FIBRE INTEGRATION AND SOME OF ITS APPLICATIONS

By

STEPHEN HALPERIN AND WERNER H. GREUB

INTRODUCTION. It is a classical result that the Euler Poincaré characteristic of a compact connected manifold can be identified with other invariants (see Theorem V, sec. 7).

The proofs of the various assertions in that theorem are scattered through the literature, and make use of widely varying techniques. It is possible, however, to establish these results within the single context of differential forms, the exterior derivative and the integral. This article is a survey of this approach (without proofs).

The main tool (in addition to standard machinery) is the fibre integral. This operator is an actual integral, and induces the «homological fibre integral» (defined in terms of a spectral sequence). Although its existence has been in the folklore for a quarter century, or more, there appears to be no explicit account of it in the literature.

0. NOTATION. All manifolds in this paper are C^∞ and all maps are C^∞-maps. If M is an n-manifold, the algebra of C^∞-functions on M is denoted by $S(M)$. The tangent bundle is written $\tau_M = (T_M, \pi, M, \mathbf{R}^n)$ while $A(M)$ denotes the graded algebra of C^∞-differential forms on M. $A_C(M)$ denotes the graded ideal of forms with compact carrier. The exterior derivative is denoted by δ.

Corresponding to $A(M)$ and $A_C(M)$ are the de Rham cohomology algebras $H(M)$ and $H_C(M)$. The multiplication of differential forms induces bilinear maps

$$H_C(M) \times H(M) \to H_C(M) \quad \text{and} \quad H_C(M) \times H(M) \to H_C(M)$$

which are written

$$(\alpha, \beta) \to \alpha * \beta \quad \text{and} \quad (\beta, \alpha) \to \beta * \alpha \qquad \begin{array}{r}\alpha \in H(M) \\ \beta \in H_C(M).\end{array}$$

These make $H_C(M)$ into a left and right graded module over the ring $H(M)$.

A map $\varphi: M \to N$ induces a bundle map $d\varphi: T_M \to T_N$ (the derivative of φ) and a homomorphism homogeneous of degree zero,

$$\varphi^* : A(M) \to A(N);$$

φ^* in turn induces a homomorphism

$$\varphi^{\#} : H(M) \leftarrow H(N).$$

1. THE INTEGRAL. With each oriented n-manifold, M, there is associated the *integral*, a linear operator

$$\int_M ;\ A_C^n(M) \to \mathbf{R}.$$

It is uniquely determined by the following conditions:

i) Naturality with respect to orientation preserving diffeomorphisms.

ii) If $U \subset M$ is an open subset containing the carrier of $\Phi \in A_C^n(M)$, then

$$\int_M \Phi = \int_U \Phi$$

iii) If $\mathbf{R}^n$ is an oriented Euclidean n-space and if $f \in S_C(\mathbf{R}^n)$, then

$$\int_{\mathbf{R}^n} f\, \delta x' \wedge \dots \wedge \delta x^n = \int_{\mathbf{R}^n} f(x)\, dx' \dots dx^n$$

(Riemann integral) where the x^ν are the coordinate functions corresponding to a positive orthonormal basis of $\mathbf{R}^n$.

It follows from Stokes' theorem that

$$\int_M \delta\Psi = 0, \quad \Psi \in A_C^{n-1}(M)$$

and so integration induces a linear map

$$\int_M^{\#} : H_C^n(M) \to \mathbf{R}.$$

This is extended to $H_C(M)$ by setting it zero in

$$H_C^p(M) \ (0 \leq p \leq n-1).$$

The *Poincaré scalar product* of an oriented n-manifold, M, is the bilinear map

$$\mathcal{P} : H(M) \times H_C(M) \to \mathbf{R}$$

given by

$$\mathcal{P}(\alpha, \beta) = \begin{cases} \int_M^{\#} \alpha * \beta & \alpha \in H^p(M), \quad \beta \in H_C^{n-p}(M) \\ & \quad 0 \leq p \leq n \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha \in H^p(M)$, $\beta \in H_C^{n-p}(M)$ are represented respectively by Φ and Ψ, then

$$\mathcal{P}(\alpha, \beta) = \int_M \Phi \wedge \Psi.$$

This scalar product induces a linear map

$$D : H(M) \to H_C(M)^*$$

($H_C(M)^*$ denotes the space of (real) linear functions in $H_C(M)$). The *Poincaré duality theorem* asserts that D is a linear isomorphism for every oriented n-manifold; in particular, the bilinear function $\mathcal{P}$ is non-degenerate.

