THE GENERALIZED THEORY OF PERFECT RIESZ SPACES I

by

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In this pair of papers, the basic results of the theory of perfect sequence spaces, as found in [8] and [9], are established for more general classes of ordered vector spaces, in particular, for the perfect Riesz spaces. This lattice-theoretic approach leads to many new results and to strengthened forms of most of the known ones, in addition to providing a more general context for the theory. The recent ancestors of this investigation are [2], [3], [7], [1] and [11].

Recently, several papers by Fremlin ([4], [5]) have appeared, which complement this present pair very nicely.

Introduction. Let λ be a vector space of real valued sequences, where addition and scalar multiplication are defined as usual. In the terminology of [8], the α-dual space λ^x of λ is the set of all real valued sequences b_n which have the following property: For every $a_n \in \lambda$, the sequence $a_n b_n$ shall be absolutely summable. Each $b_n \in \lambda^x$ defines a linear functional ϕ on λ by means of

$$\phi\ (a_n) = \sum_{n=1}^{\infty} a_n b_n\ .$$

If L is a Riesz space (vector lattice), we define its α-dual space to be the set of all linear functionals ϕ on L which are bounded on the order intervals of L and which satisfy $\lim_t |\phi\ (u_t)| = 0$, whenever u_t is a downwardly directed set of elements in L with infinum 0. If L is a Riesz space of real valued sequences, in the natural ordering, which contains those sequences which vanish outside a finite set, then L^x can be identified with the α-dual space of L as we first defined it.

We shall almost always assume that the Riesz space L in question has the property that L^x separates the points of L. In this case

L can be imbedded (monomorphically) in $L^{\times\times}$ in a natural way. L is said to be perfect if $L = L^{\times\times}$.

Now follows a description of the contents of the various paragraphs. After bringing definitions and fundamental facts in §0, we prove in § 1 that direct sums, products, and certain projective limits of perfect spaces are perfect. The next paragraph introduces the basic topological apparatus. The proofs we furnish for most of the known results are new. We obtain an interesting new result about the role played by the topology T_{b*} of uniform convergence on the strongly bounded sets of the dual space L' of a locally convex Riesz space $L[T]$. In § 3, we discuss the completion of L with respect to the topology of uniform convergence on the order intervals of L'. Consequently, we obtain a variety of results, including several topological characterizations of perfect Riesz Spaces, and a theorem (3.(2)) which asserts that all semi-reflexive locally convex Riesz Spaces are perfect and complete. We conclude § 3 by showing that in many cases, the topological completion of L will again have $L^{\times}$ as its α-dual space.

In § 4, we define a generalization of the *Stufenräume* and *gestufte Räume* of Köthe's, and obtain a characterization of perfect spaces as projective limits of function spaces of type L^1. In § 5, the longest paragraph, we begin by generalizing a result of Gelfand's, which describes the compact sets in a separable Banach space, to a larger class of locally convex spaces. We then apply this to obtain a characterization of the sets which are compact (or, if the quasi-completeness assumption is omitted, for the precompact sets) for the Mackey topology, for a large class of locally convex Riesz spaces. Then several characterizations are given for the sets precompact with respect to the topology of uniform convergence on the order intervals in L'. (We assume that L' is contained in $L^{\times}$). The last paragraph (§ 6) is devoted to atomic Riesz Spaces, which we characterize algebraically and topologically. The final theorem states that, for an important class of topologies, L is atomic if and only if the same is true of its completion.

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0. *Definitions and Basic Results.*

Unless otherwise given in this section, our terminology and notation are that of [11]. If L is a Riesz space and $f, g \in L$, we write $f > g$ to mean $f \geq g$ and $f \neq g$ and define $L_+ = \{f \in L: f > 0\}$. If f_t is a generalized sequence in L such that $f_t \leq f_{t_0}$ whenever $t \geq t_0$, we write $f_t \downarrow$. In addition, if $\{f_t\}$ has an infimum f, we write $f_t \downarrow f$. We define $f_t \uparrow$ and $f_t \uparrow f$ similarly. A subset S of L is called *solid* if $f \in S$ whenever $|f| \leq |g|$ for some $g \in S$. A Riesz subspace M of L is called *monotonely closed*, if $f_t \uparrow f$, $f_t \in M$ implies $f \in M$. A solid subspace (an *ideal*) in L which is monotonely closed is called a *band*. An ideal M in L is said to be *order dense* if the smallest band containing M is L itself. A family e_α, $\alpha \in A$, of elements of L_+ satisfying $\inf (e_\alpha, e_\beta) = 0$ if $\alpha \neq \beta$ is called an *order basis* for L, if the band generated by the e_α, $\alpha \in A$, is L.

If L_1 and L_2 are two Riesz spaces, we order the set of linear mappings from L_1 to L_2 by defining $A \geq 0$ to mean $Au \geq 0$ for all $u \in (L_1)_+$. We say that such a mapping A *preserves order relations*, if $Ag = \sup A(S)$ in L_2 follows from $g = \sup S$, S a subset of L_1. A linear *isomorphism* A from L_1 to L_2 which preserves order relations is called a *Riesz space isomorphism*. A^{-1} will also have this property.

If S is a subset of the Riesz space L, then S^p denotes, as usual, the set $\{f \in L: \inf (|f|, |g|) = 0 \text{ for all } g \in S\}$. S^p is a band, and according to a famous result of F. Riesz, if L is a Dedekind complete Riesz space (a *DCR-space*), then $L = S^p \oplus S^{pp}$, so, in this case, S^{pp} is the band generated by S.

Let L be a Riesz space, and let P be a projection on L. We call P a *Riesz projection*, if for some band M in L such that $L = M \oplus M^p$, P is the projection of L onto M along M^p. If $f \in L$ has the form $f = g + h$, $g \in M$, $h \in M^p$, then g is called the *Riesz component* of f in M. Riesz projections have the following properties (cf. [6] § 2.):

(1) *The Riesz projections of the Riesz space L are just those projections P of L, which satisfy $0 \leq P \leq I$.*

Riesz projections preserve order relations.

We denote by $L^\sim$, the set of all linear functionals on L, which are bounded on the order intervals $[f, h] = \{g \in L: f \leq g \leq h\}$ of L (*order bounded linear functionals*).

(2) *With respect to its natural ordering, $L^{\sim}$ is a DCR-space, and the following formulas hold* (cf. [11] §§ 18, 19):

For $f \in L$ and $\varphi \in L^{\sim}$

a) $\varphi^{+}(|f|) = \sup(\varphi(v) : 0 \leq v \leq |f|)$

b) $|\varphi|(|f|) = \sup(\varphi(g) : |g| \leq |f|)$ $(\Rightarrow |\varphi(f)| \leq |\varphi|(|f|)$

c) $|\varphi|(f^{+}) = \max(\psi(f) : 0 \leq \psi \leq |\varphi|)$

d) $|\varphi|(|f|) = \max(\psi(f) : |\psi| \leq |\varphi|)$

e) If $\varphi_t \uparrow \varphi$, *then* $\varphi(u) = \sup \varphi_t(u)$ *for all* $u \in L_{+}$.

We designate by L^x the set of all $\varphi \in L^{\sim}$ which have the property that $\lim |\varphi(u_t)| = 0$, whenever $u_t \downarrow 0$ *(order continuous linear functionals)*. L^x is a band in $L^{\sim}$ ([11] Theorem 27.2). To show $\varphi \in L^x$ it suffices to consider $\lim |\varphi|(u_t)$, where $u \geq u_t \downarrow 0$.

Examples of L^x: If $(E, \mathfrak{M}, m)$ is a measure space where every set of infinite measure has a subset with finite non-zero measure, then for $1 < p \leq \infty$, $(L^p(E, \mathfrak{M}, m))^x = L^q(E, \mathfrak{M}, m)$ $(1/p + 1/q = 1)$. $(L^1(E, \mathfrak{M}, m))^x$ is the Banach dual space of $L^1(E, \mathfrak{M}, m)$, and is equal to $L^{\infty}(E, \mathfrak{M}, m)$ if and only if $L^{\infty}(E, \mathfrak{M}, m)$ is dedekind complete. (cf. § 4).

If L^x separates the elements of L, i.e. if ${}^{o}(L^x)$ (the polar in L) $= \{0\}$, then L is called an *admissable Riesz space.*

One sees immediately that admissable Riesz spaces are archimedean.

The following result (a consequence of ([11] Lemma 27.9)) will be used implicitly throughout this paper.

(3) *Let L be an admissable Riesz space, and let L' be an ideal in L^x. L' is order dense in L^x if and only if ${}^{o}(L') = \{0\}$.*

Let L' be an ideal in $L^{\sim}$. Corresponding to each $f \in L$ we define a linear functional f'' on L' by setting $f''(\varphi) = \varphi(f)$ for each $\varphi \in L'$. It follows from (2e) that $f'' \in (L')^x$, and from (2d) and (2b) that $|f''| = |f|''$. It is easy to check that the mapping $f \to f''$ (L into $(L')^x$) preserves order relations if and only if $L' \subseteq L^x$. If ${}^{o}(L') = \{0\}$, this mapping is 1-1, and we can consider L to be a Riesz subspace of $(L')^x$. We have the following important theorem ([11] Theorems 32.11 and 32.7).

