ON THE COMPLETENESS OF A TOPOLOGICAL GROUP OF FUNCTIONS

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§ 1 SUMMARY. The purpose of the present paper is to generalize the results of N. ADASCH [1] on the completeness of the topological vector space $L_\sigma(E, F)$ to the context of topological groups. It may be recalled that every topological vector space is an additive abelian topological group but not conversely. We first prove an auxiliary theorem that if E be a non-empty set, S a family of subsets of E satisfying the condition:

$$A, B \in S \Rightarrow \exists C \in S \quad \text{such that} \quad A \cup B \subseteq C$$

and F be an (additive) abelian topological group then the set $m(E, F)$ of all maps of E into F can be made into an (additive) abelian topological group. With the help of this auxiliary result we prove our main theorem that if E be a topological space and F be an (additive) abelian separated complete topological group and S a family of subsets of E with the property that union of any two members of S is a subset of a member of S and union of all members of S is E then the completion

$$\widehat{C_{J_S}(E, F)} \quad \text{of} \quad C_{J_S}(E, F)$$

has the following form:

$$\widehat{C_{J_S}(E, F)} = \bigcap_{\substack{B \in S \\ V \in \{V\}}} (C(E, F) + [B, V]_m)$$

where $C(E, F)$ denotes the set of all continuous maps of E into F. Here J_S denotes a suitable topology defined by S. After proving this

result we note that this generalizes the result of N. ADASCH quoted above. It may also be observed that some similar results can be established if E and F are Hausdorff uniform spaces [4]. Finally we note that our results include as special cases the results of A. P. and W. J. ROBERTSON [5] and A. GROTHENDIECK [2] on the representation of the completion of locally convex Hausdorff spaces.

§ 2. BASIC NOTIONS: — We shall adopt the following basic notions from [3].

1. In each (additive) abelian topological group F, there exists a fundamental system $\{U\}$ of closed neighbourhoods of the identity i such that:

(a) Each U is symmetric.

(b) For each U in $\{U\}$ there exists a V in $\{U\}$ such that

$$V + V \subseteq U.$$

Conversely, given an additive abelian group F with a filter base $\{U\}$ satisfying (a) and (b), there exists a unique topology u on F such that F_u is a topological group and $\{U\}$ forms a fundamental system of neighbourhoods of i.

2. A topological group F is separeted *iff* $\cap U = \{i\}$ where U varies over a fundamental system of neighbourhoods of i.

3. Let Y be a closed subset of an (additive) topological group F and $a \in F$. Then $a + Y$ and $Y + a$ are closed.

4. Every topological vector space is an additive abelian topological group.

§ 3. *Main results:*

THEOREM 1. Let E be a set and S a family of subsets of E satisfying the following condition:

$$A, B \in S \Rightarrow \exists C \in S \text{ such that } A \cup B \subseteq C.$$

Let F be an (additive) abelian topological group. Let $m(E, F)$ be the set of all maps of E into F. Then there exists an (additive) abelian group structure on $m(E, F)$ and a unique topology J_S compatible with this group structure such that $m_{J_S}(E, F)$ is a topological group.

PROOF. We first note that by the usual definition of addition of maps given by

$$(f+g)(x) = f(x) + g(x), \ x \varepsilon E$$

the set $m(E, F)$ can be given an additive group structure.

If i be the identify element of F then the map f_0 given by

$$f_0(x) = i \quad \text{for all} \quad x \varepsilon E,$$

serves as the identity element of $m(E, F)$. This group structure on $m(E, F)$ is abelian since F is abelian. We now propose to define a compatible topology on the abelian group $m(E, F)$. Since F is an (additive) abelian topological group, there exists (on account of 1 of § 2) a fundamental system $\{U\}$ of closed neighbourhoods of the identity i such that

(a) Each U is symmetric.