If $\varphi : M \to N$ is a map between compact, connected oriented n-manifolds, then the *degree* of φ is an integer, uniquely determined by the equation

$$\int_M \varphi^* \Phi = \deg \varphi \cdot \int_N \Phi \qquad \Phi \in A^n(N)$$

For example let E be an oriented Euclidean space of dimension $n+1$ and let Δ_E denote the positive normed determinant function in E. Define $\Omega \in A^n(S^n)$ (S^n the unit sphere) by

$$\Omega(y; k_1, \ldots, k_n) = \Delta_E(y; k_1, \ldots, k_n) \qquad \begin{array}{l} y \in S^n \\ k_v \in T_y(S^n). \end{array}$$

Then Ω orients S^n and

$$\int_{S^n} \Omega = \text{vol}(S^n).$$

Now let φ be a map from a compact, connected oriented n-manifold M into S^n. Regard φ as an E -valued function on M and define $\Phi \in A^n(M)$ by

$$\Phi(x;\, h_1, \dots, h_n) \triangle_E = \varphi(x) \wedge \delta\varphi(x, h_1) \wedge \dots \wedge \delta\varphi(x;\, h_n).$$

Then $\Phi = \varphi^* \Omega$ and hence

$$\int_M \Phi = \deg \varphi .\ \mathrm{vol}\ (S^n).$$

In particular, if $M = S^n$ and if φ is the identity map, then $\Phi = \Omega$ and

$$\int_{S^n} \Phi = \mathrm{vol}\ (S^n).$$

2. Smooth fibre bundles. A C^∞-*fibre bundle* is a quadruple $\xi = (E, \pi, B, F)$. where E, B, F are manifolds, $\pi : E \to B$ is a map and, for some open covering $\mathcal{U}$, of B, there are diffeomorphisms

$$\Psi_U : U \times F \xrightarrow{\cong} \pi^{-1}(U) \qquad\qquad U \in \mathcal{U}$$

which satisfy

$$\pi \Psi_U(x, y) = x \qquad\qquad x \in,\ U\ y \in F.$$

The collection (U, Ψ_U) is called a *coordinate representation for* ξ. $\pi^{-1}(x)$ is called the *fibre over* x and is written F_x (it is a closed submanifold of E); the inclusion is written $j_x : F_x \to E$.

A subset $A \subset E$ is called *fibre-compact*, if, for each compact subset $K \subset B$, $\pi^{-1}(K) \cap A$ is compact. The differential forms on E with fibre-compact carrier are a graded ideal in $A(E)$, denoted by $A_F(E)$. Clearly,

$$A_C(E) \subset A_F(E) \subset A(E)$$

and the first (resp. second) inclusion is equality if and only if B (resp. F) is compact. The ideal $A_F(E)$ is stable under the exterior derivative, and the corresponding cohomology algebra is denoted by $H_F(E)$. $H_F(E)$ is a left and right module over $H(E)$.

Let dim $F = r$, and consider the set, O, of r-forms on E which satisfy the condition

$$(1) \qquad\qquad (j_x^* \Psi)(y) \neq 0 \qquad\qquad x \in B,\quad y \in F_x$$

(O may be empty). Thus if $\Psi \in O$, then $j_x^* \Psi$ orients F_x for each $x \in B$. Two such forms are called *equivalent*, if they induce the same orientation in each F_x Each equivalence class is called an *orientation in the bundle* ξ. An r-form satisfying (1) is said to *orient* ξ; if such forms exist, the bundle is called *orientable*.

If the manifold B is oriented by an n-form Φ, and the bundle ξ is oriented by Ψ, then $\pi^* \Phi \wedge \Psi$ orients the manifold E: this orientation of E is called the *local product orientation*.

3. FIBRE INTEGRATION. Let $\xi = (E, \pi, B, F)$ be an oriented bundle with $dim\ F = r$ and $dim\ B = n$. Then we can associate with ξ a canonical linear map, homogeneous of degree $-r$, the *fibre integral*,

$$\oint_F : A_F(E) \longrightarrow A(B).$$

It is defined as follows: Let $\Omega \in A_F^{p+r}(E)$, $p \geqq 0$. Fix a point $x \in B$ and a system $h_1, \ldots, h_p$ of tangent vectors in $T_x(B)$. Then an r-form, $\Omega_{x; h_1, \ldots, h_p}$, on F_x is defined by

$$\Omega_{x; h_1 \ldots h_p}(y;\ k_1 \ldots k_r) = \Omega(y;\ l_1, \ldots l_p,\ k_1 \ldots k_r) \tag{2}$$
$$y \in F_x, \quad k_j \in T_y(F_x),$$

where $l_i \in T_y(E)$ is chosen so that $(d\pi)\, l_i = h_i$. Note that because of the skew symmetry of Ω, the right hand side of (2) is independent of the choice of the l_j.