(4) *Let L be an admissable Riesz space, and let L' be an order dense ideal in L^x. Considering L as a Riesz subspace of $(L')^x$, we denote by D the ideal generated by L in $(L')^x$. D has the following properties:*

a) *If $g \in D$ then $g = \sup (f \in L: f \leq g) = \inf (h \in L: g \leq h)$; Thus $L = D$ if and only if L is dedekind complete.*

b) $D^x = L^x$.

c) *D is order dense in L'^x.*

The admissable Riesz space L is said to be *perfect*, if $L = L^{xx}$. Perfect Riesz spaces are characterized by the *Beppo-Levi property:*

(5) ([11] Theorem 28.4) *An admissable Riesz space L is perfect if and only if follows from $0 \leq u_t \uparrow$ and* $\sup \varphi(u_t) < \infty$ *for every $\varphi \in (L^x)_+$ that* $\sup u_t$ *exists in L.*

With the aid of (5) one easily obtains

(6) *For every Riesz space L, both $L^\sim$ and L^x are perfect.*

1. *Sums, Products and Projective Limits*

Let $L_b, b \in B$ be a family of Riesz spaces. With respect to the coordinatewise ordering the Cartesian product $\prod_B L_b$ is a Riesz space. Let L be a Riesz subspace of $\prod_B L_b$. We define the α-dual space L^α of L as

$$L^\alpha = \{(\varphi_b) \in \prod_B (L_b^x) : \sum_B |\varphi_b| (|f_b|) < \infty \text{ for all } f = (f_b) \in L\}$$

The *direct sum*, $\bigoplus_B L_b$ is the subspace of $\prod_B L_b$ generated by the L_b, $b \in B$.

(1) *Let L be a Riesz subspace of $\prod_B L_b$, which contains the direct sum $\bigoplus_B L_b$. Then L^α can be identified with L^x.*

L is admissable iff each L_b, $b \in B$ is.

Proof. The second statement follows immediately from the first, so we discuss the latter. Since L contains $\bigoplus_B L_b$, every $\varphi \in L^x$ defines an element $(\varphi_b) \in \prod_B (L_b^x)$ by means of $\varphi_b(f) = \varphi(f)$, $f \in L_b$, $b \in B$.

We claim that the correspondence $\varphi \to (\varphi_b)$ is a Riesz space isomorphism from L^x to L^α. It is clear that this is a positive linear mapping from L^x to $\prod_B (L_b{}^x)$, and from 0.(2b) it follows that $|\varphi_b| = |\varphi|_b$, i.e. $|\varphi| \to (|\varphi_b|)$. We designate by $F(B)$ the directed set of finite subsets B' of B. For each $f = (f_b) \in L$ we have $(\sum_{B'} |f_b|) \uparrow_{F(B)} |f|$. Consequently, for each $\varphi \in L^x$ we have

$$(*)\ |\varphi|(|f|) = \sup (\sum_{B'} |\varphi|(|f_b|):\quad B' \in F(B)) = \sum_B |\varphi_1|(|f_b|)$$

and

$$(**)\ \inf (|\varphi(|f|) - \sum_{B'} \varphi_b(|f_b|)| : B' \in F(B)) = 0$$

From (*) it follows that $(\varphi_b) \in L^\alpha$, and from (**) that $\varphi = 0$ if $\varphi_b = 0$ for all $b \in B$, i.e. the correspondence is 1-1.

Our proof is complete if we show that each $0 \leq (\varphi_b) \in L^\alpha$. is the image of some $0 \leq \varphi \in L^x$. For a given non-negative $(\varphi_b) \in L^\alpha$, we define a positive linear functional on L by

$$\varphi(f) = \sum_B \varphi_b(f^+) - \sum_B \varphi_b(f).$$

Let $u = (u_b)$ and $u_t = (u_{tb}) \downarrow 0$ with $u_t \leq u$ be given in L. Let $\{b_1, b_2, \ldots\}$ be a countable subset of B such that $\varphi(u) = \sum_{i=1}^{\infty} \varphi_{b_i}(u_{b_i})$. Then we also have $\varphi(u_t) = \sum_{i=1}^{\infty} \varphi_{b_i}(u_{b_i t})$ for all t. For a given real $\varepsilon > 0$, we choose $N = N(\varepsilon)$ so large, that $\sum_{i=N+1}^{\infty} \varphi_{b_i}(u_{b_i}) < \varepsilon/2$. Since $(\varphi_b) \in \prod_B (L_b{}^x)$, there is a t_0 such that for $t \geq t_0$, $\varphi_{b_i}(u_{b_i t}) < \varepsilon/2N$ for $i = 1, 2, \ldots, N$. Then

$$\varphi(u_t) = \sum_{i=1}^{\infty} \varphi_{b_i}(u_{b_i t}) = \sum_{i=1}^{N} \varphi_{b_i}(u_{b_i t}) + \sum_{i=N+1}^{\infty} \varphi_{b_i}(u_{b_i t})$$
$$\leq \sum_{i=1}^{N} \varepsilon/2N + \varepsilon/2 = \varepsilon.$$

Thus $\inf_t \varphi(u_t) = 0$, i.e. $\varphi \in L^x$. Further, (φ_b) is the element associated with φ by the mapping. q.e.d.

From (1) it follows easily that $(\prod_B L_b)^x = \bigoplus_B (L_b{}^x)$ and that $(\bigoplus_B L_b)^x = \prod_B (L_b{}^x)$. Consequently, we have

(2). *Direct sums and products of perfect spaces are perfect.*

With the help of the following two lemmas, we will establish a similar result for a class of projective limits of perfect spaces.

(3) *Let B be an upwardly directed set and let $L_b, b \in B$ be a family of Riesz spaces indexed by B. Assume further, that for every pair $a < b$ in B, there exists a linear mapping A_{ab} from L_b to L_a which preserves order relations, such that for $a < b < c$ we have $A_{ab} A_{bc} = A_{ac}$. Then the projective limit $L = \varprojlim A_{ab} (L_b)$ is a monotonically closed Riesz subspace of $\prod_B L_b$.*

Proof. For $f = (f_b) \in L$, we have, by the definition of projective limit, $f_a = A_{ab} f_b$ for every pair $a < b$. Then we have also $|f_a| = |A_{ab} f_b| = A_{ab} |f_b|$, so $|f| = (|f_b|) \in L$. This shows that L is a Riesz subspace of $\prod_B L_b$. Assume, finally, that $f \in \prod_B L_b$ is the supremum of $f_t = (f_{bt}) \in L$. Then for every $a < b, f_a = \sup_t f_{at} = = \sup_t A_{ab} f_{bt} = A_{ab} (\sup_t f_{bt}) = A_{ab} f_b$, so f also belongs to L. This shows that L is monotonically closed.

(4) *A monotonically closed Riesz subspace M of a perfect space L is perfect.*

Proof. We begin by showing that $u_t \downarrow 0$ in M implies $u_t \downarrow 0$ in L: It follows from $0 \leq u_t \downarrow$ and the Dedekind completeness of L that $u_t \downarrow v$ for some $v \in L$. But since M is monotonically closed, we have $v \in M$, so $v = 0$ as we asserted.

Now it becomes obvious that the restriction of an element of L^x to M defines an element of M^x. So if the family $0 \leq v_t \uparrow$ in M has the property, that $\sup \varphi (v_t) < \infty$ for every $0 \leq \varphi \in M^x$ then also $\sup \varphi(v_t) < \infty$ for all $0 \leq \varphi \in L^x$. From 0. (5) it follows that $v = \sup v_t$ exists in L, and since M is monotonically closed $v \in M$, so with the help of 0.(5) again, we see that M is perfect.

Applying (3), (2) and (4) we have

(5) *Every projective limit of perfect spaces with respect to mappings which preserve order relations is itself a perfect space.*

2. *Locally Convex Riesz Spaces.*

A locally convex topological vector space (for notation, definitions and basic results we refer to [10])$L[T]$ such that L is a Riesz space is called a *locally convex Riesz space*, if T posesses a fundamental system of solid neighborhoods. For such a topology, the lattice operations are continuous, so the set $\{f \in L : f^- = 0\} = \{f \in L : f \geq 0\}$ is closed.

Let L be a Riesz space such that $^0(L^\sim) = \{0\}$ and let L' be an ideal in $L^\sim$ for which $^0(L') = \{0\}$ holds. We designate the topology of uniform convergence on the order intervals of L' by $T_{|s|}(L')$. It follows from 0. (2d) that $T_{|s|}(L')$ is generated by the seminorms $|\varphi|(|\cdot|)$, $\varphi \in L'$. Thus $L[T_{|s|}(L')]$ is a locally convex Riesz space.

It is not difficult to see that the following theorem holds (cf. [6] Theorem 1.14).

(1) *If $L[T]$ is a locally convex Riesz space, then its dual space L' is an ideal in $L^\sim$.*

Conversely, if L is a Riesz space and L' is an ideal in $L^\sim$ such that $^0(L') = \{0\}$, then the order intervals of L' are $T_s(L)$-compact; so $L[T_{|s|}(L')]$ is a locally convex Riesz space with dual space L'.

Since (cf. 0. (2b)) the polar of a solid set in L is a solid set in L', we see from (1) that $T_{|s|}(L')$ is weaker than every other topology T for which $L[T]$ is a locally convex Riesz space with dual space L'.

(2) *In a locally convex Riesz space $L[T]$, the Riesz projections are continuous.*

PROOF: If P is a Riesz projection, it follows from 0. (1) that $|Pf| = P|f| \leq |f|$ holds for all $f \in L$. So if U is a solid T-neighborhood of 0, $P(U) \subseteq U$.

It follows from (2) that, in a locally convex DCR-space, bands are closed. But this is also true in general.

