THE DE SITTER UNIVERSE AND GENERAL RELATIVITY

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1 — INTRODUCTION

The aim of this work is that of outline a theory of gravitation in the framework of the De Sitter space-time; therefore in the second paragraph there will be a discussion of the principal points of view on the problem of the gravitational field in the framework of a given space-time structure; because this question is strictly connected, as it will be seen, with the unitary theories of gravitation and electromagnetism, in the paragraphs 3 and 4 there will be a short outline of the unitary theories of Kaluza-Klein, Thiry-Jordan and others; in the paragraph 5 there will be a review of the Fantappié-Arcidiacono theories on the De Sitter space-time; in the paragraph 6 will be deduced a formula leading from a 5-dimensional to a 4-dimensional formulation of the gravitational laws; in the paragraph 7, finally, there will be exposed the mathematical techniques necessary for a complete description of the gravitational phenomena in the De Sitter space-time.

2 — THE GENERAL RELATIVITY AND THE MODELS OF UNIVERSE.

The problem of the role of General Relativity in constructing models of Universe arose in the study of the De Sitter space; this latter has always been of great interest since DE SITTER [1] showed that, in the case of a static and isotropic universe, with spatial sections of constant curvature, the Einstein field equations in the empty space admit only three solutions, that is the Minkowski metric, the Einstein static metric and the De Sitter metric. This result clearly is in contradiction with the Mach principle, at least in the version known as Mach-Einstein's principle, according to which there cannot be inertial properties in a space-time in absence of matter. Whereas the Einstein static universe can escape this unpleasant situation, because the presence of the cosmological constant gives rise in it to a non-null matter density, this is no longer true for the Minkowski and De Sitter metrics which correspond, also in presence of the cosmological constant, to a density of matter everywhere null.

The contradiction with the Mach principle is here particularly troublesome because Einstein based himself fundamentally over it in constructing his General Relativity theory. A way for escape this situation has been showed for the first time, we think, by DE SITTER [2] himself which said that there were gravitational effects only if the coefficients of the metric changed from the values typical of the De Sitter universe. So De Sitter implicitly put forward the idea of a gravitation conceived as a «differential» effect on a space-time with metric not satisfying the Mach-Einstein principle; this involved that not always the space-time curvature were of gravitational origin, but De Sitter do not showed the mathematical means able to exprime this point of view. It is a matter of course, however, that a particular theory of the «differential» gravity in one of the foregoing space-times has been yet developed in the case of the Minkowski space-time and it coincides with the ordinary General Relativity theory; but a similar work has never been tried for the De Sitter metric. Moreover, with the methods then used in General Relativity, when every model of Universe was described by its ds^2, it was impossible to treat these questions; namely in the same period many theorists searched vainly for a unitary theory of gravitation and electromagnetism, acting only on the metric coefficients.

In 1952 FANTAPPIE' [3] throwed new light on these questions, dealing with them exclusively from the group-theoretical point of view and conceiving a «model of Universe» as entirely defined by a r-parameters transformations group G_r which determines both the geometry of the model under consideration and its physical laws, defined uniquely by the property of maintaining the same form under transformations of G_r. Particularly Fantappié showed that the connection between the group of motions into itself of the De Sitter space and of the Minkowski space was very close; on the basis of these results it looked, starting only from purely group-theoretical considerations, for a physics in the De Sitter space, as well as it is possible to construct a physics in the Minkowski space, i.e. the ordinary Restricted Relativity, starting only from its group of motions into itself, the Lorentz group.

This program has been completed by G. Arcidiacono which, owing to interesting mathematical techniques, constructed a «Restricted Relativity» in the De Sitter space, named «Projective Relativity»; Arcidiacono himself proposed to construct, for every model of Universe endowed with a group of motions into itself, a «Restricted Relativity» on a purely group-theoretical basis, in a fashion similar to that of «Projective Relativity». One of the most interesting results of the «Projective Relativity» is the circumstance that in it the electromagnetic field and the hydrodynamical field are fused in a unique field; this involves that a theory of the «differential» gravitation in the De Sitter space would be automatically a unified field theory.

It is suitable to remember, afterwards, that in 1951 A. H. TAUB [4] has find solutions of the Einstein field equations in absence of matter with non-null curvature, in absence of the cosmological constant. Following the foregoing ideas, the next step would be, then to study such metrics from the group-theoretical point of view for constructing the «Restricted Relativities» over them, so that one could obtain various kinds of possible physics.

Now it is necessary to point out how the two conceptions of the «models of Universe», the one based exclusively on the ds^2, and the one based exclusively on group-theoretical considerations, are both defective because each of them takes into account only one side of the problem; there is the need, then, for a third point of view which unifies the two precedings. This point of view is based on the ideas of CARTAN [5] which, generalizing the concept of space, inserted the Riemannian geometry into a group-theoretical conception. Following Cartan, a Riemannian (and not) variety V_4 can be imagined as constituted by infinite tangential spaces, each of which has a geometry based on a group G, by Cartan called «holonomous» geometry; these infinite tangential spaces are joined together according to a certain connection law which allows to deduce both the curvature and the torsion (local properties) of V_4, through the use of infinitesimal closed cycles on the variety under consideration, and, through the use of finite closed cycles, the «holonomy group» (global property) of V_4, i.e. the group of displacements associated with such cycles; vice versa once known the holonomy group, the connection law is uniquely determined. So, starting from an holonomous geometry based over a group G and choosing a particular holonomy group, it is always possible to construct an anholonomous geometry corresponding to it.

This point of view agrees perfectly with the foregoing conception of the gravitation, so providing the mathematical means to formulate it correctly; the direct construction, however, of an anholonomous geometry is, generally speaking, rather difficult and therefore we will use the Fantappié-Arcidiacono methods which allow to translate the 4-dimensional geometry of the De Sitter space in a 5-dimensional Euclidean space on which one can, through the use of the ordinary methods of the tensorial calculus, construct an anholonomous geometry having as holonomy group the restricted 5-dimensional rotations group (5-dimensional analogue of the General Relativity); through the formula showed in sixth paragraph, then, we will be in condition of translate the 5-dimensional results in a 4-dimensional space.