The function

$$(x;\ h_1, \ldots, h_p) \longrightarrow \int_{F_x} \Omega_{x; h_1 \ldots h_p}$$

is p-linear, skew symmetric in the h_i and smooth in x. Thus it defines a p-form, $\oint_F \Omega$, on B,

$$\Big(\oint_F \Omega\Big)(x;\ h_1, \ldots h_p) = \int_{F_x} \Omega_{x; h_1 \ldots h_p}.$$

The operator $\oint_F$ is extended to forms of degree less than r by setting it equal to zero in this case.

The fibre integral has the following properties:

i) It is linear and homogeneous of degree $-r$.

ii) $$\oint_F \pi^* \Phi \wedge \Omega = \Phi \wedge \oint_F \Omega \qquad \begin{array}{l}\Phi \in A(B)\\ \Omega \in A_C(E).\end{array}$$

In particular, the linear map $\oint_F$ is surjective.

iii) It restricts to a surjective linear map $\oint_F : A_C(E) \to A_C(B)$.

iv) It commutes with the exterior derivative

$$\delta \circ \oint_F = \oint_F \circ \delta.$$

v) If B is oriented and E is given the local product orientation, then

$$\int_B \oint_F \Omega = \int_E \Omega \qquad \Omega \in A_C^{n+r}(E)$$

(Fubini theorem).

It follows from iv) that $\oint_F$ induces linear maps

$$\oint_F^{\#} : H_F(E) \to H(B) \quad \text{and} \quad \oint_F^{\#} : H_C(E) \to H_C(B).$$

If B is oriented then the Fubini theorem shows that the second of these is dual to

$$\pi^{\#} : H(E) \longleftarrow H(B)$$

with respect to the Poincaré scalar products. Moreover if F is compact then $H_F(E) = H(E)$ and $\oint_F^{\#} : H(E) \to H(B)$ is dual to $\pi_C^{\#} : H_C(E) \leftarrow H_C(B)$.

4. Vector bundles and sphere bundles. Let $\xi = (E, \pi, B, F)$ be a real oriented vector bundle of rank $r \geq 2$ over an n-manifold B. Since F is contractible, it follows from Poincaré duality that the map $\int_F^{\#} : H_C(F) \to \mathbf{R}$ is a linear isomorphism. This in turn implies that the map

$$\oint_F^{\#} : H_P(E) \to H(B)$$

is an isomorphism. The inverse isomorphism, written Th, is called the *Thom isomorphism* for ξ and the class $\theta_\xi = Th(1)$ is called the *Thom class* of ξ. It is an element of $H_F^r(E)$. Using property ii), sec. 3, we see that

$$Th(\alpha) = (\pi^{\#}\alpha) * \theta_\xi \qquad \alpha \in H(B).$$

Any closed r-form on E with fibre-compact carrier and fibre integral equal to 1 represents θ_ξ. Moreover, for every neighbourhood U of the zero cross-section there is a representative of θ_ξ with carrier in U.

Next, choose a Riemannian metric in ξ and let $\xi_S = (E_S, \pi_S, B, S)$ denote the associated sphere bundle. Since S is compact, the fibre integral determines a short exact sequence

$$\text{(3)} \qquad 0 \longrightarrow \ker \int_S \longrightarrow A(E_S) \xrightarrow{\int_S} A(B) \longrightarrow 0.$$

On the other hand, since $\dim S \geq 1$ we have $\int_S \circ \pi_S^* = 0$ and so π_S^* may be regarded as a map into $\ker \int_S$. The fact that $H^+\left(\ker \int_S\right) = 0$ implies that π_S^* induces an isomorphism $H(B) \xrightarrow{\cong} H\left(\ker \int_S\right)$. Hence, (3) yields an exact triangle

$$\begin{array}{ccc} H(B) & \xrightarrow{\pi_S^*} & H(E_S) \\ & \nwarrow \partial \quad \swarrow \left(\int_S\right)^{\#} & \\ & H(B) & \end{array}$$

where ∂ is a linear map, homogeneous of degree r. The class $\chi_\xi = \partial(-1)$ is called the *Euler class of* ξ. We have the relation

$$\partial\alpha = (-1)^{p-1}\alpha \cdot \chi_\xi \qquad \alpha \in H^p(B)$$

and it follows that the sequence

$$\text{(4)} \qquad \longrightarrow H^p(B) \xrightarrow{D} H^{p+r}(B) \xrightarrow{\pi_S^{\#}} H^{p+r}(E_S) \xrightarrow{\left(\int_S\right)^{\#}} H^{p+1}(B) \longrightarrow$$

is exact, where

$$D\alpha = \alpha \cdot \chi_\xi \qquad \alpha \in H(B)$$

(4) is called the *Gysin sequence* for the vector bundle ξ.

