BASES IN THE SPACE OF ANALYTIC DIRICHLET TRANSFORMATIONS

By

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1. Introduction and Terminology: Let **C** denote the field of complex numbers equipped with the usual topology. Denote by X the family of all transformations f: $\mathbf{C} \to \mathbf{C}$, such that

$$(1.1) \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \ s = \sigma + it \ \varepsilon \ \mathbf{C},$$

where $0 < \lambda_1 < \dots < \lambda_n \to \infty$ with n, and further

$$(1.2) \qquad \limsup_{n\to\infty} \frac{\log |a_n|}{n} = -\infty ;$$

$$(1.3) \qquad \limsup_{n\to\infty} \frac{\log \lambda_n}{n} = D < \infty$$

Condition (1.2) implies that (1.1) is absolutely convergent and conditions (1.2) and (1.3) imply that (1.1) represents a holomorphic function analytic throughout the complex plane (see [4], p. 29-33). We endow X with two topologies T_1 and T_2 so as to make it a locally convex topological vector space (l. c. TVS). Indeed, let for each $f \varepsilon X$, define

$$M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|$$

Then $\{M(\sigma, \dots) : \sigma \text{ is real}\}$ defines a family of semi-norms (in fact norms) on X, satisfying the axiom of separation, and suppose T_1 is the locally convex Hausdorff topology generated by this family on X. Next, let for each $f \varepsilon X$, define

$$p(\sigma, f) = \sum_{n=1}^{\infty} |a_n| \, e^{\lambda_n}$$

Then $\{p(\sigma, ...): \sigma \text{ is real}\}$ again defines a family of seminorms (indeed, norms) on X, such that this family generates a locally convex Hausdorff topology, which we henceforth denote by T_2, on X. One of us (PKK) has proved earlier (see [3]) that X equipped with T_2 is a Fréchet space and hence a barrelled space (see also [2]). Invoking the Cauchy-Ritt inequality for $f(s)$ given as in (1.1), namely $|a_n) \leq \leq M(\sigma, f) \exp(-\sigma \lambda_n)$, for all real σ, one can easily show that for any $k > 0$

$$(1.4) \qquad M(\sigma, f) \leq p(\sigma, f) \leq C(k) M(\sigma + k, f)$$

Inequalities (1.4) yield that T_1 is equivalent to T_2, i. e., $T_1 \simeq T_2$. Therefore we may interchange the role of T_1 and T_2 according to the convenience in our analysis; in particular, we find from what has preceded above that X equipped with T_1 is a Fréchet space and so it is a barrelled space. When we wish to emphasize X equipped with a particular topology T_1 or T_2 we will hereafter write X as (X, T_1) or (X, T_2) respectively, otherwise merely X will denote from now and onwards a l. c. TVS equipped with T_1.

In this paper we shall be concerned with the characterisations of certain types of bases in X. A sequence $\{f_n\} \subset X$ is said to be a *base* in X if each $f \varepsilon X$ can be uniquely expressed as (with respect to scalars in $\mathbf{C}$)

$$(1.5) \qquad f = \sum_{n=1}^{\infty} c_n f_n,$$

where $c_n's$ are the uniquely determined coefficients in $\mathbf{C}$ and the convergence of the infinite series being with respect to the topology on X.

If $e_n \varepsilon X$, $e_n(s) = e^{s\lambda_n}$, $n \geq 1$, then each $f \varepsilon X$ can be expressed by (1.1) with coefficients c_n satisfying (1.2) i. e.

$$(1.6) \qquad \limsup_{n \to \infty} \frac{\log |c_n|}{\lambda_n} = -\infty$$

Therefore $\{e_n\}$ is a base in X. Now

$$e^{e^s} = \sum_{n=0}^{\infty} \frac{e^{ns}}{n!}$$

The sequence $\{f_n\}$, where $f_n(s) = e^{ns}/n!$ is in X and forms a base for X, but here

$$\lim_{n \to \infty} \frac{\log |c_n|}{\lambda_n} = 0 \tag{1.7}$$

Consequently there are at the outset two types of bases in X, for which (1.6) is true or not true. A base $\{f_n\}$ in X will be called a *genuine base* if the corresponding coefficients satisfy (1.6). A sequence $\{f_n\}$ in X is called an *absolute base* if it is a base in X and the infinite series corresponding to each $f \varepsilon X$ is absolutely convergent with respect to T_1 (or equivalently with respect to T_2). A sequence $\{f_n\}$ in X is called a *proper base* for X if it is a genuine as well as an absolute base for X.