(3) *If $L[T]$ is a locally convex Riesz space, then every band in L is T-closed.*

PROOF: Since the dual space L' of $L[T]$ satisfies $^0(L') \neq \{0\}$, L is archimedean. So if M is a band in L, $M = M^{pp}$, by ([11] Theorem 29.10). Thus M is the intersection of the sets $\{f \in L\colon \inf(|f|, |g|) = 0, g \in M^p\}$. Since each mapping $f \to \inf(|f|, |g|)$ is T-continuous, it follows that M is T-closed.

Given a Riesz space L and an ideal L' in $L^\sim$ such that $^0(L) = \{0\}$, it is important to know whether or not $T_{|s|}(L)$, the topology of uniform convergence on the order intervals of L, is admissable for $< L, L' >$. The following result is due to Kawai ([7] theorem 5.1).

(4) *Let L be a Riesz space. For an ideal L' in $L^\sim$ such that* $^o(L') = \{0\}$, *the conditions* a) — c) *are equivalent.*

a) *The order intervals of L are $T_s(L')$-compact.*

b) *L is dedekind complete and $L' \subseteq L^x$.*

c) *L is an ideal in $(L')^x$.*

PROOF. Since $L \subseteq (L')^x$, c) $\Leftrightarrow$ a) by (1). By 0. (4) b) $\Leftrightarrow$ c).

We deduce from (4) and the remark following (1) that the conditions a) — c) are necessary and sufficient for the existence of a topology T on L', admissable for $< L, L' >$, with respect to which $L'[T]$ is a locally convex Riesz space.

As an application of (4), we have:

(5) *Let L be an admissable DCR-space, and let L' be an ideal in L^x such that* $^o(L') = \{0\}$. *Then $L[T_b(L')]$ is a locally convex Riesz space.*

PROOF. The polar of a solid set being solid (cf. 0. (2d)), it suffices to show that the solid hull of a $T_s(L)$-bounded set is $T_s(L)$-bounded. Since $L'[T_{|s|}(L)]$ is a locally convex Riesz space, the solid hulls of the bounded sets in $L'[T_{|s|}(L)]$ are bounded. Applying Mackey's Theorem and (4), we obtain that these sets are precisely the $T_s(L)$-bounded sets.

In general, we can say the following:

(6) *Let L be a Riesz space. If L' is an ideal in $L^\sim$ such that* $^o(L') = \{0\}$, *then $T_{b*}(L')$, the topology of uniform convergence on the strongly* (i.e. *$T_b(L)$-bounded subsets of L', is locally solid.*

If, in addition, L' is a band in $L^\sim$, then $T_{b*}(L')$ is the strongest locally solid locally convex topology generated by $T_s(L)$-bounded sets of L'.

PROOF: To prove the first assertion, we have to show that the solid hull of a strongly bounded subset B of L' is strongly bounded. Since $T_{|s|}(L')$ is locally solid, the solid hull of every $T_{|s|}(L')$-bounded subset of L is $T_{|s|}(L')$-bounded. But by (1), $T_{|s|}(L')$ is an admissable topology for $< L, L' >$, so the same is true of the $T_s(L)$-bounded subsets of L. Let A be a solid $T_s(L')$-bounded subset of L. Then

$$\sup_{f \in A} \sup_{\varphi \in B} |\varphi| (|f|) = \sup_{f \in A} \sup_{|g| \leqq |f|} (\sup_{\varphi \in B} |\varphi (g)|)$$

$$= \sup_{f \in A} \sup_{B \in \varphi} |\varphi (f)| < \infty.$$

It follws that the solid hull of B is strongly bounded.

Since the solid hull of a set which is equicontinuous for a locally solid topology is again equicontinuous, the second statement of the theorem is equivalent to the assertion that every solid $T_s(L)$-bounded set B in L' is strongly bounded. If this were not the case, we could find a solid $T_s(L)$-bounded subset A in L such that $\sup (|\varphi| (|f|) : f \in A, \varphi \in B) = \infty$, from which it follows that there exist sequences $|f_n| \in A$ and $|\varphi_n| \in B$ such that $|\varphi_n| (|f_n|) \geq n \cdot 2^n$.

As B is $T_s(L)$-bounded, $\Sigma 2^{-n} |\varphi_n| (|f|) < \infty$ for all $f \in L$. Thus $\Sigma 2^{-n} |\varphi_n|$ exists in $L^\sim$. By assumption, L' is a band in $L^\sim$, so $\Sigma 2^{-n} |\varphi_n| = \sup_m (\sum_1^m 2^{-n} |\varphi_n|) \in L'$. But

$$\sup_p ((\Sigma 2^{-n} |\varphi_n|) (|f_p|)) \leq \sup (2^{-p} |\varphi_p| (|f_p|) \leq \sup p = \infty,$$

which is absurd, because f_p is a sequence in A, a $T_s(L')$-bounded set.

3. *Completness Properties*

Our principal goal in this section is to derive topological characterizations of those Riesz spaces which are perfect.

(1) (cf. Peressini [12]) *Let L be a Riesz space. Then $L^\sim$ is $T_{|s|}(L)$-complete.*

Thus, every perfect space L is complete with respect to $T_{|s|}(L^x)$.

Proof: Let φ_t be a generalized Cauchy sequence in $L^\sim [T_{|s|}(L)]$ with limit φ in the algebraic dual space of L. Let $0 \leq u \in L$ be given. For every $f \in [-u, u]$ we have

$$|\varphi (f)| = \lim |\varphi_t (f)| \leq \lim |\varphi_t| (|f|) \leq \lim |\varphi_t| (u) < \infty.$$

Thus φ is bounded on order intervals, i.e. $\varphi \in L^\sim$. This shows that $L^\sim$ is $T_{|s|}(L)$-complete. It follows from 2.(3) that L^x is $T_{|s|}(L)$-closed in $L^\sim$ and thus is also complete. L^{xx} is a band in $(L^x)^\sim$, so, if $L = L^{xx}$, L is complete for $T_{|s|}(L^x)$.

We show now, as a kind of converse to (1), that every Riesz space which is quasi-complete with respect to some $T_{|s|}$-topology is of the form L^x for some L, i.e., is perfect. If L is a DCR-space (2a) below follows from 3.3 of [12] via 0. (4b).

(2) *Let L be a Riesz space. If L' is an ideal in $L^\sim$ such that* $^0(L') = \{0\}$, *then:*

a) *The completion of L with respect to $T_{|s|}(L')$ is $(L')^x$;*

b) *The quasi-completion of $L[T_{|s|}(L')]$ coincides with the completion;*

c) *If L' has a countable order basis, then the sequential completion of L with respect to $T_{|s|}(L')$ is $(L')^x$;*

d) *If $L[T_{|s|}(L')]$ is complete, then $L' \subseteq L^x$.*

Proof: a). By (1), $(L')^x$ is $T_{|s|}(L')$-complete. Clearly, L, considered as a subspace of $(L')^x$, is $T_s(L')$-dense in $(L')^x$. By 0. (4), L' is an order dense ideal in $(L')^{xx}$, so 2. (4) asserts that $T_{|s|}(L')$ is an admissable topology for $<(L')^x, L'>$. Thus L is also $T_{|s|}(L')$-dense in $(L')^x$, and a) follows.

b). Let $\bar{L}$ denote the quasi-completion of $L[T_{|s|}(L')]$. By a), we can consider $\bar{L}$ to be a $T_{|s|}$-dense subspace of $(L')^x$ and we need only show that $\bar{L}$ is $T_{|s|}(L')$-closed in $(L')^x$. Since the lattice operations in L are $T_{|s|}(L')$-continuous, L is a Riesz subspace of $(L')^x$, so, by Theorem 13 of [5], it is closed if it is monotonely closed. Assume $f \in (L')^x$ is such that $f_t \uparrow f$ for some family $f_t \in \bar{L}$. Then $f_s \uparrow f$, where s is a variable of the set $\{s : f_s \geq f_{t_0}\}$, f_{t_0} being a fixed member of f_t. The family f_s is contained in $[f_{t_0}, f]$ and is thus $T_{|s|}(L')$-bounded, and, by (2e), $\lim |\varphi|(|f-f_s|) = \lim |\varphi|(f-f_s) = 0$ for all $\varphi \in L'$, i.e. $f = T_{|s|}(L')\text{-}\lim f_s$. Since $\bar{L}$ is quasi-complete for $T_{|s|}(L')$, it follows that $f \in \bar{L}$, which shows that $\bar{L}$ is monotonely closed, and our proof of b) is complete.

c) If L' has a countable order basis, then whenever $f_t \uparrow f$ in $(L')^x$, we can find (cf. [11] Corollary 31.14) a sequence $f_{t_n} \in \{f_t\}$ such that $f_{t_n} \to f$. Making the appropriate changes in the proof of b), we obtain c).

d) If $L = (L')^x$ (cf.a), then $L' \subseteq (L')^{xx} = L^x$.

As a corollary to (2) we have:

(3) *Let $L[T]$ be a semi-reflexive locally convex Riesz space with dual*

space L'. *Then* $(L')^x = L$ *and also* $L' \subseteq L^x$, *with equality holding if* $L[T]$ *is reflexive.*

In any case, L *is perfect and* $L[T]$ *is complete.*

PROOF: If $L[T]$ is semi-reflexive, then L is quasi-complete for $T_s(L')$, and thus also quasi-complete for $T_{|s|}(L')$. By (2), $L[T_{|s|}(L')]$ (and therefore also $L[T]$, cf. the remarks below) is complete, $L = (L')^x$, and $L' \subseteq L^x$.