As these techniques have a remarkable resemblance with the ones used in 5-dimensional unitary theories, also for the reason that in the 5-dimensional version of the De Sitter space the electromagnetic and the hydrodynamical field are unified, it will be useful to premise some informations on the 5-dimensional unitary theories, so that we will be in condition of comparing their results with the ones which will be presented here.

3 — The Kaluza-Klein 5-dimensional unitary theory

Following the aim of construct a unitary general relativistic theory of gravitation and electromagnetism, in 1921 Kaluza [6] built up a 5-dimensional theory, improved by Klein [7] in 1926. In this theory we have a 5-dimensional Riemannian space in which x_1, x_2 and x_3 are spatial coordinates, $x_4 = ct$ is the temporal coordinate, whereas the fifth coordinate x_5, is unobservable. For this reason we can consider psysically meaningful only those transformations which do not vary x_5, i.e. those of the type

$$\bar{x}_5 = x_5 + f(x_i),\ \bar{x}_i = g_i(x_k)\ (i, k = 1, \dots, 4) \tag{1}$$

So we satisfy the principle of general relativity. Now, on the basis of the (1) we can represent in a particular fashion the vectors and the tensors of 5-dimensional space: e.g. a vector $v_\alpha(\alpha = 1, \dots, 5)$ can be represented by a space-time vector $v_i(i = 1, \dots, 4)$ plus a scalar invariant s for the fifth component; a symmetrical double tensor $T_{\alpha\beta}(\alpha, \beta = 1, ..., 5)$ by a space-time symmetrical tensor T_{ik} $(i, k = = 1, ..., 4)$ plus a vector $v_i (i = 1, ..., 4)$, given by the components of $T_{\alpha\beta}$ with only an index equal to 5, and an invariant scalar corresponding to T_{55}. Thus the metric of this 5-dimensional Riemannian space can written as

(2) $$d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = V^2\,(dx^5)^2 + 2\,\gamma_{5i}\, dx^5\, dx^i + \gamma_{ik}\, dx^i\, dx^k$$

$$\begin{pmatrix} \alpha, \beta = 1, ..., 5 \\ i,\ k = 1, ..., 4 \end{pmatrix}$$

At this point we can hope to construct a unified theory by requiring that the coefficients of the metric (2) are such that the equations of the geodesics

(3) $$\frac{dx^\alpha}{d\sigma}\,\nabla_\alpha\,\frac{dx^\beta}{d\sigma} = 0$$

where ∇_α denotes covariant derivation respect to x_α in 5-dimensional space, be coincident with the equations of motion of a body with mass m and electric charge e

(4) $$\frac{dx^k}{ds}\,\nabla_k\,\frac{dx^i}{ds} + \frac{e}{c^2\,m}\,F_k{}^i\,\frac{dx^k}{ds} = 0$$

with

$$ds^2 = g_{ik}\, dx^i\, dx^k$$

where $F_k{}^i$ is the electromagnetic tensor.

We obtain a solution to this problem putting

(5a) $$V = 1$$

(5b) $$\gamma_{i5} = \beta\,\Phi_i$$

(5c) $$\gamma_{ik} = g_{ik} + \beta^2\,\Phi_i\,\Phi_k$$

(5d) $$\left(\frac{d\sigma}{ds}\right)^2 = 1 + \frac{e^2}{c^4\,m^2\,\beta^2}$$

The condition $V = 1$ means that the lines of equations $x^i = \text{const}$ $(i = 1, ..., 4)$ are geodecics satisfying the (3); β is a universal constant and Φ_i the electromagnetic potential.

The field equations in absence of matter and electric charge are

$$R_{\alpha\beta} = 0 \tag{6}$$

where the first term of (6) is the contracted Riemann tensor of the metric (2). By use of the relations (5a), (5b), (5c), (5d), we deduce from (6)

$$R_{ik} - 1/2\, R g_{ik} = -\beta\, 2/2\, E_{ik} \tag{7a}$$

$$\nabla_k F^{ik} = 0 \tag{7b}$$

where R_{ik} and R are the contracted Riemann tensor and the curvature scalar, respectively, of the metric g_{ik} and E_{ik} is the electromagnetic energy tensor given by

$$E_{ik} = F_{ij}\, F_k^{\,j} + 1/4\, F_{jh}\, F^{jh}\, g_{ik} \tag{8}$$

The equations (7a) coincide with the Einstein field equations in presence of an electromagnetic field if we put

$$2\,\chi = \beta^2 \tag{9}$$

where χ is the Einstein constant.

When there are matter and electric charge, the field equations can be written as

$$R_{\alpha\beta} - 1/2\, \gamma_{\alpha\beta}\, R_{\mu\nu}\, \gamma^{\mu\nu} + \chi\, T_{\alpha\beta} = 0 \tag{10}$$

where $T_{\alpha\beta}$ is a generalization of the space-time energy tensor

$$T_{\alpha\beta} = T \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \tag{11}$$

From the vanishing of the divergence of the purely geometrical terms in (10) we can deduce

$$\nabla_\beta T^{\alpha\beta} = \nabla_\beta (\chi\, T\, \lambda^\beta)\, \lambda^\alpha + T\, \lambda^\beta\, \nabla_\beta\, \lambda^\alpha = 0 \tag{12}$$

and

$$\nabla_\beta (\chi\, T\, \lambda^\beta) = 0, \quad , \quad \lambda^\alpha = \frac{dx^\alpha}{d\sigma} \tag{13}$$

which is the expression of the conservation of the energy; putting (13) in (12) we obtain the equations of geodesics (3), as in usual General Relativity. By use of the conditions (5a), (5b), (5c) and (5d) the field equations (10) become

$$R_{ik} - 1/2\, Rg_{ik} = -\frac{\beta^2}{2} E_{ik} - \chi\, T \frac{dx_i}{ds}\frac{dx_k}{ds} \tag{14a}$$

$$\nabla_k F^{ik} = \frac{2\chi}{\beta} T \frac{dx^i}{ds}\frac{dx^5}{ds} \tag{14b}$$

If we put

$$T = c^2 K \tag{15}$$

where K is the density of mass-energy and c the velocity of the light in the vacuum, the (14a) become the Einstein field equations in presence of matter and electromagnetic field, whereas the (14b) become the usual Maxwell equations

$$\nabla_k F^{ik} = j^i \tag{16}$$

where j^i is the current-density four-vector.