With the aid of Stokes' theorem it is shown that

$$\pi^{\#}(\chi_\xi) = \lambda_{\#}(\theta_\xi)$$

where $\lambda : A_F(E) \to A(E)$ denotes the inclusion map. Hence if σ is any cross-section in ξ, then $\chi_\xi = \sigma^{\#}(\theta_\xi)$.

If *dim* $B =$ *dim* F and B is compact and oriented we can form the integral $\overset{\#}{\int_B} \chi_\xi$, called the *Euler number of* ξ. If is any cross-section in ξ, then

$$\overset{\#}{\int_B} \chi_\xi = \overset{\#}{\int_B} \sigma^{\#}(\theta_\xi)$$

5. Euler class and index sum. Let $\xi = (E, \pi, B, F)$ be an oriented vector bundle of rank n over a compact oriented n-manifold. Let $\sigma : B \to E$ be a cross-section with only finitely many zeros, $a_1, \dots, a_q$ (σ could be constructed with the help of the Thom transversality theorem).

To each a_μ there is assigned an integer, $j_\mu(\sigma)$, called the index of σ at a_μ. It is defined as follows: In a sufficiently small neighbourhood U_μ of a_μ we can write $\sigma(x) = (x, \tau_\mu x)$ where $\tau_\mu : U_\mu \to F$ is a smooth map with an isolated zero at a_μ. Let S_μ be a sphere about a_μ in U_μ and let $S_F^{\cdot}$ be the unit shere in F. Define $\varphi_\mu : S_\mu \to S_F^{\cdot}$ by

$$\varphi_\mu(x) = \frac{\tau_\mu(x)}{|\tau_\mu(x)|} \qquad x \in S_\mu$$

and set

$$j_\mu(\sigma) = deg\ \varphi_\mu$$

(the degree of φ_μ depends only on σ). The number

$$j(\sigma) = \sum_{\mu=1}^{q} j_\mu(\sigma)$$

is called the *index sum of* σ.

THEOREM I: Let ξ be an oriented vector bundle of rank n over a compact, oriented n-manifold B. Let σ be a cross-section in ξ with finitely many zeros. Then

$$j(\sigma) = \int_B^{\#} \chi_\xi .$$

COROLLARY I: The Euler number of ξ is an integer.

COROLLARY II: The index sum of σ is independent of σ.

If B is connected, the zeros of σ can be moved around by a diffeomorphism so as to lie in a single chart neighbourhood, U, over which the bundle is trivial. If S is the unit sphere in U we may assume that the zeros are inside S. Identify the restriction of ξ to U with $U \times F$. Write $\sigma x = (x, \tau x)$ and set

$$\varphi x = \frac{\tau x}{|\tau x|} \qquad x \in S.$$

Then

$$deg\ \varphi = j(\sigma) = \int_B^{\#} \chi_\xi$$

In particular, if the Euler number of ξ is zero, so is $deg\ \varphi$. Thus a theorem of H. Hopf yields a smooth map $\widehat{\tau} : U \to F - \{0\}$ which agrees with τ outside a compact set. Evidently $\widehat{\tau}$ determines a cross-section in ξ without zeros and we have

THEOREM II: With the hypothesis of Theorem I, assume that B is connected. Then the vector bundle ξ admits a cross-section without zeros if and only if its Euler number is zero.

THE GAUSS-BONNET-CHERN THEOREM. In this section we shall discuss the relation between the Euler class of a vector bundle and the total curvature of a Riemannian connection in this bundle. To state the theorem in intrinsic terms it is necessary to recall the notion of the Pfaffian of m skew-symmetric, linear transformations in an oriented $2m$-dimensional Euclidean space, F.

Let $\varphi_1, \ldots, \varphi_m$ be skew-symmetric linear transformations of F. Each φ_j determines the skew symmetric 2-tensor Φ_j given by

$$\Phi_j(x, y) = < \varphi_j x, y > \qquad x, y \in F.$$

Let $\triangle_F$ denote the (unique) normed determinant function in F which represents the orientation. Then a real number, $\mathcal{P}f(\varphi_1, \ldots, \varphi_m)$, is determined by

$$\Phi_1 \wedge \ldots \wedge \Phi_m = \mathcal{P}f(\varphi_1, \ldots, \varphi_m) \triangle_F.$$

It is called the *Pfaffian* of the transformations $\varphi_1, \ldots, \varphi_m$.

