A NOTE ON THE INTEGRAL REPRESENTATION FOR THE PRODUCT OF TWO GENERALIZED RICE POLYNOMIALS

By

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SUMMARY: A recent formula of Manocha [4] involving the product of two generalized Rice polynomials is shown to follow immediately from more general results in the theory of generalized hypergeometric polynomials derived earlier in a joint paper of the present author [6].

1. INTRODUCTION.

In a recent paper reviewed by us for the Zentralblatt für Mathematik [*189* (1970), pp. 342-343], Manocha [4, p. 268] has proved the following result (in corrected form):

$$(1.1)\qquad H_m^{(\alpha,\beta)}[\xi, p, x]\, H_n^{(\gamma,\delta)}[\eta, q, y] = \frac{2^{\alpha+\gamma+m+n+p+q-2}}{\pi^3}$$

$$\cdot \frac{\Gamma[\alpha+m+1]\,\Gamma[\gamma+n+1]\,\Gamma[a+\beta+\gamma+\delta+m+n+2]\,\Gamma[\xi+\eta]\,\Gamma[p]\,\Gamma[q]}{\Gamma[\alpha+\gamma+m+n+1]\Gamma[\alpha+\beta+m+1]\Gamma[\gamma+\delta+n+1]\Gamma[\xi]\Gamma[\eta]\Gamma[p+q-1]}$$

$$\cdot \int_0^1\int_0^1\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} u^{\alpha+\beta+m}\,(1-u)^{\gamma+\delta+n}\,v^{\xi-1}(1-v)^{\eta-1}$$

$$\cdot \cos^{m+n}\theta\,\cos^{\alpha+\gamma}\phi\,\cos^{p+q-2}\psi\,\exp\left[(m-n)\theta\, i + (\alpha-\gamma)\,\phi\, i + (p-q)\,\psi i\right]$$

$$\cdot H_{m+n}^{(\alpha+\gamma,\beta+\delta+1)}\left[\xi+\eta,\, p+q-1,\, \left\{xuv\,e^{(\phi+\psi-\theta)i} + y\,(1-u)\,(1-v)\,e^{-(\phi+\psi-\theta)i}\right\}\right.$$

$$\left. 2\sec\theta\,\cos\phi\,\cos\psi\right]\, d\psi\, d\phi\, d\theta\, dv\, du,$$

where $H_n^{(\alpha,\beta)}[\zeta, p, v]$ denotes the generalized Rice polynomial defined by (see [3], p. 157)

$$(1.2)\quad H_n^{(\alpha,\beta)}[\zeta, p, v] = \frac{(\alpha+1)_n}{n!}\, {}_3F_2\left[\begin{matrix} -n, \alpha+\beta+n+1, \zeta; \\ \alpha+1, p; \end{matrix}\; v\right],$$

with, as usual,

$$(\lambda)_n = \frac{\Gamma[\lambda+n]}{\Gamma[\lambda]} = \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1).$$

Note that the conditions of validity of formula (1.1), which are not stated in Manocha's paper, are $\xi > 0, \eta > 0, \alpha+\beta > -1, \gamma+ +\delta > -1, \alpha+\gamma > -1$, and $p+q > 1$. Also the polynomials $\{H_n^{(\alpha,\beta)}[\zeta, p, v] \mid n = 0, 1, 2, \ldots\}$, whose study was initiated by Khandekar [3], would reduce to the ordinary Rice polynomials (see [5], p. 108)

$$(1.3)\qquad H_n(\zeta, p, v) = {}_3F_2\left[\begin{matrix} -n, n+1, \zeta; \\ 1, p; \end{matrix}\; v\right]$$

when $\alpha = \beta = 0$.

The object of the present note is to show that the formula (1.1) is indeed a very special case of our earlier multiple integral (see [6], formula (1.3)) representing the product of two different members of the class of polynomials $\{\Phi_n[z]\}$ introduced by means of the generating relation (see, e.g., [1], p. 43)

(1.4)
$$\sum_{n=0}^{\infty} \Phi_n\left[\begin{matrix} (a_p), \lambda; \\ (b_q); \end{matrix}\; z\right] t^n = (1-t)^{-\lambda}\, {}_{p+2}F_q\left[\begin{matrix} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2}, (a_p); \\ (b_q); \end{matrix}\; -\frac{4zt}{(1-t)^2}\right],$$

where, for convenience, (a_p) is taken to abbreviate the sequence of p parameters $a_1, a_2, \ldots, a_p$, and similarly for (b_q).

From (1.4) it follows at once that

$$(1.5)\qquad \Phi_n\left[\begin{matrix} (a_p), \lambda; \\ (b_q); \end{matrix}\; z\right] = \frac{(\lambda)_n}{n!}\, {}_{p+2}F_q\left[\begin{matrix} -n, \lambda+n, (a_p); \\ (b_q); \end{matrix}\; z\right],$$

provided that the parameters are independent of n.

