# GENERALIZATION OF A THEOREM OF PATI

ZIAD S. ALI *

INTRODUCTION: In this paper we consider Nörlund summability of Fourier series and extend a theorem of Pati. Let $f(t)$ be a periodic function with period $2\pi$, which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be

$$\frac{1}{2} a_0 + \sum_{n}^{\infty} a_n \cos nt + b_n \sin nt = \frac{1}{2} a_0 + \Sigma A_n(t).$$

let

$$\Phi(t) = \Phi_x(t) = f(x+t) + f(x-t) - 2f(x)$$

$$\Phi(t) = \int_0^t |\Phi(u)|\, du, \quad \text{and} \quad \tau = \left[\frac{1}{t}\right],$$

the integral part of $\frac{1}{t}$.

In 1948 Siddiqi proved the following theorem:

THEOREM $A$ If

$$\Phi(t) = o\left(\frac{t}{\log\left(\frac{1}{t}\right)}\right), \quad \text{as} \quad t \to +0,$$

then the Fourier series of $f(t)$, at $t = x$, is summable $\left(N, \frac{1}{n+1}\right)$.

In 1961 PATI generalized Siddiqi's theorem and proved the following theorem [1]:

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THEOREM $B$ Let $(N, p_n)$ be a regular Nörlund method, defined by real, non-negative, monotonic non-increasing sequence of coefficients $\{p_n\}$, such that $P_n \to \infty$, and

$$\log n = O(P_n), \quad \text{as} \quad n \to \infty.$$

Then, if

$$\Phi(t) = o\left(\frac{t}{P_\tau}\right), \quad \text{as} \quad t \to +0,$$

then the Fourier series of $f(t)$ at $t = x$, is summable $(N, p_n)$ to $f(x)$.

2. In this note we extend PATI'S theorem as follows:

THEOREM 1. Let $(N, p_n)$ be a regular Nörlund method, defined by real, non-negative, monotonic non-increasing sequence of coefficients $\{p_n\}$, such that $P_n \to \infty$, and

$$H(n) = \int_1^n \frac{h(t)}{t}\, dt = O(P_n), \quad \text{as} \quad n \to \infty,$$

where $h(t)$ is any slowly oscillating function; then if

$$\Phi(t) = o\left(\frac{t \cdot h\left(\frac{1}{t}\right)}{P_\tau}\right), \quad \text{as} \quad t \to +0$$

then the Fourier series of $f(t)$, at $t = x$, is summable $(N, p_n)$ to $f(x)$.

THE CASE $h(t)$ EQUALS A CONSTANT IS PATI'S THEOREM.

PROOF: we have

$$S_n(x) = \sum_{\nu=1}^{n} A_n(x);$$

therefore

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \Phi(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2} t}\, dt;$$

hence

$$t_n(x) - f(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_{n-k}(x) - f(x)$$

$$= \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{1}{2\pi}\int_0^{\pi} \Phi(t)\,\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

$$= \int_0^{\pi} \Phi(t)\,\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt$$

$$= \int_0^{\pi} \Phi(t)\,K_n(t)\,dt, \quad \text{say.}$$

In order to prove the theorem we show

$$\int_0^{\pi} \Phi(t)\,K_n(t)\,dt = o(1) \quad \text{as} \quad n \to \infty.$$

Now

$$\int_0^{\pi} \Phi(t)\,K_n(t)\,dt = \int_0^{1/n} + \int_{1/n}^{1} + \int_1^{\pi} \Phi(t)\,K_n(t)\,dt$$

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

$$I_1 = \int_0^{1/n} \Phi(t)\,K_n(t)\,dt = O\left(\int_0^{1/n} |\Phi(t)|\,|K_n(t)|\,dt\right)$$

$$= O\left(n\int_0^{1/n} |\Phi(t)|\,dt\right) \quad \text{By} \quad [/]$$

$$= O\left(n\,\frac{n^{-1}h(n)}{P_n}\right)$$

$$= o(1) \quad \text{as} \quad n \to \infty, \quad \text{since } \frac{h(n)}{H(n)} = o(1)$$

as $n \to \infty$ by [3].

$$I_3 = \int_1^{\pi} \Phi(t) K_n(t)\, dt = o(1) \quad \text{as} \quad n \to \infty,$$

by the Riemann Lebesgue theorem, and the regularity of the method $(N, p_n)$,

$$I_2 = \int_{1/n}^{1} \Phi(t) K_n(t)\, dt = O\left(\frac{1}{P_n} \int_{1/n}^{1} |\Phi(t)| \frac{P_\tau}{t}\, dt\right)$$

by Tamarkin and Hille's Lemma [2].

Now

$$\frac{1}{P_n} \int_{1/n}^{1} |\Phi(t)| \frac{P_\tau}{t}\, dt$$

$$-\int_{1/n}^{1/n-1} + \int_{1/n-1}^{1/n-2} + \dots + \int_{1/2}^{1} |\Phi(t)| \frac{P_\tau}{t}\, dt.$$

By integration by parts and simplifying we obtain

$$\frac{1}{P_n} \int_{1/n}^{1} |\Phi(t)| \frac{P_\tau}{t}\, dt - o(1) = \frac{1}{P_n} \Phi(t) \frac{P_\tau}{t} \Big|_{1/n}^{1} + \frac{1}{V_n} \int_{1/n}^{1} \Phi(t) \frac{P_\tau}{t^2}\, dt.$$

Therefore

$$\frac{1}{P_n} \int_{1/n}^{1} |\Phi(t)| \frac{P_\tau}{t}\, dt = O\left(\frac{1}{P_n}\right) + O\left(\frac{1}{P_n} \cdot \frac{n^{-1} \cdot h(n)}{P_n} \cdot \frac{P_n}{n^{-1}}\right)$$

$$+ o\left(\frac{1}{P_n} \int_1^{\pi} \frac{h(t)}{t}\, dt\right) = o(1) \quad \text{as} \quad n \to \infty,$$

and the theorem follows.

THEOREM 2. Let $(N, p_n)$ be a regular Nörlund method, defined by real, non-negative, monotonic non-increasing-sequence of coefficients $\{p_n\}$, such that $P_n \to \infty$, and

$$H(n) = o(P_n)$$

as $n \to \infty$, where

$$H(n) = \int_{1}^{\pi} \frac{h(t)}{t} \, dt$$

and $h(t)$ is any slowly oscillating function; then if

$$\Phi(t) = O\left(\frac{t \cdot h\left(\frac{1}{t}\right)}{P_{\tau}}\right) \quad \text{as} \quad t \to +0,$$

then the Fourier series of $f(t)$ at $t = x$, is summable $(N, p_n)$ to $f(x)$.

PROOF: This follows from the fact that $\frac{h(n)}{H(n)} = o(1)$ as $n \to \infty$ [3].

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## BIBLIOGRAPHY

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Dr. ZIAD S. ALI
Université d'Alger
Faculté des Sciences
ALGER