A GENERALIZATION OF ECHELON SPACES

G. CROFTS

ABSTRACT

In the sense that echelon spaces are projective limits of isometries of l^r, the concept is generalized to projective limits of linear homeomorphic images of $l^r\{E\}$, where E is a locally convex space. Characterizations of those generalized echelon spaces which are nuclear spaces and those which are Schwartz spaces are given.

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The method of constructing echelon sequence spaces developed in [5] has been studied and generalized by many authors (e. g. [2] and [7]). In [7] a generalization to vector valued sequence spaces was studied. We will generalize the construction of echelon spaces to vector valued sequence spaces in a manner different from [7]. The motivation of this work was to define a construction which would allow generalizations of the characterizations of nuclear and Schwartz echelon spaces (see [8] and [1]).

The generalized echelon spaces are defined in § 3 and consist of projective limits of spaces $l^r\{E\}$, for E locally convex. The characterizations involve properties of the projective system of operators for the projective limit. In § 4 we give some examples and apply our results to $\lambda\{E\}$, where λ is a sequence space.

All notation and terminology not defined can be found in [6].

§ 1. DEFINITIONS AND NOTATION

Let E be a vector space over the real or complex field. Let $\omega(E)$ denote the collection of all sequences in E and let $\varphi(E)$ denote the collection of sequences in E which are finitely nonzero. Under coordinatewise operations $\varphi(E)$ and $\omega(E)$ are vector spaces. By a *vector sequence space over* E we mean a vector subspace of $\omega(E)$ which contains $\varphi(E)$ (see [3]).

For E as above and U an absolutely convex, absorbing subset of E, we denote by p_U the gauge determined by U on E. E_U will denote $E/p_U^{-1}(\{0\})$ with the quotient norm given from p_U.

If E has a locally convex topology assigned, then throughout the paper we will mean by $\eta(E)$ a zero neighborhood base for the topology of E, consisting of barrels.

The following vector sequence spaces over E are used throughout this paper:

i) $l^r\{E\} = \{x \in \omega(E) \mid \Sigma_{n=1}^{\infty}(p_U(x_n^r)) < \infty$, for each $U \in \eta(E)\}$, topologized by the seminorms $\Pi_U^r(x) = (\Sigma_{n=1}^{\infty}(p_U(x_n))^r)^{1/r}$, for $1 \leq r < \infty$.

ii) $l^\infty\{E\} = \{x \in \omega(E) \mid \sup\{p_U(x_n) \mid i = 1, 2, \ldots\} < \infty$, for each $U \in \eta(E)\}$, topologized by the seminorms $\Pi_U^\infty = \sup\{p_U(x_n) \mid i = 1, 2, \ldots\}$.

iii) $c_0\{E\} = \{x \in \omega(E) \mid \lim_{n\to\infty} p_U(x_n) = 0$, for each $U \in \eta(E)\}$, topologized as in ii).

If E is a normed space, then from [6], page 359, $(l^r\{E\})' = l^{r'}\{E'\}$, where $1/r + 1/r' = 1$, if $1 < r < \infty$, and $r' = \infty$ if $r = 1$. The bilinear form is given by $< a, x > = \Sigma_{i=1}^{\infty} < a_i, x_i >$, where $< a_i, x_i >$ is the bilinear form between E and E'.

For $S(E)$ a vector sequence space over E and $y = (y_n)$ an element of $S(E)$, we denote by y^i. the element of $S(E)$ which has y_i. as the ith component and 0 for all other components. $y^{(n)}$ will denote the element $\Sigma_{i=1}^{n} ye^i$ in $S(E)$.

Let $S(E)$ and $R(F)$ denote vector sequence spaces over E and F, respectively. A linear map T from $S(E)$ into $R(F)$ will be called a *diagonal map* if the action of T is $T(x) = (T_n(x_n))$, for some sequence $\{T_n\}$ of linear maps from E into F. For T a diagonal map we denote by T^e the map $T^i(x) = (T(x))e^i$, and by $T^{(n)}$ the map $T^{(n)}(x) = (T(x))^{(n)}$.