So the Kaluza-Klein theory shows a unitary and coherent geometrical scheme in which we can treat gravitational and electromagnetic field and their mutual interactions; but we have also several difficulties that we can list as in the following:

1) it is difficult to accept the existence of the fifth dimension, firts of all because we do not have any experimental evidence of it;

2) the introduction of this fifth dimension appears very artificial: indeed it is treated differently from the other coordinates, as we can see in the formulae (1);

3) the generalized energy tensor (11) is introduced in a rather arbitrary fashion and its validity is proved a posteriori only from the field equations (14a) and (14b): so the theory falls in a vicious circle and lacks physical meaning;

4) the theory is not a really unified theory, because the electromagnetic and gravitational fields act separately, do not being entirely transformable one in another.

Let us see how the Kaluza-Klein theory has been modified to give more consistent descriptions of the physical phenomena.

4 — Other 5-dimensional unitary theories.

In 1951 Thiry [8], in connection with the ideas of Jordan [9], gave a generalization of the Kaluza-Klein theory, discarding the (5a), i e the V-constancy hypothesis His equations, in the exterior case, are

$$G_{\alpha\beta} = R_{\alpha\beta} - 1/2\, \gamma_{\alpha\beta}\, R = 0 \tag{1}$$

which become, with the help of the (1,5b), (1,5c), (1,5d) and with space-time metric given by

$$g_{ij} = \gamma_{ij} - \frac{\gamma_{5i}\,\gamma_{5j}}{\gamma_{55}} \tag{2}$$

become

$$\left\{\begin{aligned} & G_{ij} = \hat{G}_{ij} - \frac{\beta^2 V^2}{2}\left[\frac{1}{4} g_{ij}\, F_{kl}\, F^{kl} - F_i^{\ k}\, F_{jk}\right] - \frac{1}{V}\,[\hat{\nabla}_j\, \partial_i\, V - g_{ij}\, \hat{\Delta} V = 0] \\ & G_{i5} = \frac{\beta}{2\,V^2}\, \hat{\nabla}_j\, (V^3\, F_i^{j}) = 0 \\ & G_{55} = \frac{1}{2}\,\hat{R} + \frac{3}{4}\,\beta^2\, V^2\, F^2 = 0 \end{aligned}\right. \tag{3}$$

where $F^2 = 1/2\, F_{kl}\, F^{kl}$, the operator $\hat{\Delta}$ means $\hat{\Delta} = \hat{\nabla}_j\,(g^{ij}\,\partial_i)$, $\hat{\nabla}_j$ being the covariant derivative respect to the metric g_{ij} and the symbol $\wedge$ referring to the geometrical quantities constructed from the metrical tensor g_{ij}.

Unfortunately we do not have a unique physical interpretation of V; that adopted by Thiry and Jordan is based on the correspondence

$$V^3 = \varepsilon_0 \tag{4}$$

where ε_0 is a variable dielectric constant of the vacuum; in this theory we can represent the Einstein constant, in analogy with (1,9), as

$$\chi = \frac{\beta^2}{2\,V} \tag{5}$$

Obviously, being V variable, also χ is variable: on this basis Jordan and his group [10] have built a cosmological model whose applications range from the formation of galaxies to an explanation of the continental drift on the Earth [11].

Another interpretation, due to MARIOT and his coworkers [12], assumes V as a scalar meson field; so the third of the (3), written in presence of matter, should become a sort of Klein-Gordon equation.

The Thiry-Jordan theory suffers of the same difficulties of the Kaluza-Klein theory plus an additional difficulty of interpretation for the V-field: however from the physical point of view there is a substantial advancement in the point 4) of the previous Section, i.e. this theory is more unified; namely a concentration of neutral matter can give rise, in the Thiry-Jordan theory, to a non-null electromagnetic field.

In the early 1930 the difficulty connected with the fifth dimension in the Kaluza-Klein theory was been avoided by VEBLEN and HOFFMANN [13] who pointed out that the five coordinates could be interpreted as homogeneous projective coordinates of a 4-dimensional space-time with projective connection. After some time, VEBLEN [14], SCHOUTEN and VAN DANTZIG [15], following the ideas of CARTAN [16], gave a formal sistemation to the theory of projective connections. Substantially a 4-dimensional space with projective connection is a space which, in the infinitesimal neighbours of every of its points, is a projective space and is also provided of a law of projective representation between neighbours of two of its points infinitely close. In this space we have a field of quadrics

$$Q = g_{AB}\, dx^A\, dx^B = 0 \qquad (A, B = 1, \ldots, 5) \tag{6}$$

with $g_{AB} = g_{AB}(u^i, u^5)$, u^i and u^5 being curvilinear homogeneous coordinates; for every point P the quadric (6) is the absolute of a local non-euclidean metric. The conservation of the quadric field requires that

$$\nabla_A\, g_{BC} = 0 \tag{7}$$

where the covariant derivative ∇_A is built from the connection coefficients

$$\pi^{A}_{BC} = \left\{ {A \atop B\,C} \right\} = \frac{1}{2} g^{AK} (\partial_C\, g_{BK} + \partial_B\, g_{KC} - \partial_K\, g_{CB}) \tag{8}$$