Now let $\xi = (E, \pi, B, F)$ be an oriented vector bundle of rank $2m$. Introduce a Riemannian metric and Riemannian connection, ∇, in ξ and denote by R the corresponding curvature tensor. It assigns to each point $x \in B$ and to each pair h_1, h_2 of tangent vectors of B at x a skew linear transformation

$$R(x;\ h_1, h_2) : F_x \longrightarrow F_x.$$

Hence we can form the Pfaffian,

$$\mathcal{P}f(R(x;\ h_1, h_2), \ldots, R(x;\ h_{2m-1}, h_{2m})).$$

Alternating over the tangent vectors h_i yields a $2m$-form on B, $\mathcal{P}f(R)$, called the *total curvature form*. Explicitly,

$$\mathcal{P}f(R)(x;\ h_1, \ldots h_{2m}) = \frac{(-1)^m}{(4\pi)^m} \frac{1}{m!} \sum_\sigma \varepsilon_\sigma \mathcal{P}f(R(x;\ h_{\sigma(1)}, h_{\sigma(2)}), \ldots, R(x;\ h_{\sigma(2m-1)}, h_{\sigma(2m)})),$$

where σ runs over all permutations of $(1, \ldots 2m)$. The Bianchi identity implies that the total curvature form is closed and hence it represents an element of $H^{2m}_{-}(B)$.

THEOREM III: (Gauss-Bonnet-Chern): The total curvature form represents the Euler class of ξ.

In the special case that B is a compact and oriented n-manifold with $n = 2m$, Theorems I and III yield.

THEOREM IV: Let σ be any cross-section in ξ with finitely many zeros. Then

$$j(\sigma) = \overset{\#}{\int_B} \chi_\xi = \int_B \mathcal{P}f(R).$$

The integral on the right hand side of this equation is called the *total curvature of* ξ; thus Theorem IV shows that the total curvature is an integer, independent of the connection.

The following proof of Theorem III is due to Chern. Let (E_S, π_S, B, S) be the unit sphere bundle of ξ. We explicitly construct a $(2m-1)$-form, Ω, on E_S so that

$$\int_S \mathfrak{F}\,\Omega = -1 \quad \text{and} \quad \delta\,\Omega = \pi_S^* \, \mathcal{P}f(R);$$

these relations show that $\mathcal{P}f(R)$ represents χ_ξ.

First define the vertical subbundle $V_E \subset T_E$ by $V_z(E) = T_z(F_{\pi z})$. It can be identified with the pull-back of ξ to E via the isomorphisms $V_z(E) = T_z(F_x) = F_x$, $x = \pi z$. Restricting V_E to the manifold E_S yields a vector bundle η over E_S which inherits a metric and a connection from ξ. Moreover, each $z \in E$ belongs to F_x $(x = \pi z)$ and hence to $V_z(E)$. This correspondence defines a canonical cross-section in V_E. Its restriction to E_S is denoted by σ, and satisfies

$$|\sigma z| = 1 \qquad z \in E_S.$$

Let the connection and curvature in η be denoted by $\widehat{\nabla}$ and $\widehat{R}$; regard $\widehat{R}(z, k_1, k_2)$ as an element of $\wedge^2 F_x$, $x = \pi z$. Let Δ_x denote the positive normed determinant function in F_x. Then for each odd integer p $(1 \leq p \leq 2m-1)$ define a covariant tensor field, Ψ_p, of degree $2m-1$, on E_S by

$$\Psi_p(z, k_1, \ldots k_{2m-1}) \cdot \Delta_x = \sigma(z) \wedge \widehat{\nabla}_\sigma \sigma(z; k_1) \wedge \ldots$$
$$\ldots \widehat{\nabla}_\sigma(z;\, k_p) \wedge \widehat{R}(z;\, k_{p+1}, k_{p+2}) \wedge \ldots \wedge \widehat{R}(z;\, k_{2m-2}, k_{2m-1}).$$