A linear map B from a normed space G into a normed space F is said to be *nuclear* if there are sequences $\{b_n\} \subset G'$, and $\{y_n\} \subset F$, such that $B(x) = \Sigma_{n=1}^{\infty} < b_n, x > y_n$ and $\Sigma_{n=1}^{\infty} ||b_n|| \, ||y_n|| < \infty$ (see [8]). The *nuclear norm* of B is given by

$$\nu(B) = \inf \{\Sigma_{n=1}^{\infty} ||b_n|| \, ||y_n|| \mid \text{the } \{b_n\} \text{ and } \{y_n\} \text{ satisfy the properties above}\}.$$

If B is a continuous linear map, we will use $\beta(B)$ to denote the operator norm of B.

A locally convex space E is said to be a *nuclear space* if for each $U \in \eta(E)$ there is a $V \in \eta(E)$, $V \subset U$, such that the canonical map from $\widehat{E}_V$ into $\widehat{E}_U$ is nuclear, where $\widehat{E}_U$ is the completion of E_U (see [9]). A locally convex space E is said to be a *Schwartz space* if for each $U \in \eta(E)$ there is a $V \in \eta(E)$, $V \subset U$, such that the canonical map from $\widehat{E}_V$ into $\widehat{E}_U$ is compact (see [10]).

§ 2. Diagonal Maps

Throughout this section we assume E and F are normed spaces with $1 \leq r < \infty$. Characterizations of those diagonal maps from $l^r\{E\}$ into $l^r\{F\}$ which are nuclear and those which are precompact are presented. These characterizations are those expected from the comparable results in l^r (see [11] and [1]).

2.1. Proposition: A diagonal map T from $l^r\{E\}$ into $l^r\{F\}$ is nuclear if and only if for each i, T_i. is a nuclear and $\Sigma_{i=1}^{\infty} \nu(T_i) < \infty$, where ν is the nuclear norm of maps from E to F. Also, the nuclear norm of T, $\nu(T)$, is given by $\Sigma_{i=1}^{\infty} \nu(T_i)$.

Proof: We first assume T is nuclear. Hence T has the form

$$T(x) = \Sigma_{n=1}^{\infty} < x, \; a > y^n,$$

for some $\{a^n\} \subset (l^r\{E\})'$ and some $\{y^n\} \subset l^r\{F\}$ with $\Sigma_{n=1}^{\infty} ||a^n||_{r'}$ $||y^n||_r < \infty$. (As before $1/r + 1/r' = 1$, for $1 < r < \infty$, and $r' = \infty$, if $r = 1$). Since $(l^r\{E\})' = l^{r'}\{E'\}$, each a_i^n is in E'. Using the notation of § 1 we have

$$T^i(x) = T(xe^i) = \Sigma_{n=1}^{\infty} < xe^i, \; a^n > y^n = \Sigma_{n=1}^{\infty} < x_i, \; a_i^n > y^n e^i.$$

Hence

$$T_i(x_i) = \Sigma_{n=1}^{\infty} < x_i, a_i^n > y_i^n,$$

with

$$\Sigma_{n=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_E \leq \Sigma_{n=1}^{\infty} ||a^n||_{r'} ||y^n||_r < \infty.$$

Thus each T_i is nuclear.

From the definition $\nu(T_i) \leq \Sigma_{n=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_F$. Thus

$$\Sigma_{i=1}^{\infty} \nu(T_i) \leq \Sigma_{i=1}^{\infty} \Sigma_{n=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_F =$$
$$= \Sigma_{n=1}^{\infty} \Sigma_{i=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_F \leq \Sigma_{n=1}^{\infty} ||a^n||_{r'} ||y^n||_r < \infty,$$

using Holder's inequality to obtain the second inequality. This shows the necessity of the condition and also yields $\Sigma_{i=1}^{\infty} \nu(T_i) < \nu(T)$.