From these coefficients we can construct a projective curvature tensor

$$R^{\;A}_{BLM} = \partial_L\, \pi^{A}_{BM} - \partial_M\, \pi^{A}_{BL} + \pi^{A}_{RL}\, \pi^{R}_{BM} - \pi^{A}_{KM}\, \pi^{K}_{BL} \tag{9}$$

If this tensor vanishes, we have a projectively-flat space, i.e. a space with constant curvature; following the terminology of Cartan, we can the tensor (9) «curvature-torsion tensor», because, as it demonstrated, the 4-dimensional torsion is built from some components of the projective curvature tensor

$$2\, T^{i}_{kh} = R^{i}_{5hk} \tag{10}$$

Naturally from the relation (10) we infer that in the projective connection T^{i}_{hk} do not constitutes a tensor, at least in general.

SCHOUTEN [17] tried up to rewrite Kaluza-Klein's theory in this projective formalism, but unsuccessfully, because the theory is ambiguous with respect to the choice of the connection, leaving, in ultimate analysis, the unitary problem determined only by arbitrary conditions of the type of the (1,5). However others make use of this theory, as FLINT [18] who proposed the constant β of the (1,5) to be given by

$$\beta = \frac{e}{m_0\, c^2} \tag{11}$$

where e and m_0 are the charge and the mass of the electron, respectively, and c the velocity of the light in the vacuum. Also HOFFMANN [19] built up a similar theory, making use of a projective connection, giving results like those of Thiry-Jordan.

In 1963 DE WITT [20] pointed out that the Kaluza-Klein theory is a particular case of a more general class of theories which attempt to combine the general coordinate transformation group (which is connected with the gravitational field) with a Yang-Mills group, i.e. a continuous transformations group whose parameters are continuous functions of the space-time points; the Yang-Mills groups are a generalization of the gauge group, connected with the electromagnetic field.

5 — THE FANTAPPIE'-ARCIDIACONO THEORY.

In 1952 FANTAPPIE' [21], with his theory of «physical Universes», showed that, if we define as «physical Universe« a system governed by the same laws for all the observers, this implies the existence of a group, for which these laws are invariant. This group, conceived as group of motions into itself, defines in a unique fashion the geometry of the space, whereas, conceived as invariance group, it fix the admissible form of the physical laws. In 1954 FANTAPPIE' [22] applied these ideas to the study of the possible generalizations of the Lorentz inhomogeneous group with 10 parameters and showed that, for a 4-dimensional space-time and without changing the number of parameters, this is possible only with the group of motions into itself of the De Sitter space-time with constant curvature and metric given by

$$ds^2 = c^2\, d\tau^2 - \exp\,(2\, c\tau/r)\,(d\xi_1{}^2 + d\xi^2{}_2 + d\xi^2{}_3) \tag{1}$$

Now, applying the transformations

$$\begin{cases} Z_0 = r\ \sinh\,(c\,\tau/r) + (\xi^2/2r)\ \exp\ (c\tau/r) \\ Z_i = \xi_i\ \exp\ (c\tau/r) \qquad\qquad (i = 1, \dots, 3) \\ Z_4 = r\ \cosh\ (c\tau/r) - (\xi^2/2r)\ \exp\ (c\tau/r) \end{cases} \tag{2}$$

where $\xi^2 = \xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2$, it is easy to show [23] that the De Sitter space-time can be represented as an hypersurface of equation

$$- Z_0{}^2 + Z_1{}^2 + Z_2{}^2 + Z_3{}^2 + Z_4{}^2 = 0 \tag{3}$$

imbedded in a flat 5-dimensional space; thus, from this point of view, the De Sitter group becomes the group of transformations leaving unchanged (3), i.e. the group of rotations in a flat 5-dimensional space; this shows the nature of the generalization respect to the Lorentz group: the homogeneous part of this latter, indeed, is isomorphous, from the complex point of view, to the group of rotations in a flat 4-dimensional space.

A simple physical interpretation of the De Sitter group was given by ARCIDIACONO [24] who introduced the distinction between «Absolute Universe» (De Sitter space-time) and «Relative Universes» of every observer in which it places the physical evenements; now it is clear that every observer can perceive the Universe only as if it were flat, with geodesic lines appearing as straight lines: this means that every «Relative Universe» is a geodetic representation of the De Sitter space-time on a tangent hyperplane, giving up the so-called Beltrami metric

$$\begin{aligned} &(r^2 + \varrho^2)^2\, ds^2 = r^2\,[(r^2 + \varrho^2) \sum_i (dx^i)^2 - (x_i\, dx^i)^2] \\ &\varrho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \qquad (i = 1, \dots, 4) \end{aligned} \tag{4}$$

Now the transformations of the De Sitter group become projections from the center of the quadric (3) and sections with the tangent hyperplane: thus we can speak of «projective transformations« and «Projective Relativity«. But these terms have also a more deep signification: indeed we can represent the Beltrami metric in a projective fashion [25], making use of a 4-dimensional projective space, in which the physical space-time is the region external to the absolute of Cayley-Klein, of equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + r^2 = 0 \tag{5}$$