We remark in passing that several specialized or limiting forms of the polynomial set (1.5) occur throughout the literature. To the references given in our earlier paper [6] it would seem appropriate to add the late Professor Chaundy's paper [2] where several generating relations including (1.4) are discussed.

2. Derivation of (1.1).

We recall that the main formula (1.3) in our earlier paper [6] provides for the product

$$\Phi_m\left[\begin{matrix}(a_p),\lambda; \\ & x \\ (b_q)+1;\end{matrix}\right]\Phi_n\left[\begin{matrix}(a'_p),\lambda'; \\ & y \\ (b'_q)+1;\end{matrix}\right]$$

a $(p+q+2)$-ple integral representation involving the polynomial

$$\Phi_{m+n}\left[\begin{matrix}(a_p)+(a'_p),\lambda+\lambda'; \\ & z \\ (b_q)+(b'_q)+1;\end{matrix}\right],$$

where

$$z=\frac{2^{q-1}\cos\theta_1\ldots\cos\theta_q}{\cos\theta}\Bigg[x\,u\,u_1\ldots u_p\,e^{(\sum_{\sigma=1}^{q}\theta_\sigma-\theta)i}$$

$$+y\,(1-u)\,(1-u_1)\ldots(1-u_p)\,e^{(\sum_{\sigma=1}^{q}\theta_\sigma-\theta)i}\Bigg],$$

and $\lambda>0$, $\lambda'>0$, $a_\nu>0$, $a'_\nu>0$, $\nu=1,2,\ldots,p$, $b_\sigma+b'_\sigma>-1$, $\sigma=1,2,\ldots,q$.

Its special case $p=q-1=1$, which would serve our purpose here, was indeed stated in our paper (see [6], formula (2.1)) in the form:

$$\text{(2.1)}\qquad \Phi_m\left[\begin{matrix}a,\lambda; \\ & x \\ b_1+1,b_2+1;\end{matrix}\right]\Phi_n\left[\begin{matrix}a',\lambda'; \\ & y \\ b_1'+1,b'_2+1;\end{matrix}\right]=$$

$$= \frac{\Gamma[a+a']\,\Gamma[\lambda+\lambda']\,2^{m+n+b_1+b'_1+b_2+b'_2}}{\Gamma[a]\,\Gamma[a']\,\Gamma[\lambda]\,\Gamma[\lambda']}$$

$$\cdot \frac{\Gamma[b_1+1]\,\Gamma[b'_1+1]\,\Gamma[b_2+1]\,\Gamma[b'_2+1]}{\pi^3\,\Gamma[b_1+b'_1+1]\,\Gamma[b_2+b'_2+1]}$$

$$\cdot \int_0^1 \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} u^{\lambda+m-1}\,(1-u)^{\lambda'+n-1}$$

$$\cdot\, v^{a-1}\,(1-v)^{a'-1} \cos^{m+n}\theta \cos^{b_1+b'_1}\phi \cos^{b_2+b'_2}\psi \exp[(m-n)\,\theta\, i + (b_1-b'_1)\,\phi\, i + (b_2-b'_2)\,\psi\, i]$$

$$\cdot\,\Phi_{m+n}\left[\begin{matrix} a+a',\ \lambda+\lambda'\,; \\ b_1+b'_1+1, b_2+b'_2+1\,; \end{matrix} \left\{xuv\, e^{(\phi+\psi-\theta)i} + y(1-u)(1-v)e^{-(\phi+\psi-\theta)i}\right\} \frac{2\cos\phi\cos\psi}{\cos\theta}\right] d\psi\, d\phi\, d\theta\, dv\, du,$$

provided $\lambda > 0$, $\lambda' > 0$, $a > 0$, $a' > 0$, $b_1 + b'_1 > -1$, and $b_2 + b'_2 > -1$.

Now compare (1.2) with the special case $p = q - 1 = 1$ of (1.5) to obtain the relationship

$$(2.2) \qquad H_n^{(\alpha,\beta)}[\zeta, p, v] = \frac{(\alpha+1)_n}{(\alpha+\beta+1)_n}\,\Phi_n\left[\begin{matrix}\zeta, \alpha+\beta+1\,; \\ \alpha+1, p\,; \end{matrix}\ v\right].$$

In view of the relationship (2.2), formula (1.1) under the aforementioned conditions would follow at once from our earlier result (2.1) when the parameters are appropriately specialized, that is, when in (2.1) we replace

(i) a, a' by ξ, η;
(ii) λ, λ' by $\alpha+\beta+1, \gamma+\delta+1$;
(iii) b_1, b'_1 by α, γ;
(iv) b_2, b'_2 by $p-1$, $q-1$;

and multiply both sides by the constant

$$\frac{(a+1)_m\,(\gamma+1)_n}{(\alpha+\beta+1)_m\,(\gamma+\delta+1)_n}\,.$$

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