If $L[T]$ is reflexive, then $L'[T_{|s|}(L)]$ is semi-reflexive, so by what we have just proved, $L' = L^x$.

According to ([10] 18,4.(4)), the $T_{|s|}(L)$-completeness of L^x implies the completeness of $L^x[T]$ for every locally convex vector space topology T which is stronger than $T_{|s|}(L)$ and weaker than $T_b(L)$. (Such topologies have fundamental neighborhood systems at 0 whose members are polars of $T_s(L)$-bounded sets in L and are thus $T_{|s|}(L)$-closed by ([10] 20,7.(6))). We make special note of two cases.

(4). $L^x[T_k(L)]$ *and* $L^x[T_b(L)]$ *are complete.*

For the next result (cf. [2], Theorem 5.2), we strengthen the hypothesis on L.

(5) *If* L *is admissable Dedekind complete Riesz space, then* $L^x[T_s(L)]$ *is sequentially complete.*

Proof: Let ϕ_n be a weak Cauchy sequence in L^x and let T be a continuous linear map from $L^x[T_s(L)]$ to l^1. Since l^1 is weakly complete $\{T\phi_n\}$ is relatively weakly compact. It will follow from 6. (1b) that $\{\phi_n\}$ must also be relatively compact and hence convergent in $L^x[T_s(L)]$.

Consequently, perfect spaces are $T_s(L^x)$-sequentially complete. It follows from (2c) that the converse statement holds if L has a countable order basis. A counterexample to the general converse is provided by the space $\omega_{A,}0$ of all real valued functions on the uncountable set A, which have countable support.

We close this section by showing that the dual space L^x of an admissable Riesz space L has the following stability property.

(6) *Let* L *be an admissable Riesz space and let* T *be a locally solid locally convex topology on* L *which is finer than* $T_{|s|}(L^x)$ *and weaker than*

$T_{b^*}(L)$ (cf. 2.(6)). *If* $\bar{L}$ *denotes the completion of* L *with respect to* T, *then* $(\bar{L})^x$ *can be identified with* L^x.

Proof: Let B be a solid T-equicontinuous subset of L^x and let f be an element of L^{xx}. It follows from 0. (4a, c) that there exists a family $0 \leq u_t \uparrow$ in L such that $u_t \uparrow |f|$. Since the set $\{u_t\}$ is $T_s(L^x)$-bounded, and since, by assumption, B is strongly bounded, we have

$$\sup (|\varphi| (|f|) : \varphi \in B) = \sup_t (\sup (|\varphi| (u_t) : \varphi \in B) < \infty.$$

Thus B is $T_s(L^{xx})$-bounded. It follows now from the remarks preceding (4) that L^{xx} is complete for the topology of uniform convergence on the equicontinuous subsets of T in L^x, so $\bar{L}$ can be identified with the closure of L in L^{xx} with respect to this topology. Since the lattice operations in L are T-continuous, $\bar{L}$ is a Riesz subspace of L^x.

We complete the proof of (5) by showing that if $\bar{L}$ is a Riesz subspace of L^{xx} satisfying $L \subseteq \bar{L} \subseteq L^{xx}$, then $(\bar{L})^x = L^x$. Let $u_t \downarrow 0$ in $\bar{L}$ be given. We assert that $u_t \downarrow 0$ in L^{xx}: Indeed, let $0 \leq v \in L^{xx}$, be such that $v \leq u_t$ for all t. For every element $0 \leq v \in L^{xx}$, there exists a family $0 \leq v_s \uparrow$ in L such that $v_s \uparrow v$ (This follows from 0. (4a,c)). But since $v_s \in L \subseteq \bar{L}$ and $0 \leq v_s \leq v \leq u_t$ for all t implies $v_s = 0$, it follows that $v = 0$. So $u_t \downarrow 0$ in L^{xx}. Further, $\bar{L}$ is $T_s(L^{xxx}) = T_s(L^x)$-dense in L^x, because this is true of L. Thus we can conclude that $L^x = L^{xxx} \subseteq (\bar{L})^x$. On the other hand, if $u_t \downarrow 0$ in L, than an argument almost identical to the one just given shows that $u_t \downarrow 0$ in $\bar{L}^x$ Further, L is $T_s((\bar{L})^x)$-dense in $\bar{L}$: For if $\varphi \in (\bar{L})^x$ vanishes on L, then for every $0 \leq v \in \bar{L}$, since there exists a family $v_t \in L$ such that $v - v_t \downarrow 0$ in L^{xx}, and thus also in $\bar{L}$ as we have shown, we have $|\varphi(v)| = \lim |\varphi(v - v_t)| = 0$, so $\varphi = 0$. It follows that $(\bar{L})^x \subseteq L^x$, and hence that $(\bar{L})^x = L^x$.

4. *Step spaces and g-spaces. Representation Theorems.*

Let L be an admissable DCR-space and let $\{f_n\} \neq \{0\}$ be a countable subset of L. We define $L_s(f_n)$ as the ideal generated by $\{f_n\}$ in L and $L_g(f_n)$ as $(L_s(f_n))^x$. $L_s(f_n)$ is called the *step space (Stufenraum) of the steps* f_n and $L_g(f_n)$ is called the *g-space (gestufter Raum)* associated with $L_s(f_n)$. Since $L_s(f_n)$ is also generated by the set $\{|f_1|, \sup(|f_1|, |f_2|), \sup(|f_1|, |f_2|, |f_2|), \ldots\}$ we can (and will) always assume that the steps f_n form an increasing sequence of positive elements. $L_g(f_n)$ is perfect, by 0.(6). Being an ideal in L, $L_s(f_n)$ is Dedekind complete. But we can prove more.

(1) *The spaces* $L_s(f_n)$ *and* $L_g(f_n)$ *are both perfect. With respect to the topology* $T_{|s|}[(L_s(f_n))]$, $L_g(f_n)$ *is an* (F)*-space whose dual space is* $L_s(f_n)$.

PROOF. Let g be an element of $L_s(f_n)$. Then for some n', $|g| \leq f_{n'}$, so $|\varphi|(|g|) \leq |\varphi|(f_{n'})$. Hence, the topology $T_{|s|}(L_s(f_n))$ is generated by the countable family of seminorms $|\cdot|(f_n)$ and so is metrizable. By 3.(1), $L_g(f_n)$ is complete for this topology. Thus $L_g(f_n)[T_{|s|}(L_s(f_n))]$ is an (F)-space and by 2. (4a), its dual space is $L_s(f_n)$.

It remains to show that $L_s(f_n) = (L_g(f_n))^x$. According to 2.(4c), $L_s(f_n)$ is an order dense ideal in $(L_g(f_n))^x$. Let $0 \leq u \in (L_g(f_n))^x$ be given and let $0 \leq u_t \in L_s(f_n)$ be such that $u_t \uparrow u$. As a weakly bounded ($\{u_t\} \subseteq [0, u]$) set in the dual space of an (F)-space, $\{u_t\}$ is relatively weakly compact ([10] 21,5.(4)). But u_t is $T_s(L_g(f_n))$-convergent to u in $(L_g(f_n))^x$, so $u \in L_s(f_n)$. Thus $(L_g(f_n))^x = L_s(f_n)$.

If $\{f_n\}$ ($\neq \{0\}$) contains only a single element u, our proof of (1) shows that $L_g(u)[T_{|s|}(L_s(u))]$ is a (B)-space with norm $|\cdot|(u)$. We denote this (B)-space by $L^1(u)$ and its dual space, which we have shown to be $L_s(u)$, by $L^\infty(u)$. This change of notation is justified by the following theorem.

(2). *There exists a locally compact space* E *and a Radon measure* m *on* E *such that* $L^1(u)$ *can be mapped linearly, isometrically and order isomorphically onto* $L^1(E, m)$.

PROOF. The norm $||\cdot|| = |\cdot|(u)$ satisfies $||\,|\varphi|\,|| = ||\varphi||$ for all $\varphi \in L^1(u)$ and $||\varphi + \psi|| = ||\varphi|| + ||\psi||$ for $\varphi, \psi \in (L^1(u))_+$, so $L^1(u)$ is an (AL)-space, to which the representation theorem given in [13] applies.

