CHARACTERIZATION OF ALL ORDER RELATIONS IN DIRECT SUMS OF ORDERED GROUPS

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ABSTRACT

If one forms the direct sum of ordered groups then there exist only a few well-known methods to make this direct sum into an ordered group again such that the original orders in the components are preserved. In this paper a method is given to construct all order relations of this kind. This is done for partial orders as well as for full orders. From these results one can easily deduct several corollaries concerning the «location» of these orders.

1. INTRODUCTION

It is well known (see e.g. [1]) that one can make the direct sum G of the fully ordered groups G_i $(i = 1, 2, ..., n)$ into a partially or fully ordered group by defining various order relations. Of particular interest are naturally those orders which induce the original order in each G_i, the most natural among them being the *pointwise* order $((g_1, ..., g_n) \geq^* 0$ if $g_i \geq 0$ for all $i = 1, ..., n$ — a partial order) and the two full orders: $(g_1, ..., g_n) > 0$ if the first (last) non-zero component is > 0, often called the *lexicographic (antilexicographic)* order.

As far as the author knows there exists no characterization of all partial and full orders in direct sums of ordered groups inducing the original order in each summand. This will be done in this paper by constructing all orders of this kind. It turns out that in general there exist further partial orders, which are necessarily extensions of the pointwise order, and there exist further full orders «between» the «extremal» cases of the lexicographic and the antilexicographic order.

In [2] *H. H. Teh* developed a method to construct all full orders in abelian torsion-free groups. By means of [2] and the present paper one can therefore construct all full order relations in direct sums of torsion-free abelian groups, built up from the orders of the summands.

Since $G_1 \oplus \ldots \oplus G_n = (G_n \oplus \ldots \oplus G_{n-1}) \oplus G_n$ we can restrict our considerations to the case $n = 2$.

2. THE MAIN THEOREM

It is a fundamental fact for the theory of ordered groups that each order relation can be described by a normal subsemigroup of the considered group which contains no element other than the identity together with its inverse. Therefore it is sufficient to construct these subsemigroups (the *positive cones)* in order to get all (partial) orders. For $i = 1, 2$ P_i denotes the given positive cone of G_i consisting of all nonnegative elements in G_i, P_i^-. denotes the complement of P_i in G_i. For our purposes it is more natural to write the groups additively; this does not imply commutativity.

THEOREM 1. *Let G_1 and G_2 be ordered groups and G their direct sum. Then one gets all partial order relations in G extending P_1 and P_2 by the following method: choose any sets $X_1 \subseteq P_1^-$, $X_2 \subseteq P_2 - \{0\}$, $A_1 \subseteq P_1 - \{0\}$, $A_2 \subseteq P_2^-$ and form the normal subsemigroups $\overline{X}_1$, $\overline{A}_1$ generated by X_1 and A_1 in G_1 and $\overline{X}_2$, $\overline{A}_2$ generated by X_2, A_2 in G_2, respectively. Take certain subdirect sums $\overline{X}$ of $\overline{X}_1$ and $\overline{X}_2$ and $\overline{A}$ of $\overline{A}_1$ and $\overline{A}_2$ (on which we shall impose the relation (*)). Let $X(\bar{x})$ be the «final» set related to $\bar{x} \in \overline{X}$, consisting of all $g \in P_1^- \oplus P_2$, $g \geq^* \bar{x}$, where $\geq^*$ again denotes the pointwise order in G, and let $A(\bar{a})$ be defined in an analogue manner. Define X and A by $X := \bigcup X(\bar{x})$ and $A := \bigcup A(\bar{a})$, where the unions are taken for all $\bar{x} \in \overline{X}$ and all $\bar{a} \in \overline{A}$.*

Then one gets all partial orders by the positive cones $P := P_1 \oplus \oplus P_2 \cup X \cup A$ if and only if $\overline{X}$ and $\overline{A}$ are in a way chosen such that the following condition holds:

(*) *if $x = (x_1, x_2) \in X$, $a = (a_1, a_2) \in A$, $-x_1 > a_1$, then $a + x \in X$*

and $x + a \varepsilon X$ *and if* $-x_1 \leq a_1$ *then* $x + a$ *and* $a + x$ *are contained in* $P_1 \oplus P_2 \cap (A - \{0\})$.

