ON GENERALIZATIONS OF SOME RESULTS AND THEIR APPLICATIONS

By

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ABSTRACT

In the earlier portion of this paper, the author has derived two integrals involving generalized MEIJER functions of two variables which further evaluate the contour integrals by applying inversion theorems of LAPLACE and MELLIN transforms. Its mid-portion deals with the evaluation of an integral on generalized hypergeometric function and generalized MEIJER function in two arguments. Lastly, a result on an integral for the product of two generalized hypergeometric and JACOBI polynomials has evaluated and its result has been further utilized to establish an expansion formula. All these results incorporate generalizations of many known, new and interesting particular cases scattered throughout literature in special-functions.

1. INTRODUCTION

Generalized MEIJER Function of Two Variables

A generalization of MEIJER's G-function [(4), p. 207, (1)] has been introduced by SHARMA [(9), pp. 26-40] as follows

$$(1.1) \qquad S\begin{bmatrix} x \\ y \end{bmatrix} \equiv S\left[\begin{array}{c|c|c} \begin{bmatrix} b \;, & 0 \\ A - p, & B \end{bmatrix} & (a) \;;\; (b) & \\ \begin{pmatrix} q \;, & r \\ C-q, & D-r \end{pmatrix} & (c) \;;\; (d) & \begin{matrix} x \\ y \end{matrix} \\ \begin{pmatrix} k \;, & l \\ E-k, & F-l \end{pmatrix} & (e) \;;\; (f) & \end{array}\right] =$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi\,(\xi + \eta)\ \Psi\,(\xi, \eta)\ x^{\xi}\, y^{\eta}\, d\xi\ d\eta$$

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where

(i) A, B, C etc., are non-negative integers and satisfy the following inequalities:

$$D \geqslant 1,\ F \geqslant 1,\ A \geqslant 1,\ B \geqslant 1,\ o \leqslant p \leqslant A,\ o \leqslant q \leqslant C,\ o \leqslant k \leqslant E,$$
$$o \leqslant r \leqslant D, o \leqslant l \leqslant F,\ A + C \leqslant B + D \text{ and } A + E \leqslant B + F;$$

(ii) notation (a) stands for the set of A-parameters:

$$a_1, a_2, \ldots, a_p;\ a_{p+1}, \ldots, a_A$$

with similar interpretations for (b), (c), (d), (e) and (f);

(iii) L_1 and L_2 are suitable contours;

$$\text{(iv)} \begin{cases} \Phi(\xi+\eta) = \dfrac{\prod_{j=1}^{p} \Gamma(a_j + \xi + \eta)}{\prod_{j=p+1}^{A} \Gamma(1 - a_j - \xi - \eta) \prod_{j=1}^{B} \Gamma(b_j + \xi + \eta)}, \\ \Psi(\xi, \eta) = \dfrac{\prod_{j=1}^{q} \Gamma(1 - c_j + \xi) \prod_{j=1}^{r} \Gamma(d_j - \xi) \prod_{j=1}^{k} \Gamma(1 - e_j + \eta) \prod_{j=1}^{l} \Gamma(f_j - \eta)}{\prod_{j=q+1}^{C} \Gamma(c_j - \xi) \prod_{j=r+1}^{D} \Gamma(1 - d_j + \xi) \prod_{j=k+1}^{E} \Gamma(e_j - \eta) \prod_{j=l+1}^{F} \Gamma(1 - f_j + \eta)} \end{cases}$$

and the double integral converges if

$$\text{(v)} \begin{cases} 2(p + q + r) > A + B + C + D, \\ 2(p + k + l) > A + B + E + F, \\ |\arg(x)| < [p + q + r - 1/2\,(A + B + C + D)]\,\pi, \\ |\arg(y)| < [p + k + l - 1/2\,(A + B + E + F)]\,\pi, \\ \qquad\qquad \text{or} \\ A + C < B + D,\ A + E < B + F, \\ \text{or else } A + C = B + D,\ A + E = B + F \text{ with } |x| < 1, |y| < 1. \end{cases}$$

KAMPÉ DE FÉRIET'S FUNCTION

The double hypergeometric function of higher order has been studied by KAMPÉ DE FÉRIET, J., [(1), p. 150]:

$$(1.2)\qquad F\left[\begin{array}{c|c|c} m & a_m & \\ l & b_l ; b_l' & x \\ n & c_n & y \\ p & d_p ; d_p' & \end{array}\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\prod_{j=1}^{m}(a_j)_{r+s} \prod_{j=1}^{l}\{(b_j)_r (b_j')_s\}\, x^r y^s}{r!\, s!\, \prod_{j=1}^{n}(c_j)_{r+s} \prod_{j=1}^{p}\{(d_j)_r (d_j')_s\}}$$

where, for absolute convergence of the series, $|x| < 1, |y| < 1$; $m + l \leqslant n + p + 1$ and a_m represents the sequence $a_1, a_2, \ldots, a_m$, $\prod_{j=1}^{m}(a_j)_r$ stands for the product $(a_1)_r (a_2)_r \ldots (a_m)_r$ and so on.

By appropriately specializing the parameters, KAMPÉ DE FÉRIET'S function (1.2) can be reduced to APPELL'S functions [(1)] F_1, F_2, F_3 and F_4, WHITTAKER'S function of two variables, etc. and also the generalized hypergeometric function of one variable.