Let an admissable Riesz space L and an element $0 < \varphi \in L^x$ be given. We denote by f_φ the restriction of the element $f \in L^{xx}$ to $L^\infty(\varphi)$, the ideal in L^x generated by φ. The correspondence $J_\varphi : f \to f_\varphi$ is a linear mapping from L^{xx} to $L^1(\varphi)$ which preserves order relations. (cf.0.(2b,e)). Thus, since L is admissable, the mapping $J : f \to (f_\varphi) \in \prod_{0<\varphi\in L^x} L^1(\varphi)$ is an imbedding of L^{xx} into the product Riesz space $\prod_{0<\varphi\in L^x} L^1(\varphi)$. For $0 < \varphi < \psi \in L^x$ we denote by $J_{\varphi\psi}$ the linear and order relation preserving mapping from $L^1(\psi)$ to $L^1(\varphi)$ which assigns to each element its restriction to $L^\infty(\varphi)$. For $0 < \varphi < \psi < \pi$ we have $J_{\varphi\pi} = J_{\varphi\psi} J_{\psi\pi}$, as is obvious from the definitions, and for each $f \in L^{xx}$, $f_\varphi = J_{\varphi\psi} f_\psi$. Conversely, let $(f_\varphi) \in \prod_{0<\varphi\in L^x} L^1(\varphi)$ have the property that $f_\varphi = J_{\varphi\psi} f_\psi$ holds for every pair $0 < \varphi < \psi \in L^x$. Then we can define an element $f \in (L^x)^\sim$ by setting $f(0) = 0$, $f(\varphi) = f_\varphi(\varphi)$ for every $0 < \varphi \in L^x$ and $f(\varphi) = f(\varphi^+) - f(\varphi^-)$ for an arbitrary $\varphi \in L^x$. (Note that for $\varphi, \psi \in (L^x)_+$, $f(\varphi + \psi) = f_{\varphi+\psi}(\varphi + \psi) = f_{\varphi+\psi}(\varphi) + f_{\varphi+\psi}(\psi) = f_\varphi(\varphi) + f_\psi(\psi) = f(\varphi) + f(\psi)$). We assert that $f \in L^{xx}$. Indeed, if φ_0 and $\varphi_0 \geq \varphi_t \downarrow 0$ are given in L^x, then

$$\inf_t |f(\varphi_t)| = \inf_t |f_{\varphi_t}(\varphi_t)| \leq \inf |f_{\varphi_0}(\varphi_t)| = 0.$$

Thus $f \in L^{xx}$ and, further, $Jf = (f_\varphi)$. This shows that $J(L^{xx})$ is the projective limit $\varprojlim (J_{\varphi\psi} L^1(\psi),\ 0 < \varphi < \psi \in L^x)$. Let

$\varphi_1, \ldots, \varphi_n \in (L^x)_+$ and $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}_+$ be given. Then

$$U = \{f \in L^{xx} : \varphi_i(|f|) < \varepsilon_i,\ i = 1, \ldots n\}, \text{ resp.}$$

$V = \{(f_\varphi) \in J(L) : \varphi_i(|f_{\varphi_i}|) < \varepsilon_i,\ i = 1, \ldots n\}$ is a representative 0-neighborhood in $L^{xx}[T_{|s|}(L^x)]$, resp. in $J(L)$ with the topology induced by the product topology. But $J(U) = V$ and $J^{-1}(V) = U$. So we conclude that:

(3) *If L is an admissable Riesz space, then $L^{xx}[T_{|s|}(L^x)]$ can be identified with the topological projective limit*

$$\varprojlim (J_{\varphi\psi} L^1(\psi),\ 0 < \varphi < \psi \in L^x).$$

From (3), (2) and 1.(5) we obtain the following characterization of perfect spaces.

(4) *A Riesz space L is perfect if and only if it is a projective limit, with respect to mappings which preserve order relations, of spaces of type $L^1(E,m)$, E a locally compact space and m a Radon measure on E.*

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THE GENERALIZED THEORY OF PERFECT RIESZ SPACES II

by

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This paper is a continuation of [14].

5. *Compact Sets.*

Let L be an admissable DCR-space. In a recent paper [4] Fremlin gave a list of properties characterizing the relatively $T_s(L)$-compact subsets of L^x. (cf. also [1]). We mention a few:

(1). *Let L be an admissable Dedekind complete Riesz space. A subset S of L^x is relatively $T_s(L)$-compact if and only if one of the following holds;*

a) *The solid hull of S is relatively $T_s(L)$-compact,*

b)*) *If T is a weakly continuous positive linear mapping of L^x into the normed sequence space l^1, then $T(S)$ is weakly compact,*

c) *If P_n is a decreasing sequence of Riesz projections on L^x with infimun* 0, *then for every $f \in L$*

$$\lim \; (\sup \; (|P_n \varphi (f)| : \varphi \in S) = 0,$$

d) *If $u_t \downarrow 0$ in L, then* $\lim \; (\sup \; (| \varphi | \; (u_t) : \varphi \in S)) = 0$.

In this section we shall derive some characterizations of the relatively compact sets of the topologies $T_k(L)$ and $T_{|s|}(L)$. We start with some general considerations.

Let $E[T]$ be a locally convex space with dual space E'. A subset S of E is called *begrenzt* if for every sequence φ_n in E' which converges weakly to 0, $\lim \; (\sup \; (| \varphi_n (f) | : f \in S)) = 0$.

It is asserted in ([2] Lemma 14.2) that every relatively $T_k(E')$-compact subset of E is *begrenzt*, but this is not always the case. (For a counterexample, consider the relatively compact set (convergent

*) The condition b) appears in the proof of 8. in [4].

sequence) $\{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, \ldots), \ldots\}$ of $l^1 [T_k(\varphi)] = l^1 [T_s(\varphi)]$ ($T_s(\varphi)$ is metrisable), where, as usual, φ denotes the space of (real) sequences having only finitely many terms different from 0. For a sequence in φ choose $(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots$.)

We call a sequence φ_n in E' *acceptable,* if φ_n is $T_s(E)$-convergent to 0 and the absolutly convex hull of $\{\varphi_n\}$ is relatively $T_s(E)$-compact. A set S in E is called *limited* if $\lim (\sup |\varphi_n(f)| : f \in S) = 0$ holds for every acceptable sequence $\varphi_n \in E'$. Every *begrenzt* set is limited. Conversely, if $E[T]$ is barrelled, or if $E' [T_k(E)]$ is quasi-complete, then every limited set in E is *begrenzt,* because in these cases, the absolutely convex hull of every weakly compact subset in E' is again compact. (cf. Krein's Theorem ([9] 24, 5(4))).

(2) *If $E[T]$ is a locally convex space with dual space E', then every subset of E which is totally bounded for $T_k(E')$ is limited.*

Proof. Let S be a totally bounded subset in $E[T_k(E')]$ and let φ_n be an acceptable sequence in E'. For every $\varepsilon > 0$, the set $V = \{f \in L : \sup_n |\varphi_n(f)| < \varepsilon/2\}$ is a $T_k(E')$-neighborhood of 0, so there exists a finite family $f_1, \ldots f_q$ in S such that $S \subseteq \bigcup_1^q (f_i + V)$. If we choose n_0 such that for $n \geq n_0$, $|\varphi_n(f_i)| < \varepsilon_0/2$ for all $i = 1, \ldots, q$, then we have $\sup (|\varphi_n(f)| : f \in S) < \varepsilon$: Indeed, for an arbitrary $f \in S$, there exists an f_i, $1 \leq i \leq q$ such that $f - f_i \in V$, so, in particular, $|\varphi_n(f - f_i)| < \varepsilon/2$. Thus

$|\varphi_n(f)| \leq |\varphi_n(f - f_i)| + |\varphi_n(f_i)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. We conclude that $\lim (\sup (|\varphi_n(f)| : f \in S)) = 0$, i.e., that S is limited.

Under some additional hypotheses, the converse of (2) holds:
(3) *Let $E[T]$ be a quasi-complete locally convex space, whose dual space E' can be given a metrizable locally convex topology weaker than $T_k(E)$. Then a subset S of E is relatively compact for $T_k(E')$ if and only if it is limited.*

Proof. If S is relatively $T_k(E')$-compact, it is limited by (2). Assume that S is limited. We show first that S is relatively $T_s(E')$-compact. For this it suffices, according to a theorem of Grothendieck's (cf. [9] 24,6.(1)), to show that $\lim_m \lim_n \varphi_m(f_n) = \lim_n \lim_m \varphi_m(f_n)$ whenever a sequence f_n in S and a sequence φ_m in some absolutely convex and compact subset in E' are given, such that all of these limits exist. Assuming we have such sequences φ_m and f_n, we need only show that $\lim_p \lim_n \varphi_p(f_n) = \lim_n \lim_p \varphi_p(f_n)$ for some subsequence φ_p of φ_m.

So, using the generalized Šmulian theorem ([9] 24,1.(3)), we choose a subsequence φ_p of φ_m which converges to an element $\varphi \in E'$. What we have to verify now becomes $\lim_p \lim_n (\varphi_p - \varphi)(f_n) = 0$. But $\varphi_p - \varphi$ is an admissable sequence, so we can assert even more: $\lim_p \sup_n (|(\varphi_p - \varphi)(f_n)|) = 0$. We conclude that S is relatively $T_s(E')$-compact.

We prove next that S is relatively countably compact for $T_k(E')$. Let f_m be a sequence in S and let f be a $T_s(E')$-adherent point of f_m in E. Let us assume that f is not a $T_k(E')$-adherent point of f_m. Then, by the definition of the Mackey topology, there exists an absolutely convex and $T_s(E)$-compact subset B of E' and a positive real number ε such that $\max(|\varphi(f_m - f)|: \varphi \in B) > \varepsilon$ for all but a finite number of indices m, which we can omit from consideration. For each m, we chose $\varphi_m \in B$ such that $|\varphi_m(f_m - f)| > \varepsilon$. Let F' denote the $T_s(E)$-closed subspace of E' generated by the set $\{\varphi_m: m = 1,2, ...\}$ and let F designate the quotient space $E/(F')^o$ **). If $g \in E$, we denote the corresponding element of F by $\bar{g}$. The topology $T_s(F)$ on F' coincides with the topology induced by $T_s(E)$ so $\{\bar{f}_m: m = 1,2,...\}$ and thus also $\{\bar{f}_m - \bar{f}: m = 1,2,...\}$ is limited for $< F, F' >$. Further, the latter set is relatively $T_s(F')$-compact and has $\bar{0}$ as a $T_s(F')$-adherent point. Now, since F' is $T_s(F)$-separable, we can find a subsequence $\bar{f}_n - \bar{f}$ of $\bar{f}_m - \bar{f}$ which is $T_s(F')$-convergent to $\bar{0}$ (cf. [9] 24,1.(5)).