PROOF. First we prove that each partial order extending P_1 and P_2 is of the kind $P = P_1 \oplus P_2 \cup X \cup A$. From the equation

$$g = (g_1, g_2) = (g_1, 0) + (0, g_2)$$

one can derive: if both g_1 and g_2 (or, equivalently, $(g_1, 0)$ and $(0, g_2)$) are ≥ 0 in the original order in G_1 and G_2, respectively, then g is $\geq (0, 0)$ in each order in G; if both g_1 and g_2 are strictly negative, then g is also strictly negative. Therefore $P_1 \oplus P_2 \subseteq P$ and $P \cap (P_1^- \oplus P_2^-) = \phi$. This shows that $P = P_1 \oplus P_2 \cup X \cup A$, where $X \subseteq P_1^- \oplus (P_2 - \{0\})$ and $A \subseteq (P_1 - \{0\}) \oplus P_2^-$.

Take $X_1 \oplus X_2 := P \cap ((P_1^- \oplus (P_2 - \{0\})) \cup ((P_1 - \{0\}) + P_2^-))$.

Since P is a normal subsemigroup of G, one verifies easily that X_1 and X_2 are also normal subsemigroups of G_1 and G_2, respectively. Therefore $X_1 = \overline{X}_1, X_2 = \overline{X}_2$ and $\overline{X}$ is aubdirect sum of X_1, X_2. If $\bar{x} \varepsilon \overline{X}$, $g \varepsilon P_1 \oplus P_2$, $g \geq^* x$, then $g \varepsilon P$ and also $g \varepsilon (P_1^- \oplus (P_2 - - \{0\})) \cup ((P_1 - \{0\}) \oplus P_2^-)$, which can be checked immediately. Therefore $X(\bar{x}) = \overline{X}$ for all $\bar{x} \varepsilon \overline{X}$ yielding $X = \overline{X}$. According to the construction of X_1 and X_2 each element εP with strictly negative first and strictly positive second component is contained in X, which gives the first part of (*). An analogous argument for the dually defined A gives then

$$P = P_1 \oplus P_2 \cup X \cup A$$

and the condition (*).

Conversely, we prove that each $P := P_1 \oplus P_2 \cup X \cup A$ as described in the theorem is a partial order relation in G extending the original order relations in G_1 and G_2.

In order to prove that P is a subsemigroup of G we first observe that by definition of X and A and by the properties of P_1 and P_2 the sets X, A, $P_1 \oplus P_2$ are subsemigroups of G. If $g \varepsilon P_1 \oplus P_2$, $x \varepsilon X$, $a \varepsilon A$, then $x \varepsilon X(\bar{x})$, $a \varepsilon A(\bar{a})$ for some $\bar{x} \varepsilon \overline{X}$, $\bar{a} \varepsilon \overline{A}$, hence $\bar{x} \leq x \leq x + g$, $\bar{x} \leq x \leq g + x$, $\bar{a} \leq a \leq a + g$ and $\bar{a} \leq a \leq g + a$. Therefore $x + g$ and $g + x$ are elements of $P_1 \oplus P_2 \cup X$, $a + g$ and $g + a$ are contained in $P_1 \oplus P_2 \cup A$. Let $x = (x_1, x_2)$ be εX and $a = (a_1, a_2) \varepsilon A$. If $-x_1 > a_1$, then by (*) $x + a$ and $a + x$ are εX, if $-x_1 \leq a_1$, then, again by (*), $x + a$ and $a + x$ are contained in $P_1 \oplus P_2 \cup A$; therefore $X + A \subseteq P$ and $A + X \subseteq P$, which proves that P is a subsemigroup of G.

The normality of $\overline{X}_1, \overline{X}_2, \overline{A}_1, \overline{A}_2$ implies the normality of $\overline{X}$ and $\overline{A}$ hence of $X(\bar{x})$ and $A(\bar{a})$ and therefore X and A are normal, too. It is easy to see that this implies the normality of P.

In order to calculate $P \cap -P$ we remark that $P_1 \oplus P_2 \cap - (P_1 \oplus P_2) = \{(0, 0)\}$, $X \cap -X = A \cap -A = -(P_1 \oplus P_2) \cap X = P_1 \oplus P_2 \cap -X = -(P_1 \oplus P_2) \cap A = P_1 \oplus P_2 \cap -A = \phi$. Let g be $\varepsilon X \cap -A$. Then $g = (x_1, x_2) =: x = (-a_1, -a_2) =: -a$ with $x \varepsilon X$ and $a \varepsilon A$. Hence $x + a = (0, 0)$. But, since $-x_1 = a_1$, it follows from (*) that $x + a \varepsilon (P_1 \oplus P_2 \cap A - \{0\})$, a contradiction, therefore $X \cap -A$ is empty. If $g \varepsilon -X \cap A$, then $-g \varepsilon X \cap - A = \phi$, again a contradiction. This shows that $P \cap -P = \{(0, 0)\}$.