R-FUNCTION

AL-SALAM W. A. and CARLITZ L. [(2), p. 911, (1.1)] have discussed some properties of the R-function defined in the form

$$(1.3)\qquad R(\lambda, \mu, \nu, x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda + n + 1)_n\, x^n}{n!\, \Gamma(\mu + n + 1)\, \Gamma(\nu + n + 1)}.$$

This function was suggested by the well-known formula for the product of two BESSEL-function $J_\mu(x)\, J_\nu(x)$; they have also given [(2), p. 911, (1.2) and (1.3)]:

$$(1.4)\qquad J_\mu(2x)\, J_\nu(2x) = x^{\mu+\nu} R(\mu + \nu, \mu, \nu, x^2),$$

$$(1.5)\qquad R(2\mu, \mu, \mu - 1/2, x^2) = \frac{x^{-2\mu}}{\sqrt{\pi}} J_{2\mu}(4x)$$

and the expansion-formula [(2], p. 912]:

$$(1.6)\qquad \Gamma\left[\begin{matrix}\lambda + 1; \\ \mu + 1, \nu + 1\end{matrix}\right] {}_pF_{q+1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p \\ \mu + 1, \nu + 1, \beta_1, \ldots, \beta_{q-1}\end{matrix}; -4xy\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda + 2n)\, \Gamma(\lambda + n)}{n!} x^n R(\lambda + 2n, \mu + n, \nu + n, x)$$

$${}_{p+2}F_{q+1}\left[\begin{matrix}-n, \lambda + n, \quad \alpha_1, \ldots, \alpha_p; y \\ 1/2(\lambda + 1), 1/2(\lambda + 2), \beta_1, \ldots, \beta_{q-1}\end{matrix}\right]$$

where $\Gamma\begin{bmatrix} a_p; \\ b_q \end{bmatrix}$ denotes the product of the type

$$\frac{\Gamma(a_1)\,\Gamma(a_2)\,...\,\Gamma(a_p)}{\Gamma(b_1)\,\Gamma(b_2)\,...\,\Gamma(b_q)}$$

GENERALIZED HYPERGEOMETRIC POLYNOMIAL

Recently, the autor has made an attempt in the unification and extension of most of the well-known sets of polynomials which have attracted considerable attention in different applied field by introducing a more generalized hypergeometric polynomial [(10), p. 79, (2.1)] in the form

$$(1.7)\qquad F_n(x) = x^{(\delta-1)n}\,{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\, a_p; \\ b_q\end{matrix}\; \lambda x^c\right]$$

where δ and n are positive integers, and

$$\Delta(\delta,-n) = \frac{-n}{\delta},\ \frac{-n+1}{\delta},\ ...,\ \frac{-n+\delta-1}{\delta};$$

a_p (b_q) for p (q) parameters $a_1, ..., a_p$ $(b_1, ..., b_q)$.

The polynomial (1.7) is terminating and unrestricted. The terminating nature of the polynomial is governed by the numerator parameters $\Delta(\delta, -n)$. The parameters a_p and b_q are all independent of x but these can be functions of x such that the polynomial always remains well defined.

The present paper has been mainly divided into three parts. In the first part, the integrals involving generalized MEIJER function of two variables have derived. These integrals have been further used to evaluate the contour integrals with the applications on inversion theorems of LAPLACE and MELLIN transforms. In the second part, an integral for generalized hypergeometric function and extended MEIJER'S G-function in two arguments has been evaluated which includes several results on images of some functions in various transforms. In the third part, the result on integral for the product of two generalized hypergeometric and JACOBI polynomials has obtained. An expansion-formula for the product of two polynomials has been further established with the help of this integral and orthogonality property of the Jacobi polynomials.

The main object of this paper is to establish some generalized formulae necessary for the unification and extension for a variety of familiar results appearing in the theory of special-functions which may also prove to be useful in the analysis of certain problems in mathematical physics.

2. *Some Results on Integrals involving Generalized Functions of Two Variables.*

Here we shall first evaluate the two integrals involving the extended MEIJER'S G-functions of two variables on basis of interchan ging the order of integration and term-by-term integration with the help of known EULERIAN integrals of the first and second kinds.

INTEGRALS

First Integral:

$$(2.1)\quad \int_0^\infty x^{\lambda-1} e^{-\mu x} S\left[\begin{array}{c|c|c} \begin{bmatrix} p\ ,\ o \\ A-p\ ,B \end{bmatrix} & (a)\ ;\ (b) & \alpha x^{\varrho} \\ \begin{pmatrix} q\ ,\ r \\ C-q\ ,D-r \end{pmatrix} & (c)\ ;\ (d) & \beta x^{\varrho} \\ \begin{pmatrix} k\ ,\ l \\ E-k\ ,\ F-l \end{pmatrix} & (e)\ ;\ (f) & \end{array}\right] dx$$

$$= (2\pi)^{1/2(1-\varrho)} \frac{\varrho^{\lambda-1/2}}{\mu^{\lambda}} S\left[\begin{array}{c|r|c} \begin{bmatrix} p+\varrho\ ,o \\ A-p\ ,B \end{bmatrix} & \Delta(\varrho,\lambda),(a)\ ;\ (b) & \dfrac{\alpha\,\varrho^{\varrho}}{\mu^{\varrho}} \\ \begin{pmatrix} q\ ,\ r \\ C-q\ ,D-r \end{pmatrix} & (c)\ ;\ (d) & \\ \begin{pmatrix} k\ ,\ l \\ E-k\ ,F-l \end{pmatrix} & (e)\ ;\ (f) & \dfrac{\beta\varrho^{\varrho}}{\mu^{\varrho}} \end{array}\right]$$