So the De Sitter group becomes the group of projectivities transforming into itself the quadric (5). Homogeneous coordinates can also be introduced by the position

$$x_i = r \frac{\overline{x}_i}{\overline{x}_5} \tag{6}$$

Thus the quadric (5) becomes

$$\overline{x}_1^2 + \overline{x}_2^2 + \overline{x}_3^2 + \overline{x}_4^2 + \overline{x}_5^2 = 0 \tag{7}$$

and the De Sitter transformations become the rotations in a flat 5-dimensional projective space; thus, from the projective point of view, we can give a physical interpretation to the mathematical property known from the requirement of invariance for the quadric (3). The projective space defined as the region external to the quadric (5) is known as Castelnuovo's space-time and only in this space all mathematical relationships receive a physical interpretation; naturally it is convenient to use the 5-dimensional formulation with homogeneous coordinates defined by (6), because we can use the tensorial calculus of the 5-dimensional Euclidean space. The problem of the elimination of the fifth coordinate can be solved by using the Weierstrass normalization condition

$$\overline{x}_A \, \overline{x}_A = r^2 \qquad (A = 1, \ldots, 5) \tag{8}$$

from which, by use of (6), we can obtain

$$\overline{x}_5 = r / A \tag{9a}$$

$$A = 1 + \alpha_i \, \alpha_i, \; \alpha_i = \frac{x_i}{r} \tag{9b}$$

The validity of (8) is proved by the fact that, making use of it and of (6), we can go from the 5-dimensional flat Euclidean metric

$$ds^2 = d\bar{x}_1{}^2 + d\bar{x}_2{}^2 + d\bar{x}_3{}^2 + d\bar{x}_4{}^2 + d\bar{x}_5{}^2 \tag{10}$$

to the Beltrami metric (4). Namely, differentiating (9a) with the help of (9b), we obtain

$$d\bar{x}_5 = - A^{-3}\, \alpha_i\, dx_i \tag{11}$$

From the (6) we can derive

$$r\, d\bar{x}_i = x_i\, d\bar{x}_5 + \bar{x}_5\, dx_i \tag{12}$$

By substituting (9a), (11) and (12) into (10), we obtain

$$A^4\, ds^2 = A^2\, (dx_i\, dx_i) - (\alpha_i\, dx_i)^2 \tag{13}$$

which is precisely the Beltrami metric (4). From the (9a) and making use of the Euler theorem for homogeneous functions, the derivatives respect to the homogeneous coordinates $\bar{x}_A$ can be expressed in terms of derivatives respect to the inhomogeneous coordinates x_i; the formulae are

$$\text{(14)} \quad \begin{cases} \bar{\partial}_i = A\, \partial_i + \dfrac{n\, x_i}{r^2\, A}\,. \\[2ex] r\, \bar{\partial}_5 = - A\, x_s\, \partial_s + \dfrac{n}{A}\,. \end{cases}$$

where n is the degree of homogeneity of the function to be derived.

With these mathematical methods ARCIDIACONO [26] was able to construct a «Projective Relativity« with a large number of physical applications. Here we do not treat the problem of the mechanics in Projective Relativity; we will note only some interesting features as, e.g., the new formula of variation of the mass with the velocity and the physical possibility of hyper-c velocities (perhaps connected with tachyons?) [27]; besides, there is the interesting fact that the linear and angular 4-momentum are synthetized, in Projective Relativity, in a unique 5-dimensional tensor; as a consequence of this, the linear and angular momentum cannot be conserved separately and every material point of mass m_0 has an intrinsic energy and an intrinsic polar moment of inertia given by

$$E_0 = m_0\, c^2 \ ; \ I_0 = m_0\, r^2 \tag{15}$$

Much interesting, for our purposes, is the study of Maxwell's equations in Projective Relativity; these equations, when the indices equal to 5 are distinguished from the others, ans putting

$$J_{ik5} = \Omega_{ik} \; ; \; H_{i5} = C_i \tag{16}$$

can be written, in dual form, as

$$\left\{ \begin{array}{l} A \text{ rot } H_{kl} + \dfrac{n}{A\, r^2} (x_i H_{kl} + x_k H_{li} + x_l H_{ik}) = J_{ikl} \\[2ex] A \, (\text{rot } C_i - \dfrac{x_s}{r} \partial_s H_{ik}) - \dfrac{n}{A\, r^2} (x_i C_k - x_k C_i + H_{ik}\, r) = \Omega_{ik} \\[2ex] A \, (\text{div } H_{ik} + \dfrac{x_s}{r} \partial_s C_k) = \dfrac{n}{A\, r^2} (x_i H_{ik} + C_k\, r) \\[2ex] A \; \partial_s C_s = - \dfrac{n}{A\, r^2} (x_s C_s) \end{array} \right. \tag{17}$$

For a physical interpretation of the (17), let us go to the ordinary Restricted Relativity, with $r \to \infty$; then we obtain

$$\left\{ \begin{array}{l} \text{Rot } H_{ik} = J_{ikl} \\ \text{Div } H_{ik} = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} \text{Rot } C_i = \Omega_{ik} \\ \text{Div } C_i = 0 \end{array} \right. \tag{18}$$

The first group of the (18) coincides with the usual Maxwell equations, whereas the second group is coincident with the equations of relativistic hydrodynamics of incompressible fluids [28], C_i and Ω_{ik} being the hydrodynamical current and the vortex field, respectively. Thus the equations (17) represent, in a unified fashion, a «magnetohydrodynamical« field. The De Sitter-Castelnuovo space-time offers thus the possibility of constructing a totally unified theory of the material (hydrodynamical) and electromagnetic fields. This is shown clearly by the structure of the energy-momentum tensor derived from the (17)

$$T_{AB} = H_{AC} H_{CB} + 1/4\, H_{CD} H_{CD}\, \delta_{AB} \tag{19}$$

which can be written in the form

$$(20)\quad \begin{aligned} T_{ik} &= (H_{is} H_{sk} + 1/4\, H_{rs} H_{rs}\, \delta_{ik}) - (C_i C_k - 1/2\, C_s C_s\, \delta_{ik}) \\ T_{\alpha 5} &= C_4 H_\alpha + (\vec{E} \wedge \vec{C})_\alpha \\ T_{45} &= i\,(\vec{C} \times \vec{H}) \\ T_{55} &= 1\ 2\,[(E^2 - H^2) + (C_4{}^2 - C^2)] \end{aligned}$$

where E_α and H_α are the electric and magnetic field, respectively, with $E^2 = E_\alpha E_\alpha$ and $H^2 = H_\alpha H_\alpha$. The components of the tensor (20) with an index equal to 5 represent the interaction between electromagnetic and hydrodynamical fields, a circumstance not present in any of the unified field theories.

