THE DERIVATIVES OF A MEROMORPHIC FUNCTION AND NEVANLINNA'S DEFICIENT VALUES

by

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1. Throughout this paper I consider $f(z)$ to be a transcedental mermorphic function. Let l be a finite positive integer ($l \geq 1$). Let $f^{(l)}(z)$ stand for the l —th derivative of $f(z)$. It is assumed that the reader is familiar with the symbols $m(r,f)$, $N(r,f)$, $\bar{N}(r,a)$, $S(r)$ and $T(r,f)$ frequently occuring in the Nevanlinna theory of meromorphic functions and so $m(r,f^{(l)})$, $N(r,f^{(l)})$, $\bar{N}(r,f^{(l)})$ and $T(r,f^{(l)})$ stand as above with $f^{(l)}(z)$ instead of $f(z)$. Let

$$\delta(\alpha) = \delta(\alpha, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,\alpha)}{T(r,f)}; \; \Theta(\alpha) = \Theta(\alpha, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r,\alpha)}{T(r,f)},$$

and

$$\theta(\alpha) = \theta(\alpha, f) = \overline{\lim_{r\to\infty}} \frac{N(r,\alpha) - \bar{N}(r,\alpha)}{T(r,f)},$$

when one has $\theta(\alpha) + \delta(\alpha) \leq \Theta(\alpha)$.

I wish to prove a number of results on the various relations amongst $T(r,f)$, $T(r,f^{(l)})$, $N(r,f)$, $N(r,f^{(l)})$, $m(r,f)$ and $m(r,f^{(l)})$, and also amongst the deficient values defined above. A major role is played by lemma 1 of this paper and also its intermediate steps. The second inequality in (i) of lemma 1 is roughly contained in lemma 4 of Hayman [2] but I include its proof for the sake of completeness.

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2. LEMMA 1: Let $f(z)$ be a meromorphic function in the plane and let

$$F(z) = \sum_{\mu=1}^{q} \left(\frac{1}{f(z) - a_\mu} \right).$$

Then for all $r \to \infty$

$$\sum_{\mu=1}^{q} m(r, a_\mu) + N(r, 1/f^{(l)}) + O(\log r) \leq T(r, f^{(l)})$$

(i) $$\leq O(\log r) + m(r, f) + (l+1) N(r, f),$$

provided $f(z)$ is of finite order ϱ; for $r \to \infty$, excluding a set of finite length, we also have (for every order)

$$\sum_{\mu=1}^{q} m(r. a_\mu) + N(r, 1/f^{(l)}) + O(\log \{r\, T(r, f)\}) \leq T(r, f^{(l)})$$

(ii) $$\leq O((\log \{r\, T(r, f)\}) + m(r, f) + (l+1) N(r, f).$$

This lemma includes the result of Ullrich ([9], p. 207).

PROOF: We have:

$$\sum_{\mu=1}^{q} m(r, a_\mu) = \sum_{\mu=1}^{q} m\left(r, \frac{1}{f(z) - a_\mu}\right)$$

$$\leq m(r, F) + O(1)$$

$$\leq m(r, F f^{(l)}) + m(r, 1/f^{(l)}) + O(1).$$

But

$$m(r, F f^{(l)}) \leq m(r, F f^{(v-1)}) + m(r, f^{(l)}/f^{(l-1)})$$

$$\leq m(r, F f^{(l-2)}) + m(r, f^{(l-1)}/f^{(l-2)}) + m(r, f^{(l)}/f^{(l-1)})$$

$$\leq m(r, F f^{(1)}) + \sum_{i=2}^{l-1} m(r, f^{(i)}/f^{(i-1)})$$

$$\leq m(r, F f^{(1)}) + (l-1)\, S(r),$$

by lemma 4 of Hayman ([2], p. 13). If $f(z)$ is of finite order then $S(r) = O(\log r)$, therefore (see p. 32 [3])