provided ϱ is a positive integer $> o$, $Re\,(\mu) > o$ and conditions of validity:

$$
\text{(i)} \quad \begin{cases} 2\,(p+q+r) > A+B+C+D, \\ 2\,(p+k+l) > A+B+E+F, \\ |\arg(\alpha)| < [p+q+r-1/2\,(A+B+C+D)]\,\pi, \\ |\arg(\beta)| < [p+k+l-1/2\,(A+B+E+F)]\,\pi, \\ Re\,[\lambda+\varrho d_{h_1}+\varrho f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l \end{cases}
$$

or

$$
\text{(ii)} \quad \begin{cases} A+C < B+D,\ A+E < B+F, \\ \text{or else } A+C = B+D,\ A+E = B+F \text{ with } |\alpha| < 1,\ |\beta| < 1, \\ Re\,[\lambda+\varrho d_{h_1}+\varrho f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l. \end{cases}
$$

Proof: —

On substituting the contour integral (1.1) for $S\begin{bmatrix} \alpha x^{\varrho} \\ \beta x^{\varrho} \end{bmatrix}$ in the integrand of (2.1), interchange the order of integration, we have

$$
(2.2) \quad \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Phi(\xi+\eta)\,\Psi(\xi,\eta)\left\{\int_0^{\infty} e^{-\mu x}\,x^{\lambda+\varrho\xi+\varrho\eta-1}\,dx\right\}\alpha^{\xi}\,\beta^{\eta}\,d\xi\,d\eta.
$$

Regarding the interchange of the order of integration, it is observed that the x-integral is absolutely convergent if $Re\,(\lambda+\varrho\xi+ +\varrho\eta) > o$, $Re\,(\mu) > o$, the double contour integral converges absolutely under the conditions referred to earlier, and convergence of the repeated integral follows from that of integral (2.1). Hence the interchange of the order of integration is justified.

Now interpret the x-integral by making use of the known EULERIAN integral [(4), p. 12, (33)]:

$$
\int_0^{\infty} e^{-st}\,t^{\alpha-1}\,dt = \frac{\Gamma(\alpha)}{s^{\alpha}},\quad Re\,(\alpha) > o,\ Re\,(s) > o,
$$

and GAUSS's multiplication theorem [(4), p. 4, (11)]:

$$
(2.3) \quad \Gamma\,(mz) = (2\pi)^{\frac{1}{2}(1-m)}\,m^{mz-1/2}\prod_{i=0}^{m-1}\Gamma\left(z+\frac{i}{m}\right),\ m = 2, 3, 4, \ldots,
$$

we obtain

$$(2.4) \qquad (2\pi)^{\frac{1}{2}(1-\varrho)} \frac{\varrho^{\lambda-1/2}}{\mu^{\lambda}}$$

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Phi(\xi+\eta)\,\Psi(\xi,\eta)\prod_{i=0}^{\varrho-1}\Gamma\left(\frac{\lambda+i}{\varrho}+\xi+\eta\right)\frac{\varrho^{\varrho\xi}}{\mu^{\varrho\xi}}\,\frac{\varrho^{\varrho\eta}}{\mu^{\varrho\eta}}\,\alpha^{\xi}\,\beta^{\eta}\,d\xi\,d\eta$$

where the contour L_1 is in the ξ-plane and runs from $-i\infty$ *to* $+i\infty$ with loops to ensure, if necessary, that the poles of $\Gamma(d_j-\xi)$, $j = 1, 2, \ldots, r$ lie to the right and the poles of $\Gamma(1-c_j+\xi)$, $j = 1, 2, \ldots, q$ and $\Gamma(a_j+\xi+\eta)$, $j = 1, 2, \ldots, p$, $\Gamma\left(\frac{\lambda+i}{\varrho}+\xi+\eta\right)$, $\{i = o, 1, \ldots, (\varrho-1)\}$ to the left of the contour.

Similarly the contour L_2 in the η-plane consists of the portion of the imaginary axis from $-i\infty$ *to* $+i\infty$ along with the necessary loops so as to ensure that the poles of $\Gamma(f_j-\eta)$, $j = 1, 2, \ldots, l$, lie to the right and the poles of $\Gamma(1-e_j+\eta)$, $j = 1, 2, \ldots, k$ and $\Gamma(a_j + \xi+\eta)$, $j = 1, 2, \ldots, p$, $\Gamma\left(\frac{\lambda+i}{\varrho}+\xi+\eta\right)$, $\{i = o, 1, \ldots, (\varrho-1)\}$ to the left of the contour.

Therefore, on interpreting (2.4) in view of (1.1), we get the value of (2.1).

Second Integral:

$$(2.5) \quad \int_0^{\infty}\frac{x^{\lambda-1}}{(1+x)^{\mu}} S\left[\begin{array}{c|c|c} \begin{bmatrix} p\ ,\ o \\ A-p\ ,\ B \end{bmatrix} & (a)\ ;\ (b) & \alpha\left\{\dfrac{x}{1+x}\right\}^{\varrho} \\ \begin{pmatrix} q\ ,\ r \\ C-q\ ,D-r \end{pmatrix} & (c)\ ;\ (d) & \\ \begin{pmatrix} k\ ,\ l \\ E-k\ ,F-l \end{pmatrix} & (e)\ ;\ (f) & \beta\left\{\dfrac{x}{1+x}\right\}^{\varrho} \end{array}\right] dx$$