We now assume each T_i is nuclear with $\Sigma_{i=1}^{\infty} \nu(T_i) < \infty$. From our assumption each T_i has the form $T_i(x_i) = \Sigma_{n=1}^{\infty} < a_i^n, x_i > y_i^n$, where $\{a_i^n\} \subset E'$, $\{y_i^n\} \subset F$, and $\Sigma_{n=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_F < \infty$. Let $\{a^n\}$ and $\{y^n\}$ be the elements of $\omega(E')$ and $\omega(F)$ given by $a^n = (a_i^n)$, $y^n = (y_i^n)$. This yields $T^i(x) = \Sigma_{n=1}^{\infty} < a^n e^i, x > y^n e^i$, with

$$\Sigma_{n=1}^{\infty} ||a^n e^i||_{r'} ||y^n e^i||_r = \Sigma_{n=1}^{\infty} ||a_i^n||_{E'} ||y_i^n||_F < \infty.$$

Hence T^i is nuclear with $\nu(T^i) \leq \nu(T_i)$.

Since $T^{(n)} = \Sigma_{i=1}^{n} T^i$, we have $T^{(n)}$ nuclear. For $n \leq m$, $\nu(T^{(m)} - T^{(n)}) \leq \Sigma_{i=n+1}^{m} \nu(T_i)$. Thus $\{T^{(n)}\}$ is a Cauchy sequence in the normed space of nuclear maps from $l^r\{E\}$ into $l^r\{F\}$. Clearly $\lim_{n\to\infty} T^{(n)}(x) = T(x)$ in the norm of $l^r\{F\}$, for each $x \in l^r\{E\}$. Hence by 3.1.3 of [8], T is nuclear from $l^r\{E\}$ into $l^r\{F\}$, with $T = \lim_{n\to\infty} T^{(n)}$ in the nuclear norm. Therefore $\nu(T) \leq \Sigma_{i=1}^{\infty} \nu(T^i) \leq \Sigma_{i=1}^{\infty} \nu(T_i)$.

2.2 Remark: If E and F are Banach spaces and either E' or F has the approximation property, then it follows from [4], page 158, and 2.1, that the space of diagonal nuclear maps from $l^r\{E\}$ into $l^r\{F\}$ is just $l^1\{E' \tilde{\otimes} F\}$, where $\tilde{\otimes}$ denotes the projective tensor product.

2.3 Proposition: A diagonal map T from $l^r\{E\}$ into $l^r\{F\}$ is precompact if and only if each T_i is precompact and $\lim_{i\to\infty} \beta(T_i) = 0$. Also, $\beta(T) = \sup\{\beta(T_i) \mid i = 1, 2, \ldots\}$.

PROOF: We first suppose that T is precompact. Let B_E denote the unit ball of E and let $B_{l^r\{E\}}$ denote the unit ball of $l^r\{E\}$. Our assumtion is that $T(B_{l^r\{E\}})$ is precompact in $l^r\{F\}$. Thus for fixed i, $\{(T(x))e^i \mid x_i \in B_E\}$ is precompact in F. This verifies that each T_i is precompact.

Let $\varepsilon > 0$ be given. For each i let $z_i \in B_E$ with $\beta(T_i) < \|T_i(z_i)\|_F + \varepsilon$. The set $\{T(ze^i) \mid i = 1, 2, \ldots\}$ is precompact in $l^r\{F\}$. Hence there is a finite set $\{y^1, y^2, \ldots, y^m\} \subset l^r\{F\}$ with the property that for each i there is a j with $\|T(ze^i) - y^j\| < \varepsilon$. Since $r < \infty$, there is an integer n_0 with $\|y^j_n\| < \nu$, each $j = 1, 2, \ldots, m$, and each $n \geq n_0$. Finally for $n \geq n_0$

$$\beta(T_n) < \|T_n(z_n)\|_F + \varepsilon \leq \|y^j_n\|_F + \|T(ze^n) - y^j\|_F + \varepsilon < 3\,\varepsilon.$$

Conversely, suppose each T_i is precompact with $\lim\limits_{i\to\infty} \beta(T_i) = 0$. Thus T^i is precompact and $T^{(n)}$ is precompact from $l^r\{E\}$ into $l^r\{F\}$. From the definitions

$$\begin{aligned}\beta(T - T^{(n)}) &= \sup\{(\Sigma_{i=n+1}^{\infty} \|T_i(x_i)\|^r)^{1/r} \mid x \in l^r\{E\},\ \|x\|_r \leq 1\}\\ &\leq \sup\{(\Sigma_{i=n+1}^{\infty} (\beta(T_i)\,\|x_i\|_E)^r)^{1/r} \mid \|x\|_r \leq 1\}\\ &\leq \sup\{\beta(T_i) \mid i \geq n+1\}.\end{aligned}$$

Since $\{\beta(T_i)\}$ converges to 0, it follows that $\{T^{(n)}\}$ converges to T with respect to the β norm. Thus T is precompact.