In this paper we shall give results pertaining to the characterisation of proper base in X and they are given in the following section.

2. In the discussion that follows, we will be frequently making use of

LEMMA 2.1: Let $\{\phi_n\} \subset X$ and suppose that $\sum_{n=1}^{\infty} \phi_n$ converges absolutely with respect to the topology on X, i. e. $\sum_{n-1}^{\infty} M(\sigma, \phi_n)$ converges for every real σ. Then given $\mu > 0$ and a real σ, there corresponds an integer p, such that for all $n \geq p$, we have

$$\log M(\sigma, \phi) < \mu \lambda_n$$

PROOF: The proof is straight forward. Indeed, let the conclusion of the lemma be false. Then there exists an increasing sequence $\{n_k\}$ of $\{n\}$, $n_k \to \infty$ as $k \to \infty$, corresponding to a given $\mu < 0$ and a real σ, such that

$$\log M(\sigma, \phi_{n_k}) \geq \mu \lambda_{n_k}$$

Therefore

$$\sum_{n=1}^{\infty} M(\sigma, \phi_n) \geq \sum_{k=1}^{\infty} e^{\mu \lambda_{n_k}},$$

and this contradicts the hypothesis of the lemma.

The main result about the characterisation of proper bases will require two intermediatry results and in this direction we first of all have

Theorem 2.1 : Let $\{\alpha_n\} \subset X$. Suppose $\{c_n\}$ be an arbitrary sequence contained in $\mathbf{C}$, such that

$$(2.1) \quad \begin{cases} \limsup_{n\to\infty} \dfrac{\log |c_n|}{\lambda_n} = -\infty \Rightarrow \text{the convergence of} \\ \sum_{n=1}^{\infty} M(\sigma, c_n \alpha_n), \text{ for each real } \sigma \end{cases}$$

Then (2.1) is equivalent to

$$(2.2) \quad \limsup_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} < +\infty, \text{ for each real } \sigma$$

PROOF : (Necessity). Let (2.1) hold good. Suppose (2.2) is not true. Hence for some σ, there corresponds a sequence $\{n_k\} \subset \{n\}$, such that

$$(2.3) \quad \log M(\sigma, \alpha_{n_k}) > k\lambda_{n_k},\ k = 1, 2, 3, \ldots$$

Define $\{c_n\}$ by

$$\log |c_n| = \begin{cases} \lambda_{n_k} - \log M(\sigma, \alpha_{n_k}),\ k = 1, 2, \ldots \\ -\infty\ n \neq n_k \end{cases}$$

Then from (2.3)

$$\limsup_{n\to\infty} \frac{\log |c_n|}{\lambda_n} = -\infty$$

and so (2.1) hold and in particular the series in (2.1) converges. Now for every σ

$$\begin{aligned} M(\sigma, c_{n_k} \alpha_{n_k}) &= c_{n_k} M(\sigma, \alpha_{n_k}) \\ &= e^{\lambda_{n_k}}, \\ M(\sigma, c_n \alpha_n) &= 0,\ n \neq n_k \end{aligned}$$

Thus

$$\frac{\log M(\sigma, c_n \alpha_n)}{\lambda_n} = \begin{cases} 1,\ n = n_k\ (k = 1, 2, \ldots) \\ -\infty,\ n \neq n_k \end{cases}$$

and this contradicts the conclusion of the lemma (indeed, take $\phi_n = c_n \alpha_n$ and $\mu = 1$)

(Sufficiency). Let σ be given. Then there exists a constant $C = C(\sigma)$ depending on a pre-assigned real value σ, such that (from (2.2))

$$\log M(\sigma, \alpha_n) \leq e^{C\lambda_n}, \text{ for } n \geq n_0 = n_0(C) \tag{2.4}$$

Let $C_1 > C$. Then there exists $n_1 = n_1(C_1)$, such that

$$|c_n| \leq e^{-C_1\lambda_n}, \text{ for } n \geq n_1. \tag{2.5}$$

Now for real σ, we get from (2.4) and (2.5)

$$\begin{aligned} M(\sigma, c_n\alpha_n) &= |c_n| M(\sigma, \alpha_n) \\ &\leq e^{(C-C_1)\lambda_n}, \; n \geq N = \max(n_0, n_1) \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} M(\sigma, c_n\alpha_n)$$

converges for every σ. The proof of the result is now complete.