$$= \frac{\Gamma(\mu-\lambda)}{\varrho^{\mu-\lambda}} S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\varrho\ ,\ o \\ A-p\ ,B+\varrho \end{bmatrix} & \Delta(\varrho,\lambda),(a)\ ;\ \Delta(\varrho,\mu),(b) & \\ \begin{pmatrix} q\ ,\ r \\ C-q\ ,\ D-r \end{pmatrix} & (c)\ ;\ (d) & \begin{array}{c}\alpha \\ \beta\end{array} \\ \begin{pmatrix} k\ ,\ l \\ E-k\ ,\ F-l \end{pmatrix} & (e)\ ;\ (f) & \end{array}\right]$$

where, for convergence, $Re\,(\mu - \lambda) > o$ and

$$\text{(i)} \quad \begin{cases} 2\,(p + q + r) > A + B + C + D \\ 2\,(p + k + l) > A + B + E + F, \\ |arg\,(\alpha)| < [p + q + r - 1/2\,(A + B + C + D)]\,\pi, \\ |arg\,(\beta)| < [p + k + l - 1/2\,(A + B + E + F)]\,\pi, \\ Re\,[\lambda + \varrho d_{h_1} + \varrho f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l \end{cases}$$

or

$$\text{(ii)} \quad \begin{cases} A + C < B + D,\ A + E < B + F, \\ \text{or else } A + C = B + D,\ A + E = B + F \text{ with } |\alpha| < 1,\ |\beta| < 1, \\ Re\,[\lambda + \varrho\, d_{h_1} + \varrho f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l, \end{cases}$$

ϱ being a positive integer $> o$.

Proof: —

To prove (2.5), we express $S \begin{bmatrix} \alpha \left(\dfrac{x}{1+x}\right)^{\varrho} \\ \beta \left(\dfrac{x}{1+x}\right)^{\varrho} \end{bmatrix}$ in its integrand by means of the double contour integral (1.1) and invert the order of integration which can readily be justified by DE LA VALLÉE POUSSIN theorem [(3), p. 504)] in view of the conditions stated in (2.5), we get

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi\,(\xi + \eta)\ \Psi\,(\xi, \eta) \left\{ \int_0^\infty \frac{x^{\lambda + \varrho\xi + \varrho\eta - 1}}{(1 + x)^{\mu + \varrho\xi + \varrho\eta}}\, dx \right\} \alpha^{\xi}\, \beta^{\eta}\, d\xi\, d\eta.$$

Now invoking the x-integral using [(4), p. 9, (2)]:

$$\int_0^\infty \frac{x^{p-1}}{(1 + x)^{p+q}}\, dx = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p + q)},\ Re\,(p) > o,\ Re\,(q) > o,$$

and (2.3), we obtain

(2.6)
$$\frac{\Gamma\,(\mu - \lambda)}{\varrho^{\mu - \lambda}} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(\xi + \eta)\ \Psi(\xi, \eta) \left\{ \frac{\prod_{i=0}^{\varrho - 1} \Gamma\left(\dfrac{\lambda + i}{\varrho} + \xi + \eta\right)}{\prod_{i=0}^{\varrho - 1} \Gamma\left(\dfrac{\mu + i}{\varrho} + \xi + \eta\right)} \right\} \alpha^{\xi} \beta^{\eta} d\xi\, d\eta$$

which yields the value of integral (2.5) on applying (1.1).

Applications of Inversion Theorems on LAPLACE and MELLIN Transforms

By applying inversion theorems of LAPLACE and MELLIN transforms [(5), p. 129, (1), p. 307, (1)] in (2.1) and (2.5) respectively the two other contour integrals can be obtained as

$$(2.7)\qquad x^{\lambda-1}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p\,, & o \\ A-p\,, & B \end{bmatrix} & (a)\,;\,(b) & \alpha\, x^{\varrho} \\ \begin{pmatrix} q\,, & r \\ C-q\,, & D-r \end{pmatrix} & (c)\,;\,(d) & \\ \begin{pmatrix} k\,, & l \\ E-k\,, & F-l \end{pmatrix} & (e)\,;\,(f) & \beta\, x^{\varrho} \end{array}\right]$$

$$= (2\pi)^{\frac{1}{2}(1-\varrho)}\,\varrho^{\lambda-1/2}\,\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu^{-\lambda}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\varrho\,, & o \\ A-p\,, & B \end{bmatrix} & \Delta(\varrho,\lambda),(a);(b) & \frac{\alpha\varrho^{\varrho}}{\mu^{\varrho}} \\ \begin{pmatrix} q\,, & r \\ C-q\,, & D-r \end{pmatrix} & (c)\,;\,(d) & \\ \begin{pmatrix} k\,, & l \\ E-k\,, & F-l \end{pmatrix} & (e)\,;\,(f) & \frac{\beta\varrho^{\varrho}}{\mu^{\varrho}} \end{array}\right] e^{\mu x}\, d\eta$$

and

$$(2.8)\qquad \frac{1}{(1+x)^{\varrho}}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p\,, & o \\ A-p\,, & B \end{bmatrix} & (a)\,;\,(b) & \alpha\left\{\frac{x}{1+x}\right\}^{\varrho} \\ \begin{pmatrix} q\,, & r \\ C-q\,, & D-r \end{pmatrix} & (c)\,;\,(d) & \beta\left\{\frac{x}{1+x}\right\}^{\varrho} \\ \begin{pmatrix} k\,, & l \\ E-k\,, & F-l \end{pmatrix} & (e)\,;\,(f) & \end{array}\right]$$