The Arcidiacono theory, however, is not yet totally unified, do not being present the gravitational field. The next step will be that of constructing a theory in which the gravitation, as in Einstein's theory, is represented by an anholonomous geometry, in the sense of Cartan, but the difference with ordinary General Relativity lies in the fact that, whereas it uses an anholonomous geometry constructed from the holonomous geometry of the Minkowski space-time, here the anholonomous geometry will be constructed from the holonomous geometry of the De Sitter-Castelnuovo space-time. Thus we can hope to arrive at a unified theory of matter (hydrodynamical), electromagnetic and gravitational fields, because, contrarily to what happens in usual General Relativity, the matter and electromagnetic fields are already unified in the holonomous base geometry. This theory will be named «Projective General Relativity».

6 — The induced 4-dimensional metric.

Following what has been said to the end of the last Section, we assume that the field equations in Projective General Relativity are

$$(1)\qquad R_{AB} - 1/2\, R g_{AB} = \chi\, T_{AB}$$

where R_{AB} and R are, respectively, the contrated curvature tensor and the invariant scalar constructed from the curvature tensor (4,9), whereas T_{AB} is the energy-momentum tensor (5,19). Namely, as we have seen in the last Section, the De Sitter space-time can be represented by a 5-dimensional projective space with Euclidean metric (from the point of view of every relative observer): then an anholonomous geometry constructed from the De Sitter space-time is equivalent to an anholonomous geometry constructed from the 5-dimensional Euclidean space; but this latter method is much easy to treat from a mathematical point of view, because we must only transfer the Riemannian geometry of the usual General Relativity from 4 to 5 dimensions. It is to be noted, however, that, being the 5-dimensional space a projective space, we must use projective connections and whence the curvature tensor (4,9): so we arrive at the field equations (1). A word about the constant χ in the right-hand side of (1): we can put it equal to usual Einstein's constant because, as in General Relativity, it connects the physical phenomena in holonomous base geometry with the gravitation.

However, if one would make use of the equations (1), there would be great difficulties, because in a curved-space theory we are not immediately able to translate mathematical proceedings from the 5-dimensional to the 4-dimensional space. This problem can be solved completely in order of translating the 5-dimensional metric g_{AB} in an equivalent, induced 4-dimensional metric $\tilde{g}_{ab}$: indeed the normalization condition (5,8) now becomes

$$g_{AB}\,\bar{x}_A\,\bar{x}_B = r^2 \tag{2}$$

Resolving the (2) with respect to $\bar{x}_5$ and with the help of the (5,6), we obtain

$$\bar{x}_5 = r/A \tag{3a}$$

$$A^2 = g_{ab}\,\alpha_a\,\alpha_b + 2\,g_{a5}\,\alpha_a + g_{55} \tag{3b}$$

These formulae are very similar to the (5,9) to which they are reduced putting

$$(4)\ g_{ab} = \delta_{ab}\ ;\ g_{a5} = 0\ ;\ g_{55} = 1$$

Differentiating (3a) and remembering that in a Riemannian space $dg_{AB} = 0$, we obtain

$$d\bar{x}_5 = -\frac{r^2\,(g_{ab}\,\alpha_b + g_{a5})\,dx_a}{A^3} \tag{5}$$

From the (5,12) and (5) we obtain the transformation of the 5-dimensional line element

(6) $$ds^2 = g_{ab}\, d\overline{x}_a\, d\overline{x}_b + 2\, g_{a5}\, d\overline{x}_a\, d\overline{x}_5 + g_{55}\, d\overline{x}_5^2$$

into the 4-dimensional one

(7) $$ds^2 = \tilde{g}_{ab}\, dx_a\, dx_b$$

where

(8) $$\boxed{A^4\, \tilde{g}_{ab} = g_{ab}\, g_{55} - g_{a5}\, g_{b5}}$$

This is the formula of the induced 4-dimensional metric; substituting in it the values (4) we obtain exactly the Beltrami metric; therefore (8) gives the displacements from this latter.

It is useful, at this point, to compare the formula (8) with the (4,2) of Jordan-Thiry; the two formulae are coincident if we make the following identifications

(9) $$g_{ij} \to A^4 \frac{\tilde{g}_{ij}}{g_{55}} \qquad \gamma_{5i} \to g_{i5}$$
$$\gamma_{ij} \to g_{ij} \qquad \gamma_{55} \to g_{55}$$

Even if it is possible to connect, from a formal point of view, the (8) with the (4,2), it is a matter of course that the field equations in the two theories have a totally different meaning, both because the (1,5) are no longer valid in Projective General Relativity, and for the different geometrical structure with which they are endowed.

As we cannot make use directly of the results of the previous 5-dimensional unitary theories, the more direst way to solve the equations (1) would be that of rewriting them, by expriming the derivatives respect to homogeneous coordinates through the derivatives respect to nonhomogeneous coordinates, with the help of (5,14) and (3b). So we could, on principle, find g_{ab}, g_{a5} and g_{55} in function of nonhomogeneous coordinates and then, through the formula (8), give a physical meaning to them. Also a crude calculation, however, shows that in this fashion the equations (1) would become rather complicated and we should not understand the function accomplished in them by the components of g_{AB}; for these reasons it seemed fruitful make use of an interesting mathematical technique developed by CATTANEO [29], known as «projections technique».

7 — The projections technique.