$$m(r, F f^{(1)}) = -m(r, f^{(1)}/f) + O(\log r)$$
$$= O(\log r).$$

Hence

$$\sum_{\mu=1}^{q} m(r, a_\mu) \leq m(r, 1/f^{(l)}) + O(\log r).$$

But

$$m(r, 1/f^{(l)}) = -N(r, 1/f^{(l)}) + T(r, 1/f^{(l)})$$
$$= -N(r, 1/f^{(l)}) + T(r, f^{(l)}) + \log |f^{(l)}(O)|$$

and we may let $f(O) \neq O$. Therefore

$$\sum_{\mu=1}^{q} m(r, a_\mu) \leq O(\log r) - N(r, 1/f^{(l)}) + T(r, f^{(l)}) + O(1),$$

and so first inequality in (i) follows.

Now

(iii) $$T(r, f^{(l)}) = N(r, f^{(l)}) + m(r, f^{(l)}).$$

But

$$m(r, f^{(l)}) \leq m(r, f^{(l)}/f) + m(r, f)$$

(iv) $$\leq S(r) + m(r, f),$$

by lemma 4, cited above. Also at a pole of $f(z)$ of order k, $f^{(l)}(z)$ has a pole of order $k + l$. Thus

(v) $$N(r, f^{(l)}) \leq (l+1) N(r, f).$$

Therefore from (iii)-(v), we find that

$$T(r, f^{(l)}) \leq S(r) + m(r, f) + (l+1) N(r, f),$$

and if $f(z)$ is of finite order, $S(r) = O(\log r)$, one finds that the second inequality in (i) is done.

If $f(z)$ is of infinite order, then

$$S(r) = O(\log \{r\, T(r, f)\}),$$

for all r but in a set E of finite linear measure (see [3]), and so

$$\begin{aligned} m(r, F f^{(l)}) &= m(r, f^{(l)}/f) + S(r) \\ &= O(\log \{r\, T(r, f)\}), \end{aligned}$$

and the previous considerations lead to (ii).

Using lemma 1, we also have:

LEMMA 2: Let $f(z)$ be a meromorphic function of finite order. If

$$\sum_{\alpha \neq \infty} \delta(\alpha, f) \geq 1 - \gamma;\ \delta(\infty, f) \geq 1 - \gamma,\ (O < \gamma < 1)$$

then

$$\text{(vi)} \qquad 1 - \gamma l \leq \varliminf_{r \to \infty} \frac{T(r, f^{(l)})}{T(r, f)} \leq \varlimsup_{r \to \infty} \frac{T(r, f^{(l)})}{T(r, f)} \leq 1 + \gamma l .$$

PROOF: From the second inequality in (i), we have

$$\begin{aligned} T(r, f^{(l)}) &\leq O(\log r) + m(r, f) + N(r, f) + l\, N(r, f),\ r \geq r_0 \\ &\leq O(\log r) + T(r, f) + l\,(1 - \delta(\infty, f) + \varepsilon)\, T(r, f), r \geq r^0 \\ &\leq O(\log r) + T(r, f) + l\,(r + \varepsilon)\, T(r, f),\ r \geq r_0. \end{aligned}$$

Therefore

$$\varlimsup_{r \to \infty} \frac{T(r, f^{(l)})}{T(r, f)} \leq 1 + \gamma l.$$

Now from the first of the inequality in (i), one has

$$\begin{aligned} \varliminf_{r \to \infty} \frac{T(r, f^{(l)})}{T(r, f)} &\geq \sum_{\mu=1}^{q} \varliminf_{r \to \infty} \frac{m(r, a_\mu)}{T(r, f)} = \sum_{\mu=1}^{q} \delta(\mu, f) \\ &\geq 1 - \gamma \geq 1 - \gamma l,\ (1 \geq 1). \end{aligned}$$

Therefore (vi) follows.