$$= \frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\mu-\lambda)}{\varrho^{\mu-\lambda}}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\varrho\,, & o \\ A-p\,, & B+\varrho \end{bmatrix} & \Delta(\varrho,\lambda)(a);\,\Delta(\varrho,\mu),(b) & \\ \begin{pmatrix} q\,, & r \\ C-q\,, & D-r \end{pmatrix} & (c)\,;\,(d) & \begin{matrix}\alpha\\ \beta\end{matrix} \\ \begin{pmatrix} k\,, & l \\ E-k\,, & F-l \end{pmatrix} & (e)\,;\,(f) & \end{array}\right] x^{-\lambda} d\lambda$$

where ϱ is a positive integer $> o$, $Re(\mu) > o$, $Re(\mu - \lambda) > o$, and conditions of validity:

$$\text{(i)} \begin{cases} 2\,(p + q + r) > A + B + C + D, \\ 2\,(p + k + l) > A + B + E + F, \\ |\arg(\alpha)| < [p + q + r - 1/2\,(A + B + C + D)]\pi, \\ |\arg(\beta)| < [p + k + l - 1/2\,(A + B + E + F)]\pi, \\ Re\,[\lambda + \varrho\, d_{h_1} + \varrho\, f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l \end{cases}$$

or

$$\text{(ii)} \begin{cases} A + C < B + D,\ A + E < B + F, \\ \text{or else } A + C = B + D,\ A + E = B + F \text{ with } |\alpha| < 1,\ |\beta| < 1, \\ Re\,[\lambda + \varrho\, d_{h_1} + \varrho\, f_{h_2}] > o,\ h_1 = 1, 2, \ldots, r;\ h_2, \ldots, l. \end{cases}$$

3. *A Result on Generalized Hypergeometric Function and Generalized* MEIJER *Function of Two Variables.*

In this section, an integral on generalized hypergeometric function and generalized MEIJER function in two arguments has been evaluated by using known expansion and integral results.

The formula to be proved is:

$$(3.1)\quad \int_0^\infty x^{u+v}\ {}_mF_{n+1}\left[\begin{matrix}\alpha_m \\ \Delta(2, u+1), \beta_{n-1}\end{matrix};\ -4x^2yz^2\right] S\left[\begin{matrix}\alpha\, x^{2\varrho} \\ \beta\, x^{2\varrho}\end{matrix}\right] dx$$

$$= \frac{\varrho^v}{(2z)^{u+v+1}} \sum_{\sigma=0}^{\infty} \frac{(\tfrac{1}{2}u + \sigma)\,\Gamma(u+\sigma)}{\sigma!}\ {}_{m+2}F_{n+1}\left[\begin{matrix}-\sigma, u+\sigma, \alpha_m;\ y \\ \Delta(2, u+1), \beta_{n-1}\end{matrix}\right]$$

$$S\left[\begin{matrix}\left[\begin{matrix}p+\varrho, & o \\ A-p+\varrho, & B\end{matrix}\right] \\ \left(\begin{matrix}q, & r \\ C-q, & D-r\end{matrix}\right) \\ \left(\begin{matrix}k, & l \\ E-k, & F-l\end{matrix}\right)\end{matrix}\ \middle|\ \begin{matrix}\Delta(\varrho, \tfrac{1}{2}v + 1/2\,(u+2\sigma) + 1/2), (a), \\ \Delta(\varrho, \tfrac{1}{2}v - 1/2\,(u+2\sigma) + 1/2); (b) \\ (c); (d) \\ (e); (f)\end{matrix}\ \middle|\ \begin{matrix}\alpha\left(\dfrac{\varrho}{2z}\right)^{2\varrho} \\ \beta\left(\dfrac{\varrho}{2z}\right)^{2\varrho}\end{matrix}\right]$$

where ϱ is a positive integer, $z > o$ and valid under the conditions:

$$
\text{(i)} \begin{cases} 2(p+q+r) > A+B+C+D, \\ 2(p+k+l) > A+B+E+F, \\ |\arg(\alpha)| < [p+q+r-1/2(A+B+C+D)]\pi, \\ |\arg(\beta)| < [p+k+l-1/2(A+B+E+F)]\pi, \\ -Re(u+2\sigma) - 3/2 < Re[v+2\varrho(d_{h_1}+f_{h_2})] < -1/2, \\ (h_1 = 1, 2, \ldots, r;\ h_1 = 1, 2, \ldots, l) \end{cases}
$$

or

$$
\text{(ii)} \begin{cases} A+C < B+D,\ A+E < B+F, \\ \text{or else } A+C = B+D,\ A+E = B+F \text{ with } |\alpha| < 1,\ |\beta| < 1, \\ -Re(u+2\sigma) - 3/2 < Re[v+2\varrho(d_{h_1}+f_{h_2})] < -1/2, \\ (h_1 = 1,2, \ldots, r;\ h_2 = 1, 2, \ldots, l). \end{cases}
$$

Proof: —

Firstly, in view of (1.5) and using LEGENDRE's duplication formula [(4), p. 5, (15)]:

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2),$$

we reduce (1.6) to the following form

$$
\text{(3.2)} \qquad x^u\, {}_mF_n + 1 \left[\begin{matrix} \alpha_m \\ \Delta(2, u+1), \beta_{n-1} \end{matrix} ; -4x^2yz^2 \right]
$$

$$
= (2z)^{-u} \sum_{\sigma=0}^{\infty} \frac{(u+2\sigma)\Gamma(u+\sigma)}{\sigma!} J_{u+2\sigma}(4xz)\, {}_{m+2}F_{n+1} \left[\begin{matrix} -\sigma, u+\sigma, \alpha_m; y \\ \Delta(2, u+1), \beta_{n-1} \end{matrix} \right].
$$