The projections technique takes into account the scomposition of the space T_x, tangential at the point x of a Riemannian variety, in two supplementary subspaces, θ_x, tangential to the coordinate line x_4, and Σ_x perpendicular to θ_x, so that

$$T_x = \Sigma_x + \theta_x \tag{1}$$

Through complex mathematical methods every geometrical object of the Riemannian variety is decomposed (i.e. it is «projected») in components lying only on Σ_x, only on θ_x or mixed. All this machinery is based on the relation

$$g_{ij} = \gamma_{ij} - \gamma_i \gamma_j \tag{2}$$

where g_{ij} is the metrical tensor of the Riemannian variety under consideration, γ_{ij} the projection on Σ_x of g_{ij} (namely, also if only one of the indexes is equal to 4, $\gamma_{ij} = 0$) and $\gamma_i = \frac{g_{i4}}{\sqrt{-g_{44}}}$ The (2) is very similar to the formula of Jordan-Thiry and, as we have seen for this latter, we can connect it with the (6,8); for doing this, it is enough let to vary the indexes in (2) from 1 to 5 and make the following substitutions

$$\boxed{g_{ij} \to g_{AB} \; ; \; \gamma_{ij} \to A^4 \frac{\tilde{g}_{ij}}{g_{55}} = \gamma_{AB} \; ; \; \gamma_i \to \frac{g_{A5}}{\sqrt{g_{55}}} = \gamma_A} \tag{3}$$

Through the (3) it is possible to write systematically the geometrical objects of the 5-dimensional Riemannian space in terms of $\tilde{g}_{ab}, g_{a5}, g_{55}$ only; naturally the formulae, also being analogous to those of Cattaneo, have a different physical meaning. We will develope the calculations only for the Christoffel symbols of the first kind; the formulae of Cattaneo, translated in our language are

$$\begin{aligned}(AB, C) = (AB, C)^* - 1/2\, [\gamma_A (\Omega_{BC} + \tilde{K}_{BC}) + \gamma_B (\Omega_{CA} + \tilde{K}_{CA}) + \\ + \gamma_C (\varrho_{AB} - \tilde{K}_{AB})]\end{aligned} \tag{4}$$

where

(5a) $(AB, C)^* = 1/2\,(\tilde{\partial}_A\,\gamma_{BC} + \tilde{\partial}_B\,\gamma_{AC} - \tilde{\partial}_C\,\gamma_{AB})$

(5b) $\tilde{K}_{AB} = \tilde{\partial}_5\,\gamma_{AB}$

(5c) $\Omega_{AB} = \tilde{\Omega}_{AB} + \tilde{\Omega}_A\,\gamma_B - \gamma_A\,\tilde{\Omega}_B$

(5d) $Q_{AB} = \tilde{Q}_{AB} + \tilde{Q}_A\,\gamma_B + \gamma_A\,\tilde{Q}_B + 2\,\gamma_A\,\gamma_B\,\bar{\partial}_5\,\gamma^5$

(5e) $\tilde{Q}_{AB} = \gamma_5\left[\tilde{\partial}_A\left(\frac{\gamma_B}{\gamma_5}\right) + \tilde{\partial}_B\left(\frac{\gamma_A}{\gamma_5}\right)\right]$

(5f) $\tilde{\Omega}_{AB} = \gamma_5\left[\tilde{\partial}_A\left(\frac{\gamma_B}{\gamma_5}\right) - \tilde{\partial}_B\left(\frac{\gamma_A}{\gamma_5}\right)\right]$

(5g) $\tilde{Q}_A = -\,\gamma_5\,\tilde{\partial}_A\left(\frac{1}{\gamma_5}\right) + \bar{\partial}_5\left(\frac{\gamma_A}{\gamma_5}\right)$

(5h) $\tilde{\Omega}_A = -\,\gamma_5\,\tilde{\partial}_A\left(\frac{1}{\gamma_5}\right) - \bar{\partial}_5\left(\frac{\gamma_A}{\gamma_5}\right)$

The symbols $\tilde{\partial}_A$ and $\tilde{\partial}_5$ mean

(6a) $\tilde{\partial}_A = \bar{\partial}_A + \gamma_A\,\gamma^5\,\bar{\partial}_5$

(6b) $\tilde{\partial}_5 = \gamma^5\,\bar{\partial}_5$

(6c) $\gamma^5 = \frac{1}{\sqrt{g_{55}}}$

where $\bar{\partial}_A$ and $\bar{\partial}_5$ are the usual derivatives respect to homogeneous coordinates defined in the (5,14); developing the calculations with the help of the (3), we obtain (naturally all the symbols are 4-dimensional, being null the components also with only one index equal to 5)

$$(a\,b,\,c)^* = A\,(ab,\,c)_p + \frac{n\,A^3}{2r^2}\,(x_a\,\tilde{G}_{bc} + x_b\,\tilde{G}_{ac} - x_c\,\tilde{G}_{ab}) -$$

(7) $$-\frac{A}{2\,r}\,[G_{a5}\,x_s\,\partial_s\,(A^4\,\tilde{G}_{bc}) + G_{b5}\,x_s\,\partial_s\,(A^4\,\tilde{G}_{ac}) - G_{c5}\,x_s\,\partial_s\,(A^4\,\tilde{G}_{ab})] +$$

$$+\frac{n\,A^3}{2\,r^2}\,(G_{a5}\,\tilde{G}_{bc} + G_{b5}\,\tilde{G}_{ac} - G_{c5}\,\tilde{G}_{ab})$$

where

$$(ab,c)_p = 2A^3[ab,c] - \frac{A^4}{2g_{55}} < ab,c > + \frac{A^4}{g_{55}}(ab,c)_{\tilde{g}}\,;\; G_{ab} = \tag{8}$$

$$= \frac{g_{ab}}{g_{55}}\,;\; G_{a5} = \frac{g_{a5}}{g_{55}}\,;\; \widetilde{G}_{ab} = \frac{\tilde{g}_{ab}}{g_{55}}$$