LEMMA 3: We also have for $f(z)$ to be of finite order, the following

(vii) $$\varlimsup_{r\to\infty} \frac{N(r, f^{(l)})}{T(r, f^{(l)})} \leq \frac{(l+1)\gamma}{1-\gamma l};$$

(viii) $$\varlimsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \leq \frac{2\gamma l}{1+\gamma l},$$

and therefore

(ix) $$k(f^{(l)}) \leq \frac{(3l+1)\gamma - (l^2 - l)\gamma^2}{1 + l^2\gamma^2}.$$

PROOF: We have

$$\varlimsup_{r\to\infty} \frac{N(r, f^{(l)})}{T(r, f^{(l)})} \leq (l+1) \varlimsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} \frac{T(\gamma, f)}{T(r, f^{(l)})}$$

$$\leq \frac{l+1}{1-\gamma l} \varlimsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} \text{ (from } (vi))$$

$$\leq \frac{(l+1)\gamma}{1-\gamma l};$$

also from (i) (first inequality) for $r \geq r_0$,

$$\frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \leq 1 - \sum_{\mu=1}^{q} \frac{m(r, a_\mu)}{T(r, f^{(l)})} = 1 - \frac{T(r,f)}{T(r, f^{(l)})} \sum_{\mu=1}^{q} \frac{m(r, a_\mu)}{T(r, f)}$$

$$\leq 1 - \frac{\sum_{\mu=1}^{q} \delta(a_\mu, f)}{1 + \gamma l + \varepsilon}$$

$$\leq 1 - \frac{1-\gamma l}{1+\gamma l+\varepsilon} \quad (r \geq r_0).$$

Hence

$$\varlimsup_{r\to\infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \leq \frac{2\gamma l}{1+\gamma l},$$

and so (vii) and (viii) are proved. (ix) follows from (vii) and (viii) by adding them.

Corollary: Let $\gamma = O$ (so that $\delta(\infty, f) = 1, \sum_{\alpha \neq \infty} \delta(\alpha, f) = 1$), then

$$\text{(x)} \qquad \overline{\lim_{r \to \infty}} \frac{N(r, f^{(l)}) + N(r, 1/f^{(l)})}{T(r, f^{(l)})} = O.$$

REMARK: Let $f(z)$ be a meromorphic function of non-integral order ϱ, then ([4], [3])

$$\text{(xi)} \qquad \overline{\lim_{r \to \infty}} \frac{N(r, f^{(l)}) + N(r, 1/f^{(l)})}{T(r, f^{(l)})} - O,$$

since the order of $f^{(l)}(z)$ is not changed ($f^{(l)}(z)$ is also of order ϱ). Hence if $\delta(\infty, f) = 1, \sum_{\alpha \neq \infty} \delta(\alpha, f) = 1$, and $f(z)$ is a meromorphic function then the order of $f(z)$ is an integer, for otherwise (xi) will hold good and we shall arrive at a contradiction on account of (x). This result was proved by Shah and Singh {8].

REMARK: Lemma 2 includes Theorem 2 of Shah and Singh [7], since taking $\gamma = O$, xe get

$$T(r, f^{(l)}) \sim T(r, f),$$

and if $l = 1$, this result is of Shah and Singh. Lemmas 2 and 3 also include lemma 1 of Edrei and Fnchs [1] (for $l = 1$).

LEMMA 4: Let $f(z)$ be meromorphic of finite order and let $\alpha_1 \neq \alpha_2$; $|\alpha_1| < \infty$, $|\alpha_2| < \infty$. Let

$$\overline{\lim_{r \to \infty}} \frac{N(r, \alpha_i)}{T(r, f)} = O; \ (i = 1, 2).$$

Then

$$T(r, f^{(l)}) \sim (l + 1) \ T(r, f).$$

REMARK: This result also includes Theorem 3 of Shah and Sing [7].
PROOF OF LEMMA 4: We have

$$\begin{aligned} T(r, f^{(l)}) &\geq N(r, f^{(l)}) \\ &= N(r, f^{(l-1)}) + \overline{N}(r, f^{(l-1)}) \\ &= N(r, f^{(1)}) + \overline{N}(r, f^{(1)}) + \overline{N}(r, f^{(2)}) + \ldots + \overline{N}(r, f^{(l-1)}) \\ &= N(r, f) + \overline{N}(r, f) + \overline{N}(r, f^{(1)}) + \ldots + \overline{N}(r, f^{(l-1)}) \\ &\geq (l+1)\ \overline{N}(r, f). \end{aligned}$$