Alternating over the k_p yields a $(2m-1)$-form, Φ_p, on E_S. A straightforward computation shows that there are scalars $c_p \in \mathbf{R}$ such that

$$\delta \sum_p c_p \Phi_p = \mathcal{P}f(\widehat{R}).$$

In particular,

$$c_{2m-1} = -\frac{1}{\operatorname{vol}(S^{2m-1})}.$$

On the other hand, since $\widehat{\nabla}$ is the pull-back of ∇, it follows that

$$\widehat{R}(z; h_1, h_2) = R(\pi z, (d\pi) h_1, (d\pi) h_2) \tag{5}$$

whence

$$\mathcal{P}f(\widehat{R}) = \pi_S^* \mathcal{P}f(R).$$

Thus we have

$$\pi_S^* \mathcal{P}f(R) = \delta \sum_p c_p \Phi_p.$$

It remains to show that $\oint_S \sum_p c_p \Phi_p = -1$. Equation (5) shows that R has no vertical dependence. Thus the restriction of Φ_μ to a fibre S_x in E_S is zero, unless $p = 2m - 1$. This implies that

$$\Big(\oint_S \sum_p c_p \Phi_p\Big)(x) = \int_{S_x} c_{2m-1} \Phi_{2m-1}.$$

Furthermore, the restriction of η to S_x is canonically $S_x \times F_x$, and the induced connection is simply the exterior derivative. Also, the restriction of σ to S_x is the inclusion $i : S_x \to F_x$ (S_x the unit sphere in F_x) regarded as a vector valued function on S_x. Thus the restriction of Φ_{2m-1} to S_x is the differential form, Φ, of sec. 1. It follows that

$$\int_{S_x} c_{2m-1} \Phi_{2m-1} = -1.$$

This completes the proof.

7. THE LEFSCHETZ CLASS. Recall that the Euler-Poincaré characteristic of a compact n-manifold M is defined by

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \, dim \, H^p(M).$$

THEOREM V: Let M be a compact connected oriented $2m$-manifold. Then

$$\int_M^{\#} \chi_{\tau_M} = \int_M^{\#} \mathcal{P}f(R) = j(X) = \chi(M)$$

where $j(X)$ is the index sum of a vector field on M with finitely many zeros and R is the curvature of a Riemannian connection in the tangent bundle τ_M.

The last equality was first shown by H. Hopf, via a triangulation. For the sake of completeness we sketch here a (different) proof of Theorem V, which identifies the Euler number of τ_M with the Euler-Poincaré characteristic of M.

Let $\triangle : M \to M \times M$ be the diagonal map, and let

$$\triangle^{\sim} : H(M) \to H(M \times M)$$

be the linear map dual to $\triangle^{\#}$ (with respect to the Poincaré scalar product). Then the element $\triangle^{\sim}(1) \in H^{2m}(M \times M)$ is called the *Lefschetz class of M* and is written λ_M,

$$\lambda_M = \triangle^{\sim}(1).$$

It follows directly from the definition and the Künneth isomorphism $H(M \times M) \cong H(M) \otimes H(M)$ that

$$\int_M^{\#} \triangle^{\#}(\lambda_M) = \chi(M). \tag{6}$$

On the other hand, a linear connection in τ_M determines an exponential map, $\exp : T_M \to M$. Let $\pi : T_M \to M$ be the projection and let $0 : M \to T_M$ denote the zero cross-section. Then a smooth map $\varphi : T_M \to M \times M$ is defined by

$$\varphi(h) = (\pi h, \exp h) \qquad h \in T.$$

Moreover, in a neighbourhood O of $0(M)$, φ is a diffeomorphism. Thus φ determines a map

$$(\varphi_c)_* : A_c(O) \to A(M \times M).$$

Now let $\Phi \in A_c(O)$ represent the Thom class of T_M. Since φ is fibre preserving (where $M \times M$ is regarded as a trivial bundle over M with respect to projection onto the left factor), $(\varphi_c)_*$ commutes with fibre integration. It follows that $(\varphi_c)_* \Phi$ represents the Lefschetz class of M. Moreover, since $\varphi \circ 0 = \triangle$, we have

$$\triangle^* (\varphi_c)_* \Phi = 0^* \varphi^* (\varphi_c)_* \Phi = 0^* \Phi,$$

and hence

$$(7)\qquad \overset{\#}{\int\limits_M} \triangle^{\#}(\lambda_M) = \int\limits_M \triangle^{*}(\varphi_c)_{*}\,\Phi = \int\limits_M 0^{*}\,\Phi = \overset{\#}{\int\limits_M} \chi_{\tau_M}.$$

Combining relations (6) and (7) we obtain Theorem V.

8. THE SPECTRAL SEQUENCE OF A SMOOTH BUNDLE. Let $\xi = (E, \pi, B, F)$ be a smooth fibre bundle with *dim* $B = n$, *dim* $F = r$. The *vertical subbundle* of T_E is the vector bundle V_E whose fibre at $z \in E$ is the vector space

$$V_z(E) = T_z(F_x) = ker\,(d\,\pi)_z \qquad x = \pi z.$$

A cross-section in V_E is called a *vertical vector field* on E. The space of vertical vector fields is written $X_V(E)$.