(b) For each U in $\{U\}$ there exists a V in $\{U\}$ such that

$$V + V \subseteq U.$$

For each $A \varepsilon S$ and $U \varepsilon \{U\}$ we define

$$[A, U]_m = \{f : f \varepsilon m(E, F) \ \& \ f(A) \subseteq U\}.$$

We claim that the family $\{[A, U]_m\}$, where A varies over S and U varies over $\{U\}$, forms a fundamental system of neighbourhoods of the identity element f_0 in $m(E, F)$ for a topology J_S on $m(E, F)$ which makes $m(E, F)$ a topological group.

To verify our claim let $[A, U]_m$ and $[B, V]_m$ be any two members in the family $\{[A, U]_m\}$. Then by hypothesis $\exists\, C \varepsilon S$ such that $A \cup B \subseteq C$. Also $\exists\, W \varepsilon \{U\}$ such that $W \subseteq U \cap V$. Hence for each $f \varepsilon [C, W]_m$ we have

$$f(C) \subseteq W \subseteq U \cap V.$$

This implies that $f(A) \subseteq f(C) \subseteq U$, and $f(B) \subseteq f(C) \subseteq V$. Hence we have $f \varepsilon [A, U]_m \cap [B, V]_m$. Consequently, we have

$$[C, W]_m \subseteq [A, U]_m \cap [B, V]_m.$$

Then it follows quite easily that the family $\{[A, U]_m\}$ is a filter base on $m(E, F)$.

Secondly, we note that the symmetry of U implies that each $[A, U]_m$ is symmetric.

Lastly, consider any $[B, V]_m \in \{[A, U]_m\}$. On account of (b), $\exists\, W \in \{U\}$ such that $W + W \subseteq V$. From this it follows easily that $[B, W]_m + [B, W]_m \subseteq [B, V]_m$.

In fact, if we take any $f = g + h$, where $g, h \in [B, W]_m$, then $f(B) = (g + h)(B) \subseteq g(B) + h(B) \subseteq W + W \subseteq V$.

Consequently, $f \in [B, V]_m$.

Hence by the converse part of 1 in § 2 there exists a unique topology J_S on $m(E, F)$ such that $m_{J_S}(E, F)$ is a topological group and the family $\{[A, U]_m\}$ forms a fundamental system of neighbourhoods of the identity f_0 in $m(E, F)$.

Cor 1. Let E be a topological space and S a family of subsets of E satisfying the condition of theorem 1. Let F be an (additive) abelian topological group. Let $C(E, F)$ be the set of all continuous maps of E into F. Then there exists an additive abelian group structure on $C(E, F)$ and a unique compatible topology (namely the topology induced from $m_{J_S}(E, F)$ such that $C(E, F)$ equipped with such a topology is a topological group. We shall denote this topological group by $C_{J_S}(E, F)$.

Proof follows directly from theorem 1 and the fact that every algebraic subgroup of a topological group is a topological group in the induced topology.

Cor. 2. If in cor. 1., S be the family of all finite subsets of E, then the group $C(E, F)$ with the topology induced from $m_{J_S}(E, F)$ will be denoted by $C_{J_\sigma}(E, F)$. This topology J_σ turns out to be the topology of simple convergence.

Theorem 2. Let E be a topological space and F be an (additive) abelian separated complete topological group. Let S be a family of subsets of E satisfying

(a) $A, B \in S \Rightarrow \exists\, C \in S$ such that $A \cup B \subseteq C$.

(b) $\bigcup_{A \in S} A = E$.

Then the completion of $C_{J_S}(E, F)$ in $m_{J_S}(E, F)$ up to an isomorphism may be represented as

$$\widetilde{C_{J_S}(E, F)} = \bigcap_{\substack{B \in S \\ V \in \{V\}}} (C(E, F) + [B, V]_m). \tag{1}$$

where $\{V\}$ is a fundamental system of closed neighbourhoods of the identity element i in F.