By hypothesis, there exists a metrizable locally convex topology on F' which is weaker than the topology induced by $T_k(E)$. But the latter is weaker than $T_k(F)$, so we can apply the generalized Šmulian Theorem to conclude that some subsequence φ_p of φ_n (φ_n denotes the subsequence of φ_m corresponding to the subsequence $\bar{f}_n - \bar{f}$) is $T_s(F)$-convergent to an element $\varphi \in B \cap F'$. Since $\varphi_p - \varphi$ is an admissable sequence, there exists a p_0 such that for $p \geq p_0$, $|(\varphi_p - \varphi)(\bar{f}_n - \bar{f})| < \varepsilon/2$ holds for all n. On the other hand, being a subsequence of $\bar{f}_n - \bar{f}$, the sequence $\bar{f}_p - \bar{f}$ is $T_s(F')$-convergent to 0, so there is a $p \geq p_0$ such that $|\varphi(\bar{f}_p - \bar{f})| < \varepsilon/2$. Consequently, we have

$$|\varphi(f_p - f)| = |\varphi_p(\bar{f}_p - \bar{f})| \leq |(\varphi_p - \varphi)(\bar{f}_p - \bar{f})| + |\varphi(\bar{f}_p - \bar{f})| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

**) Throughout 5. the notation B^o will denote the polar set to B in the other space of the dual pair.

which is impossible, so f is a $T_k(E')$-adherent point of f_m. Now it is apparent that S is relatively countably $T_k(E')$-compact.

We deduce from Eberlein's Theorem ([9]24,2.(1)) that S is relatively compact for $T_k(E')$.

The above theorem generalizes a result of Gelfand's about separable (B)-spaces. (cf. Phillip's paper [11] pp. 524-525). Phillips [11] has shown that the subset $\{1,0,0,...), (0,1,0,...),...\}$ of the (B)-space l^∞ is *begrenzt*, although it is not compact in the norm topology. With respect to this topology, l^∞ is a complete locally convex perfect Riesz space whose dual space properly contains $(l^\infty)^x = l^1$. When this isn't so, no such example can exist.

(4) *If L is an admissable DCR-space and L' is an order dense ideal in L^x with the property that $L[T_k(L')]$ is quasi-complete, then a subset of $L[T_k(L')]$ is relatively compact if and only if it is limited.*

PROOF. Every relatively compact subset is limited by (2). To prove the converse, it suffices, according to Eberlein's Theorem, to show that a countable subset of $L[T_k(L')]$ which is limited, is also relatively compact. Let $\{f_n\}$ be such a set. We define $E = \{f_n\}^{pp}$ and designate by E' the dual space of E with respect to the topology induced by $T_k(L')$. By 2.(2), the Riesz projection P of L onto E is continuous for the $< L,L' >$-admissable topology $T_{|s|}(L')$, so the adjoint Riesz projection P' on L' is $T_s(L)$-continuous and E' can be identified with the range of P', which is E^{oo}. It follows easily that $\{f_n\}$ is limited in the dual system $< E,E' >$ and that the topology on E induced by $T_k(L')$ coincides with $T_k(E')$, so $E[T_k(E')]$ is quasi-complete. Further, since L' is an order dense ideal in L^x (by hypothesis), each of the order intervals $[-|f_n|, |f_n|]$ $n = 1,2,...$ is compact for $T_s(L')$ by 2. (4a), and thus also for $T_s(E')$. The metrizable topology on E' induced by the semi-norms $P_n(\varphi) = \sup\ (\varphi(g) : g \in [-|f_n|, |f_n|])$ $n = 1,2,...$ is therefore weaker than $T_k(E)$, so the hypotheses of (3) are satisfied, and we conclude that $\{f_n\}$ is relatively $T_k(E')$-compact, so $\{f_n\}$ is relatively $T_k(L')$-compact. This completes the proof.

As a corollary to (4) we have

(5) *If L is a perfect Riesz space, then the relatively $T_k(L^x)$-compact sets of L are precisely the* begrenzt *sets of L for $< L,L^x >$.*

PROOF: By 3.(4), $L^x[T_k(L)]$ is complete, so by the discussion preceding (2), the *begrenzt* sets of L and the limited sets of L for $< L,L^x >$ coincide. Again by 3.(4), L is complete for $T_k(L^x)$, so (4) implies (5).

Anticipating 6.(2), we can prove:
(6) *Let L be a perfect space. The solid hull of every relatively $T_k(L^x)$-compact subset of L is again relatively $T_k(L^x)$-compact if and only if L is atomic.*

PROOF. If the solid hulls of such sets are relatively $T_k(L^x)$-compact, then, in particular, the order intervals of L are $T_k(L^x)$-compact. It follows from 6.(2) that L is atomic.

On the other, hand, if L is atomic, the proof of ([9] 30,6.(6)) can be repeated, practically without change.

(7) *Let L be an admissable Riesz space and let L' be an order dense ideal in L^x. For a subset S of L, the following conditions are equivalent:*

a) *S is totally bounded for $T_{|s|}(L')$*

b) *For every sequence φ_n in L' which is $T_s(L)$-convergent to 0 and bounded by some $\varphi \in L'$* (i.e. $|\varphi_n| \leq \varphi$), *we have*

$$\lim (\sup |\varphi_n(f)| : f \in S) = 0.$$

c) *If A is a continuous positive mapping from $L[T_{|s|}(L')]$ into the (B)-space $L^1(0,1)$ of Lebesgue summable functions on the interval* (0,1), *then the image $A(S)$ of S under A is relatively compact.*

PROOF:

a) $\Rightarrow$ b). Let φ and φ_n be given as in b). By 2.(1), the order interval $[-\varphi,\varphi]$ is compact for the topology $T_s(L)$ and also for the stronger topology $T_s(L^1(\varphi))$, so these two topologies coincide on this set, which implies that φ_n is $T_s(L^1(\varphi))$-convergent to 0, i.e. φ_n is an *acceptable* sequence (cf. before (2)). The canonical mapping J_φ of L into $L^1(\varphi)$ is $T_{|s|}$-continuous, so $J_\varphi(S)$ is totally bounded in $L^1(\varphi)$, so by (2),

$$\lim \sup (|\varphi_n(f)| : f \in S) = \lim \sup (|\varphi_n(J_\varphi f)| : f \in S) = 0.$$

b) $\Rightarrow$ a). Let $\varphi \in (L')_+$ and $\varepsilon > 0$ be given. Let φ_n be an acceptable sequence in $L^\infty(\varphi)$ for $< L^1(\varphi), L^\infty(\varphi) >$. Then φ_n is also convergent to 0 for the weaker topology $T_s(L)$. By the theorem of Banach-Mackey ([9] 20,11.(3)), the set $\{\varphi_n : n = 1,2,...\}$ is norm-bounded in $L^\infty(\varphi)$, i.e. is contained in $\lambda[-\varphi,\varphi]$ for some $\lambda > 0$. It follows now from b), that $J_\varphi(S)$ is limited in $L^1(\varphi)$. By (4), $J_\varphi(S)$ is relatively compact, so there exist elements $f_1, \dots f_m$ in S such that

$$S \subseteq \bigcup_{i=1}^{m} \{f \in L : \varphi(|J_\varphi f - J_\varphi f_i|) < \varepsilon\} = \bigcup_{i=1}^{m} \{f \in L : \varphi(|f - f_i|) < \varepsilon\}.$$

Since φ and ε were arbitrarity chosen, we conclude that S is totally bounded for $T_{|s|}(L')$.

a) $\Rightarrow$ c): A continuous image of a totally bounded set is totally bounded. Since $L^1(0,1)$ is complete, the totally bounded set $A(S)$ is relatively compact. (cf. [9] 5,6.(2a)).

c) $\Rightarrow$ b) Assume that b) does not hold, i.e., that for some $\varphi \in (L')_+$ there exists a sequence $\varphi_n \in [-\varphi, \varphi]$ converging weakly to 0 and a real number $\varepsilon > 0$ such that for some sequence $f_n \in S$, $|\varphi_n(f_n)| > \varepsilon$ holds for all n. We designate by $\bar{\varphi}$ resp. $\bar{\varphi}_n$ the component of φ resp. φ_n in $\{f_n : n = 1,2,\dots\}^{p_0}$, and by $\bar{f}_n$ the component of f_n in $[-\bar{\varphi}, \bar{\varphi}]^{p_0}$. Let J denote the mapping which associates each $f \in L \subseteq L^{xx}$ with its restriction to $L^\infty(\bar{\varphi})$. In any measure space (E, m) representing $L^1(\bar{\varphi})$ and $L^\infty(\bar{\varphi})$, (cf. 4.(2)), E, the carrier of $\bar{\varphi}$, will be the union of the carriers of the summable functions $J f_n$, and will thus be σ-finite. So, by Prop. 6 and § 4 of [13][1]), there exists a continuous positive projection A_1 of $L^1(\bar{\varphi})$ onto the closed Riesz subspace X of $L^1(\varphi)$ generated by $\{J\bar{f}_n : n = 1,2,\dots\}$. By Theorems 11 and 8 of [13], there exists a linear isometric order isomorphism A_2 of X onto a closed Riesz subspace Y of $L^1(0,1)$. Thus the mapping $A = A_2 A_1 J$ is a continuous positive mapping of $L[T_{|s|}(L')]$ into $L^1(0,1)$.