The method used in the previous paragraph also shows that $\beta(T) \leq \sup\{\beta(T_i) \mid i = 1, 2, \ldots\}$. The reverse inequality is clear.

2.4 REMARK: If E and F are Banach spaces and either E' or F has the approximation property, then from [4, page 168] or [9, page 114] and 2.3, we may conclude that the space of diagonal compact maps of $l^r\{E\}$ into $l^r\{F\}$ is just $c_0\{E' \mathbin{\tilde{\tilde{\otimes}}} F\}$, where $\tilde{\tilde{\otimes}}$ denotes the bi-equicontinuous tensor topology.

§ 3. ECHELON SPACES OVER E

After defining echelon spaces over E we will characterize those which are nuclear spaces and those which are Schwartz spaces. Throughout this section E is a locally convex space.

3.1 DEFINITION: Let $1 \leq r < \infty$ and let $\{T^k\}$ be a sequence of diagonal maps from $l^r\{E\}$ into $l^r\{E\}$ satisfying for each i and k:

i) $T_i^k : E \to E$ is a linear homeomorphism.

ii) $T_i^{k+1}(U) \subset T_i^k(U)$, for each $U \in \eta(E)$, $\eta(E)$ some zero neighborhood base for E.

Such a sequence of maps will be called a sequence of *echelon type*.

3.2 LEMMA: If $\{T^k\}$ is of echelon type, then $T^{k+1}(l^k\{E\}) \subset T^k(l^k\{E\})$, for each k.

PROOF: Let $\eta(E)$ be a fundamental system of zero neighborhoods satisfying (ii) of 3.1. Given $U \in \eta(E)$ and i and k arbitrary, $p_{T_i^k(U)}$ is a continuous semi-norm on E. It is straightforward to check that $p_{T_i^k(U)}(x_i) = p_U((T_i^k)^{-1}(x_i))$, for $x_i \in E$. Thus $x \in \omega(E)$ is in $T^k(l^r\{E\})$ if and only if $\Sigma_{i=1}^{\infty}(p_{T_i^k(U)}(x_i))^r < \infty$, for each $U \in \eta(E)$. Property (ii), of 3.1, yields $p_{T_i^{k+1}(U)}(x_i) \geq p_{T_i^k(U)}(x_i)$, for each $x_i \in E$. The result easily follows.

For $\{T^k\}$ a sequence of echelon type we have $T^k(l^r\{E\})$ a vector sequence space in $\omega(E)$, for each k. Hence $\bigcap_k T^k(l^r\{E\})$ is a vector sequence space which we shall call an *echelon space over* E, and denote by $S(E)$.

We can view $S(E)$ as the kernel of the spaces $T^k(l^r\{E\})$. We give each $T^k(l^r\{E\})$ the topology causing $T^k(l^r\{E\})$ and $l^r\{E\}$ to be linearly homeomorphic under T^k. Thus $S(E)$ can be assigned the kernel-topology determined by the topology given to $T^k(l^r\{E\})$. Using [6], page 226, and the inclusion given in 3.2, a fundamental system of zero neighborhoods for the kernel-topology on $S(E)$ is given by all sets of the form $S(E) \cap T^k(W_U)$, where $U \in \eta(E)$, k is a natural number, and

$$W_U = \{x \in l^r\{E\} \mid (\Sigma_{i=1}^{\infty}(p_U(x_i))^r)^{1/r} \leq 1\}.$$

For the remainder of the paper $S(E)$ will be understood to have this topology.

To simplify notation we shall denote by $S_{k,U}$ the normed space $S(E)_{(T^k(W_U) \cap S(E))}$, for $U \in \eta(E)$.

3.3 LEMMA: Suppose $1 \leq r < \infty$ and $S(E)$ is an echelon space over E. The completion of $S_{k,U}$ is isometric to $l^r\{\widehat{E}_U\}$, where $\widehat{E}_U$ is the completion of E_U.