PROOF. We denote the intersection set in (1) by R. It is clear that $R \subseteq m_{J_S}(E, F)$. We consider the induced topology on R and denote the topological space thus obtained by (R, J_S^*).

(i) We first prove that R is an algebraic subgroup of the (additive) abelian topological group $m_{J_S}(E, F)$.

To show this let $g, h \varepsilon R$ be arbitrary. Then

$$g, h \varepsilon C(E, F) + [B, V]_m \quad \text{for all} \quad B \varepsilon S, \ V \varepsilon \{V\}.$$

Now take any $B \varepsilon S$, $V \varepsilon \{V\}$. Then there exists $W \varepsilon \{V\}$ such that $W + W \subseteq V$. Put $g = g_1 + g_2$, $h = h_1 + h_2$, where $g_1, h_1 \varepsilon C(E, F)$; $g_2, h_2 \varepsilon [B, W]_m$.

Then $g - g_1 = g_2 \varepsilon [B, W]_m$ and $h - h_1 = h_2 \varepsilon [B, W]_m$.

Therefore,

$$(g - g_1) - (h - h_1) = g_2 - h_2 \varepsilon [B, W]_m + [B, W]_m \subseteq [B, V]_m.$$

Then by repeated application of the commutativity and associativity of the addition in the additive group $m(E, F)$ we have

$$g - h - g_1 + h_1 \varepsilon [B, V]_m.$$

Now $$g - h = (g - h - g_1 + h_1) + (g_1 - h_1) = (g_1 - h_1) + (g - h - g_1 + h_1) \varepsilon C(E, F) + [B, V]_m.$$

Now since $B \varepsilon S$ and $V \varepsilon \{V\}$ were arbitrary, it follows that $g - h \varepsilon R$. Hence R is an additive abelian subgroup of the topological group $m_{J_S}(E, F)$. Therefore (R, J_S^*) is a topological subgroup of $m_{J_S}(E, F)$ Consequently, (R, J_S^*) is an (additive) abelian topological group.

(ii) We next prove that (R, J_S^*) and $C_{J_S}(E, F)$ are both separated topological spaces. For this purpose it is enough to show that $m_{J_S}(E, F)$ is separated. In view of 2 of § 2 it is sufficient to show that

$$\bigcap_{\substack{B \varepsilon S \\ V \varepsilon \{V\}}} [B, V]_m = \{f_0\} \quad (2)$$

where f_0 is the identity element of $m(E, F)$.

In fact, if $f \varepsilon \cap [B, V]_m$ and $f \neq f_0$, then

$$f \varepsilon [B, V]_m \quad \text{for all} \quad B \varepsilon S,\ V \varepsilon \{V\}.$$

Hence

$$f(B) \subseteq V \quad \text{for all} \quad B \varepsilon S,\ V \varepsilon \{V\}.$$

Therefore,

$$f(B) \subseteq \bigcap_{V \varepsilon \{V\}} V \quad \text{for all} \quad B \varepsilon S \tag{3}$$

Now since F is separated,

$$\bigcap_{V \varepsilon \{V\}} V = \{i\} \tag{4}$$

where i denotes the identity element of F. It follows from (3) and (4) that

$$f(B) \subseteq \{i\} \quad \text{for all} \quad B \varepsilon S \tag{5}$$

Now take any $x \varepsilon E$. By condition (b) of the hypothesis, there exists $B \varepsilon S$ such that $x \varepsilon B$. Hence by (5) we have

$$f(x) = i \quad \text{for all} \quad x \varepsilon E.$$

Thus f is the identity element of $m(E, F)$. Now since in a group the identity element is unique we have $f = f_0$. But this is a contradiction. Hence it follows that (2) holds.

Thus it follows that (R, J_S^*) and $C_{J_S}(E, F)$ are each (additive) abelian separated topological groups.

(iii) We now show that $C_{J_S}(E, F)$ is dense in (R, J_S^*).