The argument given in a) $\Rightarrow$ b) shows that $\bar{\varphi}_n$ is $T_s(L^1(\bar{\varphi}))$-convergent to 0. We denote the respective dual spaces of the (B)-spaces X and Y by X' and Y'. If φ' resp. φ'_n designates the element of X' defined by φ resp. φ_n, then φ'_n and thus also $(A_2^{-1})'\varphi'_n$ are weakly convergent to 0. ($(A_2^{-1})'$ denotes the adjoint of A_2^{-1}). Since the sequence $(A_2^{-1})'\varphi'_n$ is contained in the absolutely convex and weakly compact (cf.2.(1)) set $(A_2^{-1})[-\varphi, \varphi]$, it is an acceptable sequence in Y'. But for every n

$$|((A_2^{-1})'\varphi'_n(Af_n)| = |(A_2^{-1})'\varphi'_n(A_2 A_1 J f_n)|$$
$$= |\varphi'_n(A_1 J\bar{f}_n)| = |\varphi'_n(J\bar{f}_n)| = |\bar{\varphi}_n(J\bar{f}_n)| = |\varphi_n(f_n)| > \varepsilon > 0.$$

1) This result can be shown to be valid in arbitrary L^1-spaces, and the proof can be carried out abstractly, i.e. in the framework of spaces of the form $L^1(\varphi)$. We will return to this and related questions in a later paper.

Thus $A(S)$ is not limited in $< Y,Y' >$, so by (2), it is not relatively compact in the norm topology, i.e. c) doesn't hold. We conclude that c) $\Rightarrow$ b).

If L is perfect, then L is $T_{|s|}(L^x)$-complete, by 3.(1), so a) — c) describe the relatively $T_{|s|}(L^x)$-compact sets of L. In general, if S is $T_s(L')$-compact, then a) — c) are necessary and sufficient for the $T_{|s|}(L')$-compactness of S. We can prove another result of this type, where we can even avoid the assumption that L is dedekind complete. For this, however, we need a representation of L and L'.

Let E be a locally compact Hausdorff space. We denote the set of all compact subsets of E by $\mathfrak{C}$, and we define the σ-algebra $\mathfrak{M} = \{A \subseteq E : A \cap C$ is a Borel set, for all $C \in \mathfrak{C}\}$. A (*positive*, σ-additive) measure m, defined on $\mathfrak{M}$, is called a *Radon measure*, if

$$\text{i) } m(C) < \infty \text{ for all } C \in \mathfrak{C}, \text{ and}$$

$$\text{ii) For all } F \in \mathfrak{M}, m(F) = \sup (m(C) : C \in \mathfrak{C},\ C \subseteq F) = \inf (m(O) : O \text{ open},\ O \supseteq F)$$

are satisfied. [2])

Let L be a Riesz space, with respect to the natural ordering, of (m-equivalence classes of) m-measurable real valued functions on E, such that L and its α dual space L^α, defined by

$$L^\alpha = \{g : g\chi_C \in L^1(E,m) \text{ for all } C \in \mathfrak{C} \text{ (locally summable) and } gf \in L^1(E,m) \text{ for all } f \in L\} \ (\chi_C \text{ denotes the indicator of } C)$$

have the property

(*) For every $C \in \mathfrak{C}$, there exists an element g in the space under consideration, whose representatives are a.e. positive on C, i.e. Supp$(g) \supseteq C$.

It follows from 6. and 7. of [5], that every admissable Riesz space is representable as such an L, in such a way that L^α coincides with L^x (the members of L^α defining linear functionals on L in the obvious fashion). [3])

2) If, for a measure m on $\mathfrak{M}$ satisfying i), the condition ii) holds only for sets of finite measure, then it follows easily from (iii) of ([7] 19. 25, cf. 19. 31), that there exists a Radon measure m, which agrees with m on sets of σ-finite m-measure, and which thus yields the same integration theory. Measures satisfying ii) have the convenience, that there is no distinction between null- and locally null sets.

3) In general, for a Riesz space of m-measurable functions satisfying (*), L^x can be identified with the set of *all* m-measurable functions g satisfying $gf \in L^1(E,m)$ for all $f \in L$, and it can happen that L^x property contains L^α.

(8) *Let L and L^α be as above and let L' be an ideal in L^α satisfying*(*). *A subset S of L is $T_{|s|}(L')$-compact if and only if*

a) *S is $T_s(L)$-compact, and*

b) *Given $C \in \mathfrak{C}$ and a sequence f_n in S, one can find a subsequence of f_n which converges in measure on C to an element $f \in S$.*

PROOF: Necessity. For $f \in L$, we define $f/|f|$ to be 0 where f is 0.

Then for each $0 \leq g \in L'$, $(f/|f|)g \in [-g, g]$, from which it follows that the topology $T_{|s|}(L')$ on L is induced by the seminorms $f \to \to \int |fg|\, dm$, $g \in L'$. Thus, for every $0 \leq g \in L'$, the mapping J_g: $f \to fg$ from $L[T_{|s|}(L')]$ into the (B)-space $L^1(E,m)$ is continuous.

If S is $T_{|s|}(L')$-compact, then clearly a) is satisfied. Let $C \in \mathfrak{C}$ be given, and let g be an element of $(L')_+$ which is positive on C. Since J_g is continuous, $J_g(S) = Sg$ is compact in $L^1(E,m)$. So if f_n is a sequence in S with adherent point $f \in S$, then there exists a subsequence f_p of f_n such that $\lim\limits_p \int |f_p - f|\, gdm = 0$. Hence, (cf. [3] III.3.(6)), gf_n converges in mesure to gf. Let $\varepsilon > 0$ be given, and let $\varepsilon' > 0$ be chosen so that the measure of $F_1 = \{x \in C : g(x) < \varepsilon'\}$ is less than $\varepsilon/2$. We choose p_0, so that for all $p \geq p_0$, the measure of $F_2 = \{x : |f_p(x) - f(x)|\, g(x) \leq \varepsilon\varepsilon'\}$ is less than $\varepsilon/2$. Then

$$m(\{x \in C : |f_p(x) - f(x)| \geq \varepsilon\}) \leq m(F_1) + m(F_2) < \varepsilon$$

for all $p \geq p_0$. Since ε was arbitrarily chosen, it follows that f_p m-converges to f on C, so b) holds.

Conversely, suppose that a) and b) hold for S. Let $\mathcal{F}$ be a filter in S. By a), $\mathcal{F}$ has a weak adherent point f in S. Let us assume that f is not a $T_{|s|}(L')$-adherent point of $\mathcal{F}$. Then there exist a $g \in (L')_+$, an $\varepsilon_0 > 0$, and an $A \in \mathcal{F}$ such that

I) $$\int_{E-C} |f_n - f'|\, gdm \geq \varepsilon_0 \text{ for all } f' \in A.$$

Tha mapping $J_g : f \to fg$ from L into $L^1(E,m)$ is weakly continuous: Indeed, according to ([7] 20.20, cf. 19.31), $L^\infty(E,m)$ is the dual of $L^1(E,m)$. Let $\varepsilon > 0$ and $h_1, \ldots, h_q \in L^\infty(E,m)$ be given. Then

$gh_1, \ldots, gh_q$ are contained in $\lambda[-g,g]$ for some $\lambda > 0$, and thus belong L', because L' is an ideal in L^α. Since

$$J_g(\{f \in L : |\int fgh_i\, dm| < \varepsilon, i = 1,2,\ldots, q\}) \subseteq$$

$$\{h \in L^1(E,m) : |\int hh_i\, dm| < \varepsilon, i = 1,2,\ldots, q\},$$

we conclude that J_g is weakly continuous. Consequently, $J_g(A) = Ag$ is relatively weakly compact in $L^1(E,m)$, and fg is a weak adherent point of Ag. Let $f_n g$ be a sequence in Ag, which is weakly convergent to fg (cf. [9] 24.1.(7), for example). By [5] 9. or by ([3] IV . 8. (9-10)), there exists a compact set $C \subseteq \bigcup_{n=1}^{\infty} \mathrm{Supp}\,(g f_n)$, such that

II) $$\int_{E-C} |f_n - f|\, gdm < \varepsilon_0/2$$

By b), we can find a subsequence f_p of f_n which converges in measure to some $f' \in S$ on C. To contradict I), and thus obtain the $T_{|s|}(L')$-compactness of S, it suffices to show that $f_p g \chi_C$ is norm convergent to $f'g\chi_C$: For in this case, the weak convergence of $f_p g$ to fg implies that $f'g = fg$ on C, so that for sufficiently large p,

$$\int_C |f - f|\, gdm < \varepsilon_0/2.$$

Combining this with II), we obtain

$$\int |f - |f_p\, gdm = \int_{E-C} |f_p - f|\, gdm + \int_C |f - f_p|\, gdm < \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0.$$

which is contradictory, because $f_p \in A$.

Let, therefore, $\varepsilon < 0$ be given. $\{(f_p - f') : p = 1,2,\ldots\}$ is relatively weakly compact, so, by ([5]9) or ([3] IV.8.(9-10)), there exists a $\delta > 0$ such that $m(B) < \delta$ implies

$$\int_B |f_p - f'|\, gdm < \varepsilon/2.$$

Since g is locally summable, there exists an $\varepsilon' > 0$ such that

$$\varepsilon' \int_C gdm < \varepsilon/2.$$

Let now p_0 be so chosen, that for all $p \geq p_0$, the measure of $B_p = \{x \in C : |f_p(x) - f'(x)| > \varepsilon'\}$ is less than δ. Then we have, for all $p \geq p_0$

$$\int_C |f_p - f'| \, gdm = \int_{B_p} |f_n - f'| \, gdm + \int_{C-B_p} \varepsilon' \, gdm < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So gf_n converges strongly to gf on C, which is what we wanted to show.