PROOF: Some details of this proof are omitted, but in each case the details are straightforward.

Let μ denote the quotient map of E onto E_U, with q denoting the gauge of U on E. Let p denote the gauge of $T^k(W_U) \cap S(E)$ on $S(E)$ with G denoting $p^{-1}(\{0\})$. Define $\Psi : S_{k,U} \to l^r\{E_U\}$ by $\Psi(x+G) = (\mu \circ (T_i^k)^{-1}(x_i))_i$. Ψ is clearly linear and we will show that Ψ is an isometry.

For $x + G \in S_{k,U}$, let $\|x + G\|$ denote the norm of $x + G$ in $S_{k,U}$. By definition $\|x + G\| = \inf \{p(x+g) \mid g \in G\}$. For $y \in l^r\{E\}$, let $\|y\|_U$ denote the norm of $(\mu(y_i))_i$ in $l^r\{E_U\}$. By definition $\|y\|_U = (\Sigma_{i=1}^{\infty} < \inf \{q(y_i + t) \mid t \in q^{-1}(\{0\})\} >^r)^{1/r}$. Given $g \in G$, we have $p(x+g) = p_{W_U}((T^k)^{-1}(x+g)) = (\Sigma_{i=1}^{\infty}(q((T_i^k)^{-1}(x_i)))^r)^{1/r} \geq \|(T^k)^{-1}(x)\|_U$, the second equality following since $q((T_i^k)^{-1}(g_i)) = 0$, for $g \in G$. Thus $\|x + G\| \geq \|(T^k)^{-1}(x)\|_U$.

For the reverse inequality let $0 < \varepsilon < 1$ be given. By the definitions of $S_{k,U}$ and $l^r\{E\}$, there is an n_0 for which $\|x - x^{(n_0)} + G\| < \varepsilon$. Also, for each i, $l \leq i \leq n_0$, there is a t_i in $q^{-1}(\{0\})$ with $q((T_i^k)^{-1}(x_i) + t_i) - (\varepsilon/2i) <$ norm of $\mu((T_i^k)^{-1}(x_i))$ in E_U. Define $z = (z_i)$ by $z_i = (T_i^k)^{-1}(x_i) + t_i$, $1 \leq i \leq n_0$, and $z_i = 0$ otherwise. Then $T^k(z) + G = x^{(n_0)} + G$. Using the Minkowski inequality for the last of the following inequalities, we have

$$\|x + G\| \leq \|x^{(n_0)} + G\| + \|x - x^{(n_0)} + G\| < \|T^k(z) + G\| + \varepsilon \leq \varepsilon + (\Sigma_{i=1}^{n_0}(q(z_i))^r)^{1/r} < \varepsilon + \varepsilon + \|(T^k)^{-1}(x)\|_U.$$

The arbitrariness of ε gives the desired inequality.

It is easy to check that $l^r\{E_U\}$ is isometric to a dense subspace of $l^r\{\widehat{E}_U\}$. The lemma will be proved if we can show that $\Psi(S_{k,U})$ is dense in $l^r\{E_U\}$. Let $w = (w_i) \in \varphi(E_U)$. Then $w_i = 0 + q^{-1}(\{0\})$ for each $i > n_1$, some n_1, and $w_i = y_i + q^{-1}(\{0\})$ for $1 \leq i \leq n_1$, with $y_i \in E$. Define $z = (z_i)$ by $z_i = y_i$, for $1 \leq i \leq n_1$, and $z_i = 0$ otherwise. Then $T^k(z) + G \in S_{k,U}$ and $\Psi(T^k(z) + G) = w$. The denseness of $\varphi(E_U)$ in $l^r\{E_U\}$ completes the proof.

We can now give the characterizations announced at the beginning of the section.

If k is an inteegr and $U \in \eta(E)$, let $\widetilde{T_i^k}$ denote the linear map from $\widehat{E}_U$ into $\widehat{E}_{T_i^k(U)}$ induced by T_i^k, and let $\widetilde{(T_i^k)^{-1}}$ denote the linear map from $\widehat{E}_{T_i^k(U)}$ into $\widehat{E}_U$ induced by $(T_i^k)^{-1}$.