Now from (2.9) in [3] (put $q = 2$, $a_1 = \alpha_1$; $a_2 = \alpha_2$)

$$\begin{aligned} (1 + o(1))\ T(r, f) &\leq \overline{N}\left(r, \frac{1}{f-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{f-\alpha_2}\right) + \overline{N}(r, f) \\ &= \overline{N}(r, \alpha_1)) + \overline{N}(r, \alpha_2) + \overline{N}(r, f) \\ &\leq o(T(r, f)) + \overline{N}(r, f),\ (r \geq r_0). \end{aligned}$$

Therefore, as $T(r, f) \geq \overline{N}(r, f)$.

$$\overline{N}(r, f) \sim T(r, f).$$

Hence

(xii) $$\lim_{r \to \infty} \frac{T(r, f^{(l)})}{T(r, f)} \geq l + 1.$$

From (i) (second inequality) now

$$\begin{aligned} T(r, f^{(l)}) &\leq O(\log r) + \{m(r, f) + N(r, f)\} + l\,T(r, f) \\ &\leq O(\log r) + (l+1)\,T(r, f). \end{aligned}$$

Therefore

(xiii) $$\overline{\lim_{r \to \infty}} \frac{T(r, f^{(l)})}{T(r, f)} \leq l + 1\,.$$

From (xii) and (xiii) the result is proved.

3. Let

$$\lambda(O, f^{(l)}) = 1 - \lim_{r \to \infty} \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})}\,.$$

Then we have the following

THEOREM 1: Let $| a_i | < \infty$ $(i = 1, 2 \ldots)$, then if $f(z)$ is a meromorphic function of finite order, one has

$$(1-\delta(O, f^{(l)})+\lambda(O, f^{(l)}))\sum_{\mu=1}^{q}\delta(a_\mu)\leq(l+1-\delta(\infty, f)-\theta(\infty, f))\,\lambda(O, f^{(l)}).$$

PROOF: First af all we have (using (iv))

$$T(r, f^{(l)}) = N(r, f^{(l)}) + m(r, f^{(l)})$$

$$\leq l\,N(r, f) + N(r, f) + m(r, f) + O(\log r)$$

or, $T(r, f^{(l)})/T(r, f) \leq 1 + l(1 - \Theta(\infty, f) + \varepsilon$, all $r \geq \gamma_0$.

Therefore

$$\text{(xiv)} \qquad \overline{\lim_{r\to\infty}} \frac{T(r, f^{(l)})}{T(r, f)} \leq l + 1 - (\delta(\infty, f) + \theta(\infty, f))\,l.$$

For the sake of simplicity, let

$$A = \overline{\lim_{r\to\infty}} \frac{T(r, f^{(l)})}{T(r, f)}\,;\; B = \underline{\lim_{r\to\infty}} \frac{T(r, f^{(l)})}{T(r, f)}\,.$$

From (i) we have

$$\text{(xv)} \quad q\,T(r, f) - \sum_{\mu=1}^{q} N(r, a_\mu) + N(r, 1/f^{(l)}) + O(\log r) \leq T(r, f^{(l)}).$$

For $r = r_n$.

$$q + \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})}\frac{T(r, f^{(l)})}{T(r, f)} < (B + \varepsilon) + \sum_{\mu=1}^{q}(1 - \delta(a_\mu) + \varepsilon).$$