Multiplying both sides of (3.2) by $x^v S \left[\begin{matrix} \alpha x^{2\varrho} \\ \beta x^{2\varrho} \end{matrix} \right]$ and integrating with respect to x over (o, ∞), then changing the order of integration and summation on the right, we have

$$
\text{(3.3)} \int_0^\infty x^{u+v}\, {}_mF_{n+1} \left[\begin{matrix} \alpha_m \\ \Delta(2, u+1), \beta_{n-1} \end{matrix} ; -4x^2yz^2 \right] S \left[\begin{matrix} \alpha x^{2\varrho} \\ \beta x^{2\varrho} \end{matrix} \right] dx
$$

$$
= (2z)^{-u} \sum_{\sigma=0}^{\infty} \frac{(u+2\sigma)\Gamma(u+\sigma)}{\sigma!}\, {}_{m+2}F_{n+1} \left[\begin{matrix} -\sigma, u+\sigma, \alpha_m; y \\ \Delta(2, u+1), \beta_{n-1} \end{matrix} \right]
$$

$$\int_0^\infty x^v J_{u+2\sigma}(4xz)\, S\begin{bmatrix}\alpha x^{2\varrho}\\ \beta x^{2\varrho}\end{bmatrix} dx.$$

The change of order of integration and summation involved here is justified by the application of DE LA VALLÉE POUSSION's theorem [(3), p. 500] for the conditions given with the result (3.1) when

$$\text{(i)}\ \sum_{\sigma=0}^{\infty} \frac{(u+2\sigma)\,\Gamma(u+\sigma)}{\sigma!} J_{u+2\sigma}(4xz)\ {}_{m+2}F_{n+1}\begin{bmatrix}-\sigma,\ u+\sigma,\ \alpha_m;\ y\\ \Delta(2,\ u+1),\ \beta_{n-1}\end{bmatrix}$$

converges unfiformly for $o \leqslant x \leqslant \xi$, where ξ is arbitrary;

(ii) generalized MEIJER function in two variables is a continuous function of x and y for all finite values of $x \geqslant x_0 \geqslant o$ and $y \geqslant y_0 \geqslant o$;

and

(iii) the integral on the right converges under the conditions given below in (3.4).

Now making use of the known integral [(12), p. 97, (2.3)]:

$$\text{(3.4)}\qquad \int_0^\infty x^\mu J_\nu(xy)\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p,\ o\\ A-p,\ B\end{bmatrix} & (a)\ ;\ (b) & \\ \begin{pmatrix} q\ ,\ r\\ C-q\ ,\ D-r\end{pmatrix} & (c)\ ;\ (d) & \begin{array}{c}\alpha x^{2\varrho}\\ \beta x^{2\varrho}\end{array} \\ \begin{pmatrix} k\ ,\ l\\ E-k\ ,\ F-l\end{pmatrix} & (e)\ ;\ (f) & \end{array}\right] dx$$

$$= (2\varrho)^\mu y^{-\mu-1}\, S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\varrho,\ o\\ A-p+\varrho,\ B\end{bmatrix} & \begin{array}{c}\Delta(\varrho, \tfrac{1}{2}\mu+\tfrac{1}{2}\nu+1/2),\ (a),\\ \Delta(\varrho, \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+1/2);\ (b)\end{array} & \alpha\left(\dfrac{2\varrho}{y}\right)^{2\varrho} \\ \begin{pmatrix} q\ ,\ r\\ C-q\ ,\ D-r\end{pmatrix} & (c)\ ;\ (d) & \\ \begin{pmatrix} k\ ,\ l\\ E-k\ ,\ F-l\end{pmatrix} & (e)\ ;\ (f) & \beta\left(\dfrac{2\varrho}{y}\right)^{2\varrho} \end{array}\right]$$

where ϱ is a positive integer, $y > o$ and valid under the following conditions of convergence:

(i) $\begin{cases} 2(p+q+r) > A+B+C+D, \\ 2(p+k+l) > A+B+E+F, \\ |\arg(\alpha)| < [p+q+r-1/2(A+B+C+D)]\pi, \\ |\arg(\beta)| < [p+k+l-1/2(A+B+E+F)]\pi, \\ -Re(\nu)-3/2 < Re[\mu+2\varrho(d_{h_1}+f_{h_2})] < -1/2, \\ (h_1=1,2,\ldots,r;\ h_2=1,2,\ldots,l) \end{cases}$

or

(ii) $\begin{cases} A+C < B+D,\ A+E < B+F, \\ \text{or else } A+C=B+D, A+E=B+F \text{ with } |\alpha|<1, |\beta|<1, \\ -Re(\nu)-3/2 < Re[\mu+2\varrho(d_{h_1}+f_{h_2})] < -1/2, \\ (h_1=1,2,\ldots,r;\ h_2=1,2,\ldots,l), \end{cases}$

on the right of (3.3) which completes the evaluation of (3.1).

4. *Results on Generalized Hypergeometric and* JACOBI *Polynomials*

Firstly we shall evaluate here an integral for product of two generalized hypergeometric and JACOBI polynomials in terms of KAMPÉ DE FÉRIET'S function on the procedure based on interchanging the order of integration and summation, then term-by-term integration with the help of known integral and the result will be further employed to establish an expansion formula for the product of two generalized hypergeometric polynomials in series of KAMPÉ DE FÉRIET'S functions and JACOBI polynomials using orthogonality-property.