A *decreasing filtration*, $(\mathcal{F}^q)$, of the graded differential algebra $A(E)$ is defined as follows: $\mathcal{F}^q(A^p(E))$ $(q = 0, 1, \ldots)$ consists of those p-forms Ω which satisfy

$$i(Y_0) \ldots i(Y_{p-q})\,\Omega = 0 \qquad Y_j \in X_V(E)$$

In particular,

$$A_H(E) = \sum_p \mathcal{F}^p(A^p(E))$$

is the subalgebra of $A(E)$ consisting of the forms, Ω, such that

$$i(Y)\,\Omega = 0 \qquad Y \in X_V(E).$$

It is called the *horizontal subalgebra* of $A(E)$.

The filtration $\mathcal{F}$ leads to a spectral sequence, (E_i^{pq}) $(i = 0, 1, \ldots)$ of graded algebras converging to the cohomology algebra $H(E)$; it may be identified with the standard Cartan-Leray-Serre sequence. We shall recall how the first terms of this sequence can be computed.

Choose a subbundle, H_E of T_E, so that $T_E = H_E \oplus V_E$. A cross-section in H_E is called a *horizontal vector field* (with respect to H_E). Let $A_V(E)$ denote the subalgebra of $A(E)$ consisting of those forms Ω which satisfy

$$i(X)\,\Omega = 0, \qquad X \text{ horizontal}.$$

The multiplication in $A(E)$ defines an algebra isomorphism

$$A_H(E) \otimes_{S(E)} A_V(E) \xrightarrow{\cong} A(E).$$

Moreover, H_E is the pull-back of T_B to E via π; hence, π^* induces an isomorphism

$$A(B) \otimes_{S(B)} S(E) \xrightarrow{\cong} A_H(E).$$

Thus an isomorphism

$$A(B) \otimes_{S(B)} A_V(E) \xrightarrow{\cong} A(E)$$

is determined by

$$\Phi \otimes \Psi \longmapsto \pi^* \Phi \wedge \Psi \qquad \begin{array}{c}\Phi \in A(B)\\ \Psi \in A_V(E).\end{array}$$

Under this isomorphism $\mathcal{F}^q(A^p(E))$ corresponds to

$$\sum_{j \geq q} A^j(B) \otimes A_V^{p-j}(E).$$

It follows that $\mathcal{F}^q$ is the ideal in $A(E)$ generated by $\pi^*(\sum_{j\geq q} A^j(B))$. It also follows that this isomorphism may be regarded as an isomorphism

$$A(B) \otimes_{S(B)} A_V(E) \xrightarrow{\cong} E_0 \tag{8}$$

of bigraded algebras.

The operator d_0 is obtained as follows: Observe that the Lie product of vertical vector fields is again vertical. Identify $A_V^p(E)$ with the skew symmetric maps

$$X_V(E) \times \ldots \times X_V(E) \longrightarrow S(E)$$

which are p-linear over $S(E)$. Define $\delta_V: A_V^p(E) \to A_V^{p+1}(E)$ by

$$(\delta_V \Phi)(Y_0, \ldots, Y_p) = \sum_i (-1)^i Y_i (\Phi(Y_0 \ldots \hat{Y}_i \ldots Y_p) +$$
$$+ \sum_{i<j} (-1)^{i+j} \Phi([Y_i, Y_j], Y_0 \ldots \hat{Y}_i \ldots \hat{Y}_j \ldots Y_p).$$

Then $\delta_V(\pi^* f) = 0$, $f \in S(B)$ and so we can form the operator $\omega_B \otimes \delta_V$ in $A(B) \otimes A_V(E)$. (Here $\omega_B \Phi = (-1)^p \Phi$, $\Phi \in A^p(B)$. The operator $\omega_B \otimes \delta_V$ corresponds to d_0 under the isomorphism (8).

Henceforth it is assumed that ξ satisfies the following conditions:

i) $dim\ H(F) < \infty$

ii) There is a coordinate representation $\{U_\alpha, \Psi_\alpha\}$ of ξ such that the restrictions $\Psi_{\alpha,x} : F \xrightarrow{\cong} F_x$ of Ψ_α satisfy $\Psi^{\#}_{\alpha,x} = \Psi^{\#}_{\beta,x}$, $x \in U_\alpha \cap U_\beta$.

Thus in particular we may identify $H(F)$ with $H(F_x)$.