Hence

$$q + (1 - \lambda(O, f^{(l)}) - \varepsilon)(B - \varepsilon) < B + \varepsilon + \sum_{\mu=1}^{q}(1 - \delta(a_\mu) + \varepsilon)$$

or, $$\sum_{\mu=1}^{q}\delta(a_\mu) < B\,\lambda(O, f^{(l)}) + O(\varepsilon).$$

Making $\varepsilon \to O$, we have

(xvi) $$\sum_{\mu=1}^{q} \delta(a_\mu) \leq B \lambda(o, f^{(l)}).$$

Again from (xv) we have for all $r \geq r_0$,

$$q + \frac{N(r, 1/f^{(l)})}{T(r, f^{(l)})} \frac{T(r, f^{(l)})}{T(r, f)} < A + \varepsilon + \sum_{\mu=1}^{q} (1 - \delta(a_\mu) + \varepsilon),$$

and so

$$q + (1 - \delta(O, f^{(l)}) - \varepsilon)(B - \varepsilon) \leq A + \varepsilon + \sum_{\mu=1}^{q} (1 - \delta(a_\mu) + \varepsilon),$$

and therefore

$$(1 - \delta(O, f^{(l)}))\, B \leq A - \sum_{\mu=1}^{q} \delta(a_\mu),$$

and as $1 - \delta(O, f^{(l)}) \geq O$, we find from (vxi)

$$(1 - \delta(O, f^{(l)})) \sum_{\mu=1}^{q} \delta(a_\mu) \geq (A - \sum_{\mu=1}^{q} \delta(a_\mu))\, \lambda(O, f^{(l)}).$$

This inequality leads to Theorem 1.

3. Lut S be a family of all increasing functions $\varphi(x)$, such that $\log x = o(\varphi(x))$, and that $x^\alpha/\varphi(x)$ $(\alpha - O)$ is non-decreasing $\to \infty$ with x. Let

(xvii) $$\frac{\lim}{r \to \infty} \frac{T(r, f)}{n(r, \alpha)\, \varphi(r)} > O, \ O \leq |\alpha| < \infty,$$

where $\varphi \in S$ and call α as e.v.S. (eceptional value S). Since $n(r, \alpha) \geq N(r, \alpha)/\log r$ and so if α is an e.v.S., then

$$T(r, f) < A\, n(r, \alpha)\, \varphi(r), \ (\text{all } r \geq r_0)$$

and therefore it follows that

(xviii) $$\delta(\alpha, f) = 1.$$

If $\delta(\infty, f) = \delta(\alpha, f) = 1$ $(\alpha \neq \infty)$, it follows from lemmas 2 and 3 that $\delta(\infty, f^{(l)}) = \delta(\alpha, f^{(l)}) = 1$, and also note that (xvii) $\Rightarrow$ (xviii) and we have the fact that there cannot be more than two e.v.S. for $f(z)$ since $\Sigma\, \delta(\alpha, f) \leq 2$. Let now α_1 and α_2 $(\alpha_1 \neq \alpha_2)$ be two e.v.S. for $f(z)$, does it imply that α_1 and α_2 are also e.v.S. for $f^{(l)}(z)$? The answer is contained in the following

THEOREM 2: Let $f(z)$ be a meromorphic function of finite order o, having α $(|\alpha| < \infty)$ and ∞ as e.v.S., then $f^{(l)}(z)$ has O abd ∞ as e.v.S.

PROOF: Let $\alpha = O$, so that $\delta(O, f) = 1$. Also $\delta(\infty, f) = 1$. Therefore (see Shah and Sing [8]) $T(r, f) \sim A\, r^{\varrho}$ where ϱ is an integer. Also from lemma 2.

$$T(r, f^{(l)}) \sim T(r, f).$$

Further

$$n(r, f^{(l)}) \leq (l + 1)\; n(r, f).$$

Hence

$$\underset{r \to \infty}{\underline{\lim}}\; \frac{T(r, f^{(l)})}{n(r, f^{(l)})\, \varphi(r)} \geq \frac{1}{l+1}\; \underset{r \to \infty}{\underline{\lim}}\; \frac{T(r, f)}{n(r, f)\, \varphi(r)} > O,$$

since ∞ is an e.v.S. of $f(z)$ and this implies that ∞ is an e.v.S. of $f^{(l)}(z)$, where φ is some member of S.