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The main result:

(4.1)

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\left\{{}_{p+1}F_q\left[\begin{matrix}-l, a_p \\ b_q\end{matrix}; \mu x\right]{}_{p+1}F_q\left[\begin{matrix}-m, A_p \\ B_q\end{matrix}; \lambda x\right]\right\}dx$$

$$= \frac{(\alpha-\xi)_n}{n!}\Gamma\left[\begin{matrix}\beta+n+1, \xi+1; \\ \beta+\xi+n+2\end{matrix}\right] F\left[\begin{matrix}2 \\ p+1 \\ 2 \\ q\end{matrix}\middle|\begin{matrix}\xi+1, -\alpha+\xi+1 \\ -m, A_p; -l, a_p \\ -\alpha+\xi-n+1, \beta+\xi+n+2 \\ \beta_q; b_q\end{matrix}\middle|\begin{matrix}\lambda \\ \\ \mu\end{matrix}\right]$$

where $Re(\beta) > -1$, $Re(\xi) > -1$.

Proof: —

In the known result due to SHAH [(11), p. 100, (4.1)], setting $\lambda = c = 1$ and $m = o$, we have

$$(4.2) \qquad \int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\left\{ {}_{p+1}F_q\left[\begin{matrix} -l,\ a_p \\ b_q \end{matrix};\ \mu x\right]\right\} dx$$

$$= \frac{(\alpha-\xi)_n}{n!}\,\Gamma\left[\begin{matrix}\beta+n+1,\ \xi+1; \\ \beta+\xi+n+2\end{matrix}\right] {}_{p+3}F_{q+2}\left[\begin{matrix} -l,\ \xi+1,\ -\alpha+\xi+1,\ a_p;\ \mu \\ -\alpha+\xi-n+1, \beta+\xi+n+2, b_q\end{matrix}\right]$$

where, for convergence, $Re(\beta) > -1$, $Re(\xi) > -1$.

On substituting the series for the polynomial ${}_{p+1}F_q[\lambda x]$ in (4.1), then changing the order of integration and summation which is easily seen to be justified due to the absolute convergence of the integral and summation involved in the process, we obtain

$$(4.3) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r\,(A_p)_r\,\lambda^r}{r!\,(B_q)_r} \int_0^1 x^{\xi+r}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)$$

$$\left\{ {}_{p+1}F_q\left[\begin{matrix} -l,\ a_p \\ b_q \end{matrix};\ \mu x\right]\right\} dx.$$

Now evaluating the integral of (4.3) by using (4.2), we get

$$(4.4) \qquad \frac{(\alpha-\xi)_n}{n!}\,\Gamma\left[\begin{matrix}\beta+n+1,\ \xi+1; \\ \beta+\xi+\eta+2\end{matrix}\right] \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\xi+1)_{r+s}}{r!\,s!\,(1-\alpha+\xi-n)_{r+s}}$$

$$\frac{(1-\alpha+\xi)_{r+s}}{(\beta+\xi+n+2)_{r+s}}\ \frac{(-m)_r\,(A_p)_r}{(B_q)_r}\ \frac{(-l)_s\,(a_p)_s}{(b_q)_s}\ \lambda^r \mu^s$$

which leads to the expression on the right of (4.1) by virtue of (1.2).

The expansion-formula to be established is:

$$(4.5) \qquad x^{\xi}\ {}_{p+1}F_q\left[\begin{matrix} -l,\ a_p \\ b_q \end{matrix};\ \mu x\right] {}_{p+1}F_q\left[\begin{matrix} -m,\ A_p \\ B_q \end{matrix};\ \lambda x\right]$$

$$= (\alpha+1)_{\xi} \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(\alpha+1)_u} F\left[\begin{matrix} 2 \\ p+1 \\ 2 \\ q \end{matrix}\left|\begin{matrix} \alpha+\xi+1,\ 1+\xi \\ -m,\ A_p;\ -l,\ a_p \\ 1-\xi-u, \beta+\alpha+\xi+u+2 \\ B_q;\ b_q \end{matrix}\right|\begin{matrix}\lambda \\ \mu\end{matrix}\right]$$

$$P_u^{(\alpha,\beta)}(1-2x).$$

Proof: —

Let

$$(4.6)\quad f(x) = x^{\xi} \left\{ {}_{p+1}F_q\left[\begin{matrix} -l,\ a_p \\ b_q \end{matrix};\ \mu x\right] {}_{p+1}F_q\left[\begin{matrix} -m,\ A_p \\ B_q \end{matrix};\ \lambda x\right] \right\}$$

$$= \sum_{u=0}^{\infty} C_u\, P_u^{(\alpha,\beta)}(1-2x), \quad (o < x < 1).$$

From (4.6), we obtain C_u in a purely formal manner. With that value for C_u, we then have assumed that the series on the right in (4.6) actually converges to $f(x)$, providing $f(x)$ [(8), p. 176, § 100] is sufficiently well behaved.

From (4.6), it follows formally that

$$(4.7)\quad \int_0^1 x^{\alpha+\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)$$

$$\left\{ {}_{p+1}F_q\left[\begin{matrix} -l,\ a_p \\ b_q \end{matrix};\ \mu x\right] {}_{p+1}F_q\left[\begin{matrix} -m,\ A_p \\ B_q \end{matrix};\ \lambda x\right] \right\} dx$$

$$= \sum_{u=0}^{\infty} C_u \int_0^1 u^{\alpha}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\, P_u^{(\alpha,\beta)}(1-2x)\, dx.$$

All the integrals on the right in (4.7) vanish except for the single term for which $u = n$.