To show this let $u \varepsilon R$, $[B, V]_m$ where $B \varepsilon S$, $V \varepsilon \{V\}$ be given. Now by the definition of R we have

$$u = u_1 + u_2 \quad \text{where} \quad u_1 \varepsilon C(E, F) \quad \text{and} \quad u_2 \varepsilon [B, V]_m.$$

Then $u_1 - u = -u_2 \varepsilon -[B, V]_m = [B, V]_m$ since $[B, V]_m$ is symmetric. Since $C(E, F) \subseteq R$, $u_1 \varepsilon R$. But also $u \varepsilon R$. Hence $u_1 - u \varepsilon R$ since R is an additive group. Thus $u_1 - u \varepsilon R \cap [B, V]_m$. Therefore, $u_1 \varepsilon u + (R \cap [B, V]_m)$. Thus every neighbourhood of u in (R, J_S^*) contains a point u_1 of $C(E, F)$. Hence $u \varepsilon J_S^*$ – closure of $C(E, F)$. Hence J_S^* – closure of $C(E, F)$ is R. Thus $C(E, F)$ is J_S^* – dense in R.

(iv) We finally prove that (R, J_S^*) is complete.

Let $u_\alpha \varepsilon R$ be a J_S^* — cauchy net in R. Since J_S^* is finer than J_σ^* on R and F is complete, u_α converges pointwise to a $u_0 \varepsilon m_{J_S}(E, F)$. We shall show that $u_0 \varepsilon R$. For this purpose let $[B, V]_m$, where $B \varepsilon S$ and $V \varepsilon \{V\}$ be given. Let $W \varepsilon \{V\}$ be such that $W + W \subseteq V$.

Since $u_\alpha - u_\beta \varepsilon [B, V]_R \subseteq [B, W]_m$ for all $a, \beta \geqslant \alpha_0 = \alpha_0(B, V)$ and since $[B, W]_m$ is closed in $m_{J_\sigma}(E, F)$ (consequently $u_\beta + [B, W]_m$ closed in the topological group $m_{J_\sigma}(E, F)$). We have

$$u_0 - u_\beta \varepsilon [B, W]_m \quad \text{for all} \quad \beta \geqslant \alpha_0 \tag{6}$$

Therefore, $\quad u_0 \varepsilon u_\beta + [B, W]_m = u_\beta^{(1)} + u_\beta^{(2)} + [B, W]_m$

with $\quad u_\beta^{(1)} \varepsilon C(E, F), \; u_\beta^{(2)} \varepsilon [B, W]_m$

Hence

$$u_0 \varepsilon C(E, F) + [B, W]_m + [B, W]_m \subseteq C(E, F) + [B, V]_m.$$

Thus $u_0 \varepsilon R$. Hence on account of (6) it follows that u_α converges to u_0 in the topology J_S^* on R.

Summarizing the results in (i), (ii), (iii) and (iv) we get

$$\overline{C_{J_S}(E, F)} = R = \bigcap_{\substack{B \varepsilon S \\ V \varepsilon \{V\}}} (C(E, F) + [B, V]_m)$$

Remark: From the above theorem we get the following criterion of completeness:

Theorem 3. Let E be a topological space and F be an (additive) abelian separated complete topological group. Let S be a family of subsets of E satisfying the conditions (a) and (b) of theorem 2. Then $C_{J_S}(E, F)$ is complete if and only if all maps f of E into F, which for every $B \varepsilon S$ and $V \varepsilon \{V\}$ have a representation of the form $f = g + h$, $g \varepsilon C(E, F)$, $h \varepsilon m(E, F)$ with $h(B) \subseteq V$, are continuous.

As particular cases of theorems 2 and 3 we obtain all propositions proved in [1] for separated topological vector spaces.

Observation: Let us observe that the phrase «additive group» could everywhere be replaced by «multiplicative group» with the corresponding small changes in the proofs. However, we stick to the phrase additive group bearing in mind that our results are generalizations of the corresponding results on topological vector spaces mentioned in [1].

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