Under these hypotheses there is a unique isomorphism

$$\mu : A(B) \otimes_{\mathbf{R}} H(F) \xrightarrow{\cong} H(A(B) \otimes A_V(E)), \omega_B \otimes \delta_V)$$

such that $\mu(\Phi \otimes \alpha)$ is represented by $\Phi \otimes \Psi$, where $\delta_V \Psi = 0$ and for each $x \in B$, $j_x^{*} \Psi$ represents α.

Thus we have an explicit isomorphism

$$A(B) \otimes_{\mathbf{R}} H(F) \xrightarrow{\cong} E_1.$$

Under this isomorphism $\delta_B \otimes i$ corresponds to d_1 and so an isomorphism

$$H(B) \otimes_{\mathbf{R}} H(F) \xrightarrow{\cong} E_2$$

is induced.

9. Fibre integral and spectral sequences. The development of the preceding section applies equally well to $A_F(E)$, except that $H(F)$ must be replaced by $H_C(F)$. Thus we have canonical isomorphisms

$$E_0 \cong A(B) \otimes_{S(B)} A_{V,F}(E)$$
$$E_1 \cong A(B) \otimes_{\mathbf{R}} H_C(F)$$
$$E_2 \cong H(B) \otimes_{\mathbf{R}} H_C(F).$$

Now assume that ξ is oriented, so that the operator

$$\int_F : A_F(E) \longrightarrow A(B)$$

is defined. Filter $A(B)$ by setting $\mathcal{F}^q(A(B)) = \sum_{j \geq q} A^j(B)$. The spectral sequence for this filtration is given by

$$E_0 = E_1 = A(B)$$
$$E_i = H(B),\ i \geq 2.$$

Since

$$\mathcal{F}^q(A_F(E)) = \pi^* F^q(A(B)) \wedge A_F(E)$$

it follows from property ii) of sec. 3 that $\oint_F$ preserves the filtrations. Thus it induces a homomorphism of spectral sequences.

With the identifications above the maps between the E_0, E_1 and E_2 -terms induced by $\oint_F$ are given by

$$\iota \otimes \oint_F : A(B) \otimes_{S(B)} A_{V,F}(E) \longrightarrow A(B)$$

$$\iota \otimes \int_F^{\#} : A(B) \otimes_{\mathbf{R}} H_C(F) \longrightarrow A(B)$$

and

$$\iota \otimes \int_P^{\#} : H(B) \otimes_{\mathbf{R}} H_C(F) \longrightarrow H(B)$$

The last equation shows that $\oint_F^{\#}$ coincides with the homological fibre integral of Chern and Spanier.

University of Toronto

REFERENCES

AUER, J. W., *A spectral sequence for smooth fibre bundles and fibre integration*, Ph. D. thesis, University of Toronto, 1970.

BOLTYANSKII, V. G., *Second obstructions for cross-sections*, Amer. Mat. Soc. Transl. (2), 21 (1962), 51-66.

BOREL, A. AND HIRZEBRUCH, F., *Characteristic Classes and Homogeneous Spaces I*, Amer. Journ. Math. 80 (1958), 458-538.

CHERN, S. S., *Integral formulas for characteristic classes of sphere bundles*, Proc. Nat. Acad. Sci. U.S.A. (1944), 269-273.

— *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. 45 (1944), 747-752.

— *On the curvature integral of a Riemannian manifold*, Ann. of Math. 46 (1945), 674-684.

— *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. Journ. Math. 75, (1953), 565-597.

COCKCROFT, W. H., *On the Thom isomorphism theorem*, Proc. Cambridge Philos. Soc. 58 (1962), 206-208.

EST, W. T. VAN, *A generalization of the Cartan-Leray spectral sequence*, Neder. Akad. Wetensch. Proc. Ser. A 61 (1958), 399-413.

GREUB, W. H., *Zur Theorie der linearen Übertragungen*, Ann. Acad. Sci. Fenn, Ser. A 346 (1964), 22 pp.

GREUB, W. H., HALPERIN, S. AND VANSTONE, J. R., *Connections, curvature and cohomology*, Academic Press, 1972.

GYSIN, W., *Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten*, Comment. Math. Helv. 14 (1942), 61-122.

HATTORI, A., *Spectral sequences in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo, Sec. I, 8 (1960).

HOPF, H., *Vectorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. 96 (1927).

LERAY, J., *L'homologie d'un espace fibré dont la fibre est connexe*, Jour. de Math. 29 (1950), 169-213.

SPANIER, E. H., *Homology theory of fibre bundles*, Proc. of the International Congress of Mathematicians, vol. II (1950), 390-396.