A NOTE ON SPACES OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

by

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In this note I wish to show that the space of entire functions represented by Dirichlet series with given exponents is complete with respect to the topology of compact convergence, and thus is a Montel space. *)

Let (λ_n) be a sequence of real numbers with

$$1 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \to \infty \tag{1}$$

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty \tag{2}$$

$$\liminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) = h > 0 \tag{3}$$

and X the space of all entire functions f represented by an everywhere absolutely convergent Dirichlet series

$$f(s) = \Sigma_{n\geq 1}\, a_n\, e^{\lambda_n s} \qquad (s \in \mathbf{C}). \tag{4}$$

We note that (4) defines a function $f \in X$ if and only if

$$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty. \tag{5}$$

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*) This result was earlier stated by T. HUSAIN and P. K. KAMTHAN in «Spaces of Entire Functions Represented by Dirichlet Series», Collect. Mathem. 19 (1968), p. 203, but the proof given there is incorrect.

As (a_n) is uniquely determined by f, we will write $f \sim (a_n)$ if (4) holds. If $f \sim (a_n)$, $g \sim (b_n)$, let

$$p(f, g) = \sup_{n \geq 1} |a_n - b_n|^{\frac{1}{\lambda_n}}.$$

Then X is a metric vector space (with metric p).

PROPOSITION 1: X is complete.

PROOF: If (f_m) is a Cauchy-sequence and $f_m \sim (a_n^{(m)})$, we have for $i, j \geq M_0(\varepsilon)$ and every $n \in \mathbf{N}$ (if $\varepsilon \leq 1$)

$$|a_n^{(i)} - a_n^{(j)}| < \varepsilon^{\lambda_n} \leq \varepsilon. \tag{6}$$

Thus, each $(a_n^{(m)})$ $a_m \in \mathbf{N}$ is a Cauchy-sequence and converges to

$$a_n = \lim_{m \to \infty} a_n^{(m)}.$$

We have for $m \geq M_0(\varepsilon)$ independent of n

$$|a_n^{(m)} - a_n|^{\frac{1}{\lambda_n}} \leq \varepsilon \qquad (n \in \mathbf{N}). \tag{7}$$

If $\varepsilon > 0$ is arbitrary, we choose $m \geq M_0\left(\frac{\varepsilon}{2}\right)$. Because of (5), there is $N_0 = N_0(\varepsilon)$ with

$$|a_n^{(m)}|^{\frac{1}{\lambda_n}} < \frac{\varepsilon}{2} \qquad (n \geq N_0).$$

Then we have for $n \geq N_0$:

$$|a_n|^{\frac{1}{\lambda_n}} \leq |a_n - a_n^{(m)} \leq |a_n - a_n^{(m)}|^{\frac{1}{\lambda_n}} + |a_n^{(m)}|^{\frac{1}{\lambda_n}} < \varepsilon.$$

Thus (5) holds, $f \sim (a_n)$ is well defined and $f \in X$, and, because of (7), is the limit of (f_m).

PROPOSITION 2: The metric p induces on X the topology of compact convergence.

PROOF: Let for $r > 0$

$$K_r = \{s \in \mathbf{C} \mid \text{Rea } s \leq r;\ \text{Ima } s \leq r\}.$$

a) If $p(f, 0) < \delta$ with $f \sim (a_n)$, we have for $s \in K_r$

$$|f(s)| \leq \Sigma_{n \geq 1} |a_n| \exp(\lambda_n \cdot \text{Rea } s) \leq \Sigma_{n \geq 1} \delta^{\lambda_n} \exp(\lambda_n r) = \\ = \Sigma_{n \geq 1} \exp(\lambda_n (r + \log \delta)) < \varepsilon,$$

if δ is small enough.

b) Mandelbrojt ([1], p. 199, 201) has proved that there are constants d, c_1, c_2, depending only on (λ_n), with

$$|a_n| < c_2 \cdot M(s_1) \cdot \exp(\lambda_n (c_1 - \text{Rea } s_1)) \qquad (n \in \mathbf{N}),$$

if $f \in X$ and $s_1 \in \mathbf{C}$ are arbitrary, $f \sim (a_n)$ and

$$M(s_1) = \sup_{|s-s_1| \leq d} |f(s)|$$

If $\varepsilon > 0$ is given, choose σ_1 big enough to get

$$c_2 \cdot \exp(\lambda_n (c_1 - \sigma_1)) < \varepsilon^{\lambda_n} \qquad (n \in \mathbf{N}).$$

Define $r = |\sigma_1| + d$. Then, if

$$1 > \sup_{s \in K_r} |f(s)| \geq M(s_1).$$

we have $p(f, 0) \leq \varepsilon$.

An inmediate consequence of these results is

PROPOSITION 3: X is a Montel space.

PROOF: For the definition of a Montel space, see [2], p. 74. X is a complete (and thus closed) subspace of the space of entire functions with the topology of compact convergence, which is a Montel space.

REFERENCES

1 S. Mandelbrojt, *Dirichlet series*. The Rice Institute Pamphlet, vol. XXXI, 1944.

2 A. P. Robertson, *Topological Vector Spaces*. Cambridge University Press 1964.

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