Therefore, making use of the orthogonality-property of the Jacobi polynomials [(7), p. 276, (22)]:

$$\int_0^1 x^{\alpha}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\, P_m^{(\alpha,\beta)}(1-2x)\, dx = h_n,\ \delta_{m,n},$$

$$Re(\alpha) > -1,\ Re(\beta) > -1,$$

$$\text{where } h_n = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\lambda)\,\Gamma(n+\lambda)\, n!},\quad \lambda = \alpha+\beta+1,$$

$\delta_{m,n} = o$ for $m \neq n$ and $\delta_{m,n} = 1$ for $m = n$, on the right and (4.1) on the left of (4.7), we obtain

(4.8)

$$C_n = \frac{(-\xi)_n (\alpha+1)_\xi (\alpha+\beta+2n+1)}{(\alpha+\beta+n+1)_{\xi+1} \, (\alpha+1)_n} F \left[\begin{array}{c|c|c} 2 & \alpha+\xi+1, 1+\xi & \\ p+1 & -m, A_p; -l, a_p & \lambda \\ 2 & 1+\xi-n, \alpha+\beta+\xi+n+2 & \mu \\ q & B_q; b_q & \end{array} \right].$$

In view of (4.6) and (4.8), we shall arrive at the desired result (4.5).

5. *Applications*

(*A*) Particular cases of the results derived in sections (2) and (3).

(*a*) In (2.1), (2.5), (2.7), (2.8) and (3.1):

(i) Taking $A = B = p = o$, we obtain the corresponding results involving the product of two MMEIJER's G-functions in terms of generalized MEIJER functions of two variables.

(ii) Substituting $A = E = p = k = l - 1 = F - 1 = o$ and replacing $A + C$ by A, $\mathrm{B} + D$ by B, $A + q$ by s together with appropriate changes in the parameters etc., and then allowing $\beta \to o$, the results either for the MEIJER's G-function or hypergeometric function with G-function can be derived which further reduce to a number of particular cases, some of which are known [(6), (13), p. 419, (5)-(6) and p. 15, (4 1) - (4 2)] for either $\varrho = 1$ or $\varrho = 2$, $\lambda = 1$ and others are believed to be new and interesting.

(iii) Setting $p = A = M$, $B = N$, $q = k = C = E = L$, $r = l = 1$, $D = F = p + 1$, $d_1 = f_1 = o$ and replacing b_j, $1 - c_j$, $1 - d_j$, $1 - e_j$ and $1 - f_j$ by $c_j, b_j, d_{,j} b'_j$ and d'_j respectively, we can obtain the results on KAMPÉ DE FÉRIET's functions which in turn, yield the APPELL functions, the WHITTAKERf unctions of two variables etc., and generalized hypergeometric functions of single variable including some known results [(14), (15), p. 20, (2.4) and p. 155, (4.4)].

(*b*) In (3.1), with $m = n = 4$, $\alpha_1 = \dfrac{u+1}{2}$, $\alpha_2 = \dfrac{u+2}{2}$,

$$\alpha_3 = \frac{\lambda+1}{2}, \alpha_4 = \frac{\lambda+2}{2}, \beta_1 = 1 + \lambda, \beta_2 = 1 + \mu, \beta_3 = 1 + \nu$$

and by virtue of (1.3), we obtain a very interesting result involving R-function and extended MEIJER'S G-function of two variables.

Proceeding as above, we can easily obtain the relations on images of R-function in various transforms which play an important role in INTEGRAL TRANSFORMS and OPERATIONAL CALCULUS.

(B) Particular cases of the results established in section 4. In (4.1) and (4.5): –

(i) Taking $a_1 = l + \gamma + \delta + 1,\ b_1 = 1 + \gamma,\ b_2 = 1/2,$

$$A_1 = m + \varrho + \sigma + 1,\ B_1 = 1 + \varrho,\ B_2 = 1/2,$$

and multiplying both sides by $\frac{(1 + \gamma)_l}{l!}\frac{(1 + \nu)_m}{m!}$, we have the results involving generalized SISTER CELINE polynomials [(10), p. 80 (2.2)] which generalize the known results [(10), (16), p. 87, (3.12) and p. 80, (3.5)] when $m = 0$.

Further setting $p = q = 3,\ \ a_2 = 1/2, a_3 = \mathfrak{I},\ b_3 = h, A_2 = 1/2,$ $A_3\,\eta,\ B_3 = k,$ interesting results on generalized RICE'S polynomials can be derived which lead to the results for JACOBI polynomials for $\mathfrak{I} = h$ and $\eta = k$.

(ii) Substituting $p = o,\ q = 1,\ b_1 = 1 + \varrho,\ B_1 = 1 + \sigma$ and multiplying both sides by $\frac{(1 + \nu)_l}{l!}, \frac{(1 + \sigma)_m}{m!}$, the results on generalized LAGUERRE polynomials can be obtained.

(iii) With $p = 1, q = o,\ a_1 = n + a - 1,\ A_1 = m + A - 1,\ \mu = - 1/b$ and $\lambda = - 1/B$, we obtain the results for the generalized BESSEL polynomials.

We remark in passing that the aforesaid results involving the classical sets of orthogonal and other hypergeometric polynomials are all particular cases of the generalized hypergeometric polynomial (1.7).

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