3.4 THEOREM: Suppose $1 \leq r < \infty$ and let $S(E) = \bigcap_k T^k(l^r\{E\})$ be an echelon space over E. Then $S(E)$ is a nuclear space if and only if there is an $\eta(E)$ so that given k and $U \in \eta(E)$, there is a j and a $V \in \eta(E)$ for which $\widetilde{(T_i^k)^{-1}} \circ (\widetilde{T_i^j})$ is a nuclear map from $\widehat{E}_V$ into $\widehat{E}_U$, for each i, and

$$\Sigma_{i=1}^{\infty} \nu\,(\widetilde{(T_i^k)^{-1}} \circ (\widetilde{T_i^j})) < \infty,$$

where ν is the nuclear norm for maps of $\widehat{E}_V$ into $\widehat{E}_U$.

PROOF: Let k and j be integers with $j \geq k$, and let U and V be elements of $\eta(E)$ with $V \subset U$. Let Ψ^j and Ψ^k, respectively, denote the isometry of $S_{j,V}$ into $l^r\{\widehat{E}_V\}$ and the isometry of $S_{k,U}$ into $l^r\{\widehat{E}_U\}$ described in the proof of 3.3. The following diagram then commutes:

$$\begin{array}{ccc} S_{j,V} & \xrightarrow[\text{map}]{\text{canonical}} & S_{k,U} \\ \downarrow \Psi^j & & \Psi^k \downarrow \\ l^r\{E_V\} & \xrightarrow{\widetilde{(T^k)^{-1}} \circ (\widetilde{T^j})} & l^r\{E_U\}. \end{array}$$

Thus from 3.3 and the defintion of a nuclear space, the space $S(E)$ is nuclear if and only if $\widetilde{(T^k)^{-1}} \circ (\widetilde{T^j})$ is a nuclear map from $l^r\{\hat{E}_V\}$ into $l^r\{\hat{E}_U\}$. The theorem thus follows from 2.1.

3.5 THEOREM: Suppose $1 \leq r < \infty$ and let $S(E) = \bigcap_k T^k(l^r\{E\})$ an echelon space over E. Then $S(E)$ is a Schwartz space if and only be if there is an $\eta(E)$ so that given k and $U \in \eta(E)$, there is a j a and $V \in \eta(E)$ for which $\widetilde{(T_i^k)^{-1}} \circ (\widetilde{T_i^j})$ is a compact map from $\hat{E}_V$ to $\hat{E}_U$, for each i, and

$$\lim_{i \to \infty} \beta\,(\widetilde{(T_i^k)^{-1}} \circ (\widetilde{T_i^j})) = 0,$$

where β is the operator norm for maps from $\hat{E}_V$ into $\hat{E}_U$.

PROOF: The proof is similar to 3.4 using 2.3 instead of 2.1 for the final conclusion.

3.6 REMARK: Let $S(E)$ be an echelon space on E. Since each T_1^k is a linear homeomorphism of E into E, the collection $\{T_1^k(U) \mid k = 1, 2, \ldots,$

and $U \in \eta(E)\}$ forms a zero neighborhood base for E. If k and $U \in \eta(E)$ are given, then for $j > k$ and $V \subset U$, $V \in \eta(E)$, it makes sense to speak of the canonical map of $\hat{E}_{T_1^k(V)}$ into $\hat{E}_{T_1^k(U)}$. This map is of the form $(\widetilde{T_1^k}) \circ \widetilde{(T_1^k)^{-1}} \circ (\widetilde{T_1^j}) \circ \widetilde{(T_1^j)^{-1}}$. The composition of continuous maps and a nuclear (compact) map results in a nuclear (compact) map. Thus using 3.4 (3.5) we note that E is a nuclear (Schwartz) space if $S(E)$ is a nuclear (Schwartz) space.

§ 4. Examples, and Application to $\lambda\{E\}$

In this section we show the existence of echelon spaces over E by giving some examples. These examples and the conclusions of § 3 provide knowledge about $\lambda\{E\}$.

4.1 Examples: 1. Let E be a locally convex space with $\{f_n\}$ a sequence of linear homeomorphisms of E into E. Let $\{a^k\}$ be a sequence of sequences of real numbers with $0 < a_n^k < a_n^{k+1}$, for each k and n. Define $\{T^k\}$ by $T_n^k = (1/a_n^k) f_n$. Then $S(E) = \bigcap_k T^k(l^r\{E\})$ is an echelon space over E.