It remains to show that this P extends the original order in the summands G_1 and G_2. But from $0 \leq g_1 \varepsilon G_1$ it follows that $(g_1, 0) \varepsilon P_1 \oplus P_2 \subseteq P$; the same argument holds for the second summand. This proves the theorem.

3. CORROLARIES DERIVED FROM THE MAIN THEOREM

A partial order P in a group G is a full order if and only if $P \cup -P = G$. With the help of this remark one can prove the corresponding theorem for full orders.

THEOREM 2. *The order relation* $P = P_1 \oplus P_2 \cup X \cup A$ *is a full order for* G *if and only if*

$$(**) \qquad X \cup -A = \{(g_1, g_2) \varepsilon G : g_1 < 0, g_2 > 0\} \text{ holds.}$$

PROOF. Since all $g \varepsilon G$ with negative first and positive second component are εX (if g is > 0) or εA (if g is < 0), $X \cup -A$ must be the set on the right side in (**), proving the necessity of this relation. Conversely, let (**) be valid. We have to show that $P \cup -P = G$.

$P \cup -P = (P_1 \oplus P_2 \cup X \cup A) \cup (-P_1 \oplus -P_2) \cup -X \cap -A = (P_1 \oplus P_2 \cup -P_1 \oplus -P_2) \cup (X \cup -A) \cup (-X \cup A)$. Let $g = (g_1, g_2)$ be an arbitrary element of G. If both g_1 and g_2 are positive or negative, then g belongs to $(P_1 \oplus P_2) \cup (-P_1 \oplus -P_2)$ and therefore to $P \cup -P$. If $g_1 < 0$, $g_2 > 0$ holds then $g \varepsilon X \cup -A$

by (**). If $g_1 > 0$ and $g_2 < 0$ then $-g \varepsilon X \cup -A$ by the preceding result, therefore $g \varepsilon -X \cup A$ which proves that $P \cup -P = G$.

From this one can deduct easily

COROLLARY 1. *The lexicographic order is characterized by* $X = \phi$ *while the antilexicographic order is given by* $A = \phi$.

Thus, in general, there exist other full order relations, different from the lexicographic and the antilexicographic order. For example, let G_1 be an arbitrary and G_2 be a non-archimedian ordered group. The order in G_2 induces an order in the set of the archimedian classes of G_2 in the well-known way shown in [1]. One can get full orders in G inducing the original orders in each G_i, different from the lexicographic and the antilexicographic orders in the following way: choose a fixed archimedian class K_2 in G_2 being not the last archimedian class in G_2 and denote the set of all elements of G_2 belonging to an archimedian class less or equal to K_2 by C_2, and the complement of C_2 in G_2 by C_2'. Then $P = P_1 \oplus P_2 \cup X \cup A$ with $X := P_k^- \oplus (C_2' \cap P_2)$ and $A := (P_1 - \{0\}) \oplus (C_2 \cap P_2^-)$ yields a full order of the desired kind in G (the verification is straightforward).

COROLLARY 2. *For some (archimedian) ordered groups* G_1 *and* G_2 *there exists an archimedian order in* $G = G_1 \oplus G_2$.

PROOF. If G is archimedian ordered, then G_1 and G_2 must also be archimedian ordered. Since therefore G_1 and G_2 are in essential subgroups of the additive group of real numbers, G is isomorphic to the additive group of all linear polynomials in x (say) under the map $g = (g_1, g_2) \to g_1 x + g_2 =: g(x)$. If there exists a real number ω with $g(\omega) = 0$ if and only if $g = 0$ then $g > 0$ if and only if $g(\omega) > 0$ yields an archimedian full order in G (cf. [2]).

COROLLARY 3. *All order relations inducing the original orders in* G_1 *and* G_2 *are extensions of the pointwise order* $P = P_1 \oplus P_2$.

PROOF: by theorem 1.

COROLLARY 4. *The only order relation yielding the original orders under the projection maps* π_1 *and* π_2 *onto* G_1 *and* G_2*, respectively, is the pointwise order.*

PROOF. By theorem 1 each order relation is of the form $P = P_1 \oplus P_2 \cup X \cup A$; $X \neq \phi$ implies $\pi_1(X) \subseteq P_1^-$, $A \neq \phi$ implies $\pi_2(A) \subseteq P_2^-$, both being contradictions, while the pointwise order fulfills the requirements stated in the corollary.

This corollary implies immediately

COROLLARY 5. *There exist no full order relations in $G = G_1 \oplus G_2$ yielding the original order relations in G_1 and G_2 under the projection maps, unless $G_1 = \{0\}$ or $G_2 = \{0\}$.*

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