ON THE SUBSPACES OF FRÉCHET MONTEL SPACES AND OF THE STRONG DUALS OF FRÉCHET MONTEL SPACES

by

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We abbreviate Frechét Montel space by (FM)-space and say, for short, «Montel (DF)-space» instead of «strong dual of an (FM)-space» which amounts to the same thing.

Let X be a separated real or complex locally convex space. J. Dieudonné [1] proved:

(I) If X is barrelled, and if there exists a fundamental sequence of compact convex sets of X, then X is a Montel (DF)-space.
(II) If X is barrelled or bornological, and if there exists a fundamental sequence of compact sets of X, then X is a dense subspace of a Montel (DF)-space.

The question, whether in (II) «barrelled or bornological» may be replaced by «quasi-barrelled», was answered in the affirmative by M. Mahowald and G. Gould [6]. D. J. H. Garling [2] strengthened this by

(III) If X is quasi-barrelled, and if there exists a fundamental sequence of compact sets of X, then X is a Montel (DF)-space.

The first point of this note is to show that replacing in (III) the fundamental sequence of compact sets by a fundamental sequence of precompact sets leads to a characterization of the (dense) (DF)-subspaces of a Montel (DF)-space, namely

(IV) The following three statements are equivalent.
1. *X is quasi-barrelled and there exists a fundamental sequence of precompact sets.*
2. *X is a dense (DF)-subspace of a Montel (DF)-space.*
3. *X is a (DF)-subspace of a Montel (DF)-space.*

The equivalence of 1. and 2. is similar to, but not contained in, the following recent result by H. Hogbe-Nlend ([4], Cor. 2 of Prop. 1). (V) X is a Montel (DF)-space if and only if it is the completion of a quasi-barrelled space which has a fundamental sequence of precompact sets.

In the second part of this note we shall show the following result which bears some resemblance to (IV).

(VI) The following three statements are equivalent.
1. *X is metrizable and its bounded sets are precompact.*
2. *X is a dense linear subspace of an (FM)-space.*
3. *X is a linear subspace of an (FM)-space.*

PROOF OF (IV). The fact that 1. implies 2. is an easy consequence of (V). To show that 2. implies 1., let X be a dense subspace of a Montel (DF)-space Y. Since Y is barrelled, X is quasi-barrelled (Grothendieck [3], p. 78, cor 3). Y has a fundamental sequence (A_n) of bounded sets where the sets A_n are compact. Therefore $(A_n \cap X)$ is a fundamental sequence of precompact sets of X.

The equivalence of 2. and 3. follows immediately from the fact that the completion of a (DF)-space is a (DF)-space and from part 2. of the following lemma which has some interest in itself.

LEMMA: 1. *Every (DF)-subspace of a Montel (DF)-space is quasi-barrelled.*
2. *Every closed (DF)-subspace of a Montel (DF)-space is a Montel (DF)-space.*

PROOF. 1. Every subset of the dual of a separable metrizable locally convex space is weakly sequentially separable (cf. Köthe [5], § 21, 3. (5)). If X is a (DF)-subspace of a Montel (DF)-space Y, it follows from this remark that X is a separable subset of Y. The (DF)-space X is therefore quasi-barrelled (Grothendieck [3], p. 71, Cor. 2).
2. If X is a closed (DF)-subspace of a Montel (DF)-space, then X is complete and, by 1., quasi-barrelled. Furthermore X has a fundamental sequence of compact sets. The contention now follows from (II) or (III).

PROOF OF (VI). 2. implies 1., obviously.
1. implies 2.: It follows from 1., that X is separable, as the proof of the analogous statement for (FM)-spaces shows (see Köthe [5],

§ 27, 2. (5)). Consequently every bounded set of the completion $\tilde{X}$ of X is contained in the completion of a bounded set of X (Köthe [5], § 29, 6. (1)) and is therefore compact. Hence $\tilde{X}$ is an (FM)-space.

The equivalence of 2. and 3. follows from the fact that a closed subspace of an (FM)-space is an (FM)-space.

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