Next, we have

THEOREM 2.2: Let $\{\alpha_n\} \subset X$ and let $\{c_n\} \subset \mathbf{C}$, then

$$\left\{ \begin{aligned} &\text{Convergence of } \sum_{n=1}^{\infty} M(\sigma, c_n\alpha_n), \text{ for each real } \sigma \\ &\Rightarrow \limsup_{n\to\infty} \frac{\log|c_n|}{\lambda_n} = -\infty, \end{aligned} \right. \tag{2.6}$$

implies and implied by

$$\lim_{\sigma\to\infty} \left\{ \liminf_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} \right\} = +\infty \tag{2.7}$$

PROOF: (Necessity). Let (2.6) be true and suppose that (2.7) is false. Then for some $k > 0$

$$\lim_{\sigma\to\infty} \left\{ \liminf_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} \right\} < k < +\infty$$

But from the convexity of $\log M(\sigma, f)$ with respect to σ (see [1], p. 138) we infer that if $\sigma_1 \leq \sigma_2$, then $M(\sigma_1) \leq M(\sigma_2)$.

Therefore for each real σ,

$$\liminf_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} < k$$

Hence there exists an increasing sequence $\{n_j\}$, such that

$$\log M(\sigma, \alpha_{n_j}) < k\,\lambda_{n_j}$$

Define $\{c_n\} \subset \mathbf{C}$ as follows:

$$\log |c_n| = \begin{cases} -2k\,\lambda_{n_j}, & \text{if } n = n_j \\ -\infty \quad, & \text{if } n \neq n_j \end{cases}$$

Then for a given σ

$$\sum_{n=1}^{\infty} |c_n|\, M(\sigma, \alpha_n) \leq \sum_{j=1}^{\infty} \exp\,(-k\,\lambda_{n_j}) < +\infty$$

and so $\Sigma\, M(\sigma, c_n\alpha_n)$ converges for each real. Consequently from (2.6), $\limsup_{n\to\infty} \log |c_n|/\lambda_n = -\infty$. But it is not true, sice by construction

$$\limsup_{n\to\infty} \frac{\log |c_n|}{\lambda_n} = -2k$$

This contradiction proves the first part of the theorem. (Sufficiency). Let (2.7) hold good. Assume that (2.6) is not true. Thus $\Sigma\, M(\sigma, c_n\alpha_n)$ converges for each σ, but

$$\limsup_{n\to\infty} \frac{\log |c_n|}{\lambda_n} \neq -\infty \text{ (i. e.} > -\infty)$$

There exists a sequence $\{n_k\}$, such that

$$\log |c_{n_k}| > M\,\lambda_{n_k},\ M > -\infty$$

By (2.7), one may find a real σ, such that

$$\liminf_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} > 2 - M$$

Therefore

$$\frac{\log M(\sigma,\ c_{n_k}\alpha_{n_k})}{\lambda_{n_k}} > M + 2 - M = 2,\ k \geq 1$$

but then this contradicts lemma 2.1 (take $\phi_n = c_n \alpha_n,\ \mu = 2$). The proof of the result is now complete.

Combining Theorem 2.1 and 2.2, we have now our main result of this paper, namely.

THEOREM 2.3: Suppose $\{\alpha_n\}$ is an absolute base in X. Then $\{\alpha_n\}$ is genuine if and only if (2.2) and (2.7) hold good.

REMARK: In view of (1.4), it is clear that the above theorem remains still valid, if the function $M(\sigma,\ c_n \alpha_n)$ in (2.2) and (2.7) is replaced by $p(\sigma,\ c_n \alpha_n)$.

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