow the notations and terminology of Rickart [8] and Tomiuk [9] without any further comments. For any subset S of an algebra A, we define $L(S) = \{x \in A : xS = (0)\}$ and $R(S) = \{x \in A : Sx = (0)\}$. Following Yood [10] (see also Barnes [1]) we say that an algebra A is a modular annihilator algebra if for every maximal modular left ideal M and every maximal modular right ideal N of A, (i) $R(M) \neq (0)$ and $R(A) = (0)$, and (ii) $L(N) \neq (0)$ and $L(A) = (0)$.

We shall consider the following theorems.

Theorem 1. Let A be a weakly completely continuous B*-algebra with identity. Then A is finite dimensional.

Proof. It is proved in [6; Theorem 8, p. 24] that A is dual and consequently a modular annihilator algebra. Observe that A is semisimple. The finite dimensionality of A then follows from [1; Proposition 6,3, p. 573].

Corollary. A reflexive weakly completely continuous B*-algebra is finite dimensional.

Proof. Since A is reflexive, therefore by [2, Lemma 3.8, p. 855], A has an identity element. The conclusion now follows from Theorem 1.

THEOREM 2. Let A be a B*-algebra. Then the following statements are equivalent:

(1) A is a dual algebra,

(2) for every maximal modular left ideal M of A there exists a right identity modulo M that is weakly completely continuous.

PROOF. $1 \Rightarrow 2$. The proof follows from [7; Theorem 2.1, p. 906].

$2 \Rightarrow 1$. Let A be a B^*-algebra with norm $||\cdot||$ and let u be the right identity modulo M such that u is weakly completely continuous and $A\ (1 - u) \subset M$. Since A has an approximate identity of norm 1, for an element $v \in A$ the operator norm of left multiplication by v on A is just $||v||$. Then it follows from [4; Corollary 6, p. 484] that the set J of weakly completely continuous elements of A is a closed two-sided ideal of A. If J is proper, then by [3, Théorème 2.9.5 (iii), p. 48], J is contained in some maximal modular left ideal of M of A. By hypothesis, there exists $u \in J$ such that $A\ (1 - u) \subset M$. But then $u \in J \subset M$, so that $A \subset M$, a contradiction. Therefore $J = A$. In other words, A is weakly completely continuous B*-algebra. The desired implication now follows from [6; Theorem 8, p. 24].

REMARK. A Banach algebra is said to be completely continuous if its left and right regular representations consist of completely continuous (compact) operators; see Rickart [8, p. 284]. Let A be a semi-simple completely continuous Banach *-algebra in which $x^*\ x = 0$ implies $x = 0$. It follows from [1; Theorem 7.2, p. 576 and Corollary p. 568] and the proof of Theorem 4.10.11 [8, p. 266] that A is symmectric. The question arises as to whether or not a semi-simple weakly completely continuous Banach *-algebra with the property that $x^*\ x = 0$ implies $x = 0$, is symmetric. The answer to this question is in the negative for if G is any discrete locally compact group, then the group algebre $L^1\,(G)$ is weakly completely continuous. But $L^1\,(G)$ need not be symmetric; see [5].

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