Proof: We need only check that $\{T^k\}$ is a sequence of echelon type. Clearly each T_n^k is a linear homeomorphism. For U a zero neighborhood of E, $T_n^{k+1}(U) = (1/a_n^{k+1}) f_n(U) \subset (1/a) f_n(U) = T_n^k(U)$.

2. Let $\{T^k\}$ be as in 1. and let E be a nuclear space with $\Sigma_{n=1}^{\infty} a_n^k / a_n^{k+1} < \infty$, for each k. Then $S(E) = \bigcap_k T^k(l^r\{E\})$ is a nuclear space.

Proof: If $U \in \eta(E)$, then by the nuclearity of E there is a $V \in \eta(E)$, $V \subset U$, with the canonical map $\Psi: E_V \to E_U$ a nuclear map. By definition $\widetilde{(T_n^k)^{-1}} \circ (\widetilde{T_n^j}) = (a_n^k / a_n^j)\, \Psi$, and hence $\nu(\widetilde{(T_n^k)^{-1}} \circ (\widetilde{T_n^j})) = (a_n^k / a_n^j)\, \nu(\Psi)$. Thus for $j > k$, the property of 3.4 holds.

3. Let $\{T^k\}$ be as in 1., and let E be a Schwartz space with $\lim_{n\to\infty} a_n^k / a_n^{k+1} = 0$, for each k. Then $S(E) = \bigcap_k T^k(l^r\{E\})$ is a Schwartz space.

PROOF: Similar to 2.

Let λ be a normal sequence space over the complex or real field. Suppose λ has the topology $\mathcal{J}_b(\lambda^x, \lambda)$ (see [6]). Let E be a locally convex space. By $\lambda\{E\}$ we mean $\{x \in \omega(E) \mid (p_U(x_n))_n \in \lambda$, for each $U \in \eta(E)\}$. $\lambda\{E\}$ is topologized by the semi-norms $q((p_U(x_n))_n)$, where q ranges over the continuous semi-norms of λ.

Suppose λ is the echelon space $\bigcap_k (1/a^k)\, l^r$ (see [2]) and $\{T^k\}$ is given as in 4.1 with f_n the identity map for each n. Then $\lambda\{E\}$ and $S(E)$ are equal both algebraically and topologically. The algebraic equality follows since x is in $\lambda\{E\}$ if and only if $(p_U(x_n))_n \in \lambda$, for and each $U \in \eta(E)$, if and only if $(a_n^k p_U(x_n))_n \in l^r$, for each k and each U, if and only if $(a_n^k x_n)_n \in l^r\{E\}$, for each k. However, $(a_n^k x_n)_n = (T^k)^{-1}(x)$. Writing explicitly a fundamental zero neighborhood for $S(E)$ and a fundamental zero neighborhood for $\lambda\{E\}$ will show the topological equality. Such a neighborhood in $S(E)$ is $\{x \mid (T^k)^{-1}(x) = (a_n^k x_n)_n$ satisfies $(\Sigma_{n=1}^{\infty} (p_U(a_n^k x_n))^r)^{1/r} \leq 1$, for some $U \in \eta(E)\}$ (recall the discussion after 3.3). A zero neighborhood in $\lambda\{E\}$ is $\{x \mid (a_n^k(p_U(x_n))_n)$ is in the unit ball of l^r for some $U \in \eta(E)\}$.

Using 3.4, 3.5, and 4.1 we have established:

4.3 PROPOSITION: If λ is a nuclear (Schwartz) echelon space of the form $\lambda = \bigcap_k (1/a^k)\, l^r$ and E is a nuclear (Schwartz) space, then $\lambda\{E\}$ is a nuclear (Schwartz) space.

4.4 REMARK: As stated in the introduction the motivation was to obtain theorems which were generalizations of theorems for echelon spaces. This is the only reason we consider only a sequence of functions in 3.1 and 3.2 Each of the theorems and examples are valid for the case $\{T^\alpha\}$ is a net of diagonal functions.

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George Crofts
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061