with

$$[ab,c] = \widetilde{G}_{bc}\,\partial_a A + \widetilde{G}_{ac}\,\partial_b A - \widetilde{G}_{ab}\,\partial_c A \tag{9a}$$

$$< ab,c > = \widetilde{G}_{bc}\,\partial_a g_{55} + \widetilde{G}_{ac}\,\partial_b g_{55} - \widetilde{G}_{ab}\,\partial_c g_{55} \tag{9b}$$

and $(ab,c)_{\tilde{g}}$ are the usual Christoffel symbols of the first kind constructed from the metric $\tilde{g}_{ab}$. In a similar fashion we can obtain for the others geometrical objects defined in the (5)

$$\widetilde{K}_{ab} = \frac{A^3}{r\,g_{55}^{3/2}}\left(n\,\tilde{g}_{ab} - 4A\,\tilde{g}_{ab}\,x_s\,\partial_s A - A^2\,x_s\,\partial_s\tilde{g}_{ab} + A^2\,\widetilde{G}_{ab}\,x_s\,\partial_s g_{55}\right) \tag{10a}$$

$$\widetilde{Q}_{ab} = g_{55}^{1/2}\left[A\,\partial_{(a}G_{b)} + N x_{(a}G_{b)5} - \frac{A}{r}G_{(a5}\,x_s\,\partial_s G_{b)5} + 2N\,G_{a5}\,G_{b5}\right]; \tag{10b}$$

$$\widetilde{\Omega}_{ab} = g_{55}^{1/2}\left[A\,\partial_{[a}G_{b]5} + N\,x_{[a}G_{b]5} - \frac{A}{r}G_{[a5}\,x_s\,\partial_s G_{b]5}\right]\,;\; N = \frac{n}{r^2 A} \tag{10c}$$

$$\widetilde{Q}_a = -g_{55}^{-3}\left[-\frac{A\,g_{55}^{3/2}}{2}\partial_a g_{55} + N\,x_a\,g_{55}^{3/2} + \frac{A g_{a5}\,g_{55}^{1/2}}{2r}\,x_s\,\partial_s g_{55} + \right.$$
$$\left. + N\,g_{a5}\,g_{55}^{3/2} + \frac{A\,g_{55}}{r}\,x_s\,g_{55}\,\partial_s\,g_{a5} - N\,g_{a5}\,g_{55}^2\right] \tag{10d}$$

$$\widetilde{\Omega}_a = \widetilde{Q}_a - 2\left[\frac{A}{r}\,g_{55}\,x_s\,g_{55}\,\partial_s\,g_{a5} - N\,\delta_{a5}\,g_{55}^2\right] \tag{10e}$$

Making use of the (4) we are now able to exprime the 5-dimensional Christoffel symbols of the first kind only in terms of $\tilde{g}_{ab}$, g_{a5} and g_{55}.

At this point we could make use of all the formulae of Cattaneo to exprime in term of the same variables also all others 5-dimensional geometrical objects, as the Christoffel symbols of the second kind, the Riemann curvature tensor and the Ricci tensor; we prefer do not make heavy this work with the introduction of exceedingly long and complicated formulae; we will note only that the formulae of the Christoffel symbols of the second kind do not have a great interest in order of rewriting in 4-dimensional form the field equations, because the Riemann curvature tensor (and whence the Ricci tensor) can be written in a totally covariant form, making use only of the Christoffel symbols of the first kind through the well-known formula

$$R_{ABCD} = 1/2\,(\bar{\partial}_{CA}\, g_{BD} - \bar{\partial}_{DA}\, g_{BC} + \bar{\partial}_{DB}\, g_{AC} - \bar{\partial}_{CB}\, g_{AD}) + \tag{11}$$
$$+ g^{RS}\,[(CA, R)\,(DB, S) - (DA, R)\,(CB, S)]$$

For a better evaluation of the difference between the Cattaneo formulae and the ours ones, it will be useful to premise a brief outline on the physical meaning of the symbols present in the former. We must, first of all, remember that in General Relativity the concept of physical reference system is substituted by that of «reference fluid»; to choose such a fluid implies to choose, in a Riemannian variety V_4, a congruence Γ of timelike lines, constituted by the world-lines of the particles which constitute the fluid. Mathematically this means to fix a field of vectors tangential to these lines: the γ_a of Cattaneo (4-dimensional analogue of our γ_A) are precisely the components of such a field and, maintaining the hydrodynamical analogy, they can be viewed as proportionals to the components of the fluid «velocity». It follows, then, that $\widetilde{\Omega}_{ij}$ is coincident with the «spatial vortex tensor» of this fluid, whereas Ω_{ij} gives the «space-time vortex tensor»; finally $\widetilde{K}_{ij}$ represents the «fluid deformation velocity tensor». In the formulae (7)-(10), on the contrary, we do not have the concept of «reference fluid», because the induced metric $\widetilde{g}_{ab}$ represents only formally a 4-dimensional projection of g_{AB}, being really only a 4-dimensional «formulation» of it. Besides, in the (7)-(10) there is an explicit dependence from the space-time coordinates x_a, whereas in the Cattaneo formulae there is a dependence from the γ_a, i.e. from the «velocities»; this circumstance, distinguishing the 5-dimensional from the 4-dimensional case, is strictly connected with the fact that the 4-dimensional formulation of the Restricted Relativity implies the condition $u_a\, u_a = -c^2$ (velocity condition), whereas the 5-dimensional formulation of the Projective Relativity is possible only with the condition $\bar{x}_A\, \bar{x}_A = r^2$ (coordinate condition). To conclude, through the formula (6,8) of the induced metric and the «projections technique» of Cattaneo, we are able to rewrite the field equations

(6,1) in a 4-dimensional form with a clear view of the role played in them by g_{a5} and g_{55} and without make use of the complicated Cartan methods. The next step will be that of finding solutions of the (6,1) interesting from the physical point of view.

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FOOTNOTES

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