We remark that a diagonal argument can be used to show that b) in (8) can be replaced by the stronger condition

b)′ *If f_n is a sequence in S with weak adherent point f, then for every σ-finite set $A \subseteq E$, there is a subsequence of f_n which converges in measure to f on every subset of A having finite measure.*

We close this section with an interesting application of (8). (9) *If L is an atomic* (cf. § 6.) *Riesz space, and L' is an order dense ideal in L^x, then the $T_{|s|}(L')$ and the $T_s(L')$-compact subsets of L coincide.*

PROOF: L can be identified with a Riesz subspace of $\prod_A \mathbf{R}$, containing $\bigoplus_A \mathbf{R}$, for some A. Also, $\bigoplus_A \mathbf{R} \subseteq L'$ (cf. 1.(1)), so if we give A the discrete topology, and define m to be the counting measure, (*) is satisfied by both L and L'. Since m-convergence is simply coordinatewise convergence, one sees immediately that a) of (8) implies b) of (8). The result follows from (8).

6. *Atomic Riesz Spaces.*

An element $f \neq 0$ in a Riesz space L is called an *atom*, if for any $g \in L$ satisfying $|g| \leq |f|$, $g = \lambda f$ holds for some real number λ. Thus, if f is an atom, either f or $-f$ is positive. The Riesz space L is called atomic, if L has an order basis consisting of atoms, or, what is equivalent (this follows from [11] Theorem 29.10), if L is archimedian, and every ideal in L contains an atom.

Let L be an atomic Riesz space, and let $e_\alpha, \alpha \in A$ be an order basis for L consisting of positive atoms. For each $0 \leq u \in L$ and each $\alpha \in A$, we define $\lambda_\alpha(u) = \sup(\lambda: \lambda e_\alpha \leq u)$. Since L is archimedian, $0 \leq \lambda_\alpha(u) < < \infty$ and $\lambda_\alpha(u)\, e_\alpha \leq u$. One verifies easily that $u = \sum_A \lambda_\alpha(u)\, e_\alpha$, and that the mapping

$$f \to (\lambda_\alpha(f^+) - \lambda_\alpha(f^-))_{\alpha \in A}$$

is a 1-1 mapping of L into $\prod_A \mathbf{R}$ which which preserves order relations. Thus L may be identified with a Riesz subspace of $\prod_A \mathbf{R}$ which contains $\bigoplus_A \mathbf{R}$. It follows from 1.(1), that $L^x = L^\alpha \subseteq \bigoplus_A \mathbf{R}$, so L is admissable and L^x, and every ideal in L^x is atomic. Conversely, if L is admissable and L^x is atomic, then by the preceding remark, the ideal D generated by L in L^{xx} is atomic. By 0. (4a), there exists for every $0 \leq u \in D$, a $0 \leq v \in L$ such that $v \leq u$. So if u is an atom, $u = \lambda v$, for some $\lambda \geq 1$, so $u \in L$. It follows that L is atomic. Summarizing, we have:

(1) *The Riesz space L is atomic, if and only if L is admissable and L^x is atomic.*

We now present several topological characterizations of atomic Riesz spaces.

(2) *Let L be an admissable Riesz space, and let L' be an order dense ideal in L^x. The following conditions on L are equivalent:*

a) *L is atomic;*

b) *Every order bounded sequence g_n in L, which is $T_s(T')$-convergent to 0, converges for $T_{|s|}(L')$; or equivalently,*

$$T_s(L)\text{-}\lim g_n = 0 \textit{ implies } T_s(L)\text{-}\lim |g_n| = 0;$$

c) *The topologies $T_s(L')$ and $T_{|s|}(L')$ have the same compact sets;*

d) *The order intervals of L are $T_{|s|}(L')$-precompact.*

Proof: a) $\Rightarrow$ c) by 5.(9). c) $\Rightarrow$ b) is clear. We show b) $\Rightarrow$ a). Let $u \in L_+$ be given, and let $\varphi \in L'_+$ be such that $\varphi(u) > 0$. If D designates the ideal in $(L')^x$ generated by L, then by 0.(4a), there exists a $v \in L_+$ which is less than the Riesz component (in D) of u in $\{f \in D : \varphi(|f|) =$

$= 0\}^p$, and thus also less than u. We denote by $L_\infty = L_\infty(v)$ the ideal generated by v in L, and we denote the ideal generated by v in L^{xx} by $L^\infty = L^\infty(v)$. By 0.(4b), $L^x = (L^\infty)^x = L^1(v)$. The elements of L' define members of $L^1 = L^1(v)$, and, by our choice of v, the element $\bar\varphi$ of L^1 defined by φ has the property

$$(*)\ \bar\varphi\ (|f|) = 0 \text{ implies } f = 0$$

for any $f \in L_\infty$. Taking into account 0.(4a), we see that (*) also holds for the elements of L^∞.

We shall show that L^∞ is atomic. It then follows as in the proof of (1), that L_∞ is atomic. This demonstrates that the ideal generated by u in L contains atoms. Since u was an arbitrary element in L_+, it follows that L is atomic.

L^∞ being dedekind complete (ideal in L'^x), we can make use of the Theorem of [6], according to which L^∞ is atomic, if for any sequence $g_n \in L^\infty$, $T_s(L^1)$-lim $g_n = 0$ implies $T_s(L^1)$-lim $|g_n| = 0$. But, by 4.(1), L^∞ is the dual of the (B)-space L^1, so such a sequence g_n is strongly bounded (cf. [9] 21,5.(4)), i.e. g_n is contained in $\lambda[-v,v]$ for some $\lambda > 0$. By 0.(4a), we can find sequences f_n and h_n in $[-\lambda v, \lambda v]$ in L_∞ satisfying $f_n \leq g_n \leq h_n$ and $\varphi(h_n - f_n) < 1/n$. We assert that $h_n - g_u$ is $T_s(L^1)$-convergent to 0. Indeed, by 2.(4), the order interval $[-2\lambda v, 2\lambda v]$ in L^∞ is $T_s(L^1)$-compact. If g is a weak adherent point of $h_n - g_n$, then $g \geq 0$, and, since $h_n - g_n \leq h_n - f_n$, $\bar\varphi(g) < 1/n$ for all n, i.e. $\bar\varphi(g) = 0$. By (*), $g = 0$, so our assertion follows. From $h_n = (h_n - g_n) + g_n$, we have that $T_s(L^1)$-lim $h_n = 0$ and, since the topology induced on L_∞ by $T_s(L')$ is weaker than $T_s(L^1)$, also $T_s(L')$-lim $h_n = 0$. Now, by b), $T_s(L')$-lim $|h_n| = 0$ so in particular, lim $\varphi(|h_n|) =$ lim $\bar\varphi(|h_n|) = 0$. Similarly, lim $\bar\varphi(|f_n|) = 0$, so we obtain from $|g_n| \leq |f_n| + |h_n|$, that lim $\bar\varphi(|g_n|) = 0$. Applying the argument used above, we obtain, finally, $T_s(L^1)$-lim $|g_n| = 0$, which is what we had to show. Thus a) - c) are equivalent.

d) $\Rightarrow$ b) is clear. We prove a) $\Rightarrow$ d). If L is atomic, then so is L'^x (cf. (1) and the argument preceding it). By 2.(1), the order intervals of L'^x are $T_s(L')$-compact, so, by c) and 2.(4), they are $T_{|s|}(L')$-compact. According to 3.(2), L'^x is the $T_{|s|}(L')$-completion of L. It follows that the order intervals of L are $T_{|s|}$-precompact.

In addition to Halperin and Nakano [6], Kawai [8] and (independently) Walsh [11] have given topological characterizations of atomic admissable DCR-spaces. Our condition b) can be found in [8], and our d) is intimitely related to condition (7) of Theorem 5.4 of [8] and to the condition of [11].

Let $L[T]$ be a locally convex Riesz space. It can, in general, happen, that $L[T]$ is atomic, but its completion with respect to T is not; or, conversly, that the completion is atomic although L is not. For an important class of topologies on an admissable Riesz space, neither of these situations can occur.

(3) *Let L be an admissable Riesz space, let L' be an order dense ideal in L^x and let T be a locally solid locally convex topology on L which is finer than $T_{|s|}(L')$ but coarser than $T_{b^*}(L')$* (cf. 2. (6)). *Then L is atomic if and only if the same is true of $\bar{L}$, its completion with respect to T.*

PROOF: The arguments used in the proof of 3.(6) are easily adapted to show that $L \subseteq \bar{L} \subseteq L'^x$, and that $u_t \downarrow 0$ in $\bar{L}$ implies $u_t \downarrow 0$ in L'^x. To assert that therefore $L'^{xx} \subseteq \bar{L}^x$, we show that L, and hence also $\bar{L}$, is $T_s(L'^{xx})$-dense in L'^x. But this follows immediately from the fact (cf. 0. (4)) that for each $v \in (L'^x)_+$, there exists a family $0 \leq v_t \in L$ such that $v_t \uparrow v$. Since $L' \subseteq L'^{xx}$, we conclude from $L'^{xx} \subseteq \bar{L}^x$ that L' is an order dense ideal in $(\bar{L})^x$. (2.(1), 0.(3)). So L^x and $\bar{L}$ are atomic if and only if L' is. Our assertion now follows from (1).

FINAL REMARK: We point out, that, on the basis of the theory presented here, it is easy to generalize the remaining results of § 30. of [9].

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