Now following Nevanlinna ([5], p. 105)

$$\begin{aligned} N(r, 1/f^{(l)}) &\leq N(r, f/f^{(l)}) + N(r, 1/f) \\ &< N(r, 1/f) + N(r, f^{(l)}/f) + m(r, f^{(l)}/f) + O(1) \\ &< N(r, 1/f) + N(r, f^{(l)}/f) + O(\log r), \end{aligned}$$

from lemma 4 [2]. But

$$\begin{aligned} N(r, f^{(l)}/f) &\leq N(r, f^{(l)}) + N(r, 1/f) \\ &\leq (l + 1)\, N(r, f) + N(r, 1/f). \end{aligned}$$

Therefore

$$N(r, 1/f^{(l)}) < (l + 1)\, N(r, 1/f) + N(r, f)) + O(\log r).$$

Now using $T(r, f) \sim Ar^{\varrho}$ and denoting by A_i's various constants, we have

$$n(r, 1/f) < \frac{A_1 r^{\varrho}}{\varphi_1(r)};$$

$$n(r, f) < \frac{A_2 r^{\varrho}}{\varphi_2(r)},$$

where $\varphi_1, \varphi_2 \in S$. Let $\varphi = \min(\varphi_1, \varphi_2)$. Also $\varphi(x)/x^{\alpha}$ ($\alpha > O$) is steadily decreasing. Hence

$$N(r, f) + N(r, 1/f) < A_3 \int_{r_0}^{r} \frac{x^{\varrho-1}}{\varphi(x)} dx < A_4 \frac{r^{\varrho}}{\varphi(r)}.$$

Therefore

$$N(r, 1/f^{(l)}) < \frac{A_5 r^{\varrho}}{\varphi(r)} + O(\log r)$$

$$< \frac{A_6 r^{\varrho}}{\varphi(r)},$$

and this shows rhat

$$n(r, 1/f^{(l)}) < \frac{A_7 r^{\varrho}}{\varphi(r)}.$$

Hence

$$\overline{\lim_{r \to \infty}} \frac{T(r, f^{(l)})}{N(r, 1/f^{(l)}) \varphi(r)} < O,$$

since $T(r, f^{(l)}) \sim A r^{\varrho}$. Hence O is an e.v.S. of $f^{(l)}(z)$.

Finally, I wish to add a result which says something more, and still with less restrictive hypothesis, than what Shah has proved in a paper in 1952, see for instance his theorem 5 {6].

THEOREM 3: Let $f(z)$ be a meromorphic function with $\delta(\alpha, f) = \delta(\infty, f) = 1$, $(\alpha \neq \infty)$. Then

$$\bar{n}\left(r, \frac{1}{f^{(l)} - x}\right) \sim A \varrho r^{\varrho}, \ (r \to \infty)$$

for all x, excepting when $x = \alpha, = \infty$, where A is some positive constant.

Proof: Let $\alpha = O$. Then $T(r, f) \sim T(r, f^{(l)}) \infty A r^{\varrho}$. Since

$$N(r, f^{(l)}) \leq k\, N(r, f)$$

and so $\delta(\infty, f^{(l)}) = 1$. Also from the discussion of the preceding theorem it also follows that $\delta(O, f^{(l)}) = 1$.

Applying Nevanlinna(s second fundamental theorem to $f^{(l)}(z)$ and taking $q = 3$, $a_1 = O$, $a_2 = \infty$, $a_3 = x$ (see [5], p. 69) ans solving a little, one finds that

$$T(r, f^{(l)}) \leq N\left(r, \frac{1}{f^{(l)} - x}\right) + N(r, 1/f^{(l)}) - N(r, 1/f^{(l+1)}) +$$

$$+ N(r, f^{(l)}) + S(r, f^{(l)})$$

$$\leq \overline{N}\left(r, \frac{1}{f^{(l)} - x}\right) + \overline{N}(r, f^{(l)}) + N(r, f^{(l)}) + N\left(r, \frac{1}{f^{(l)}}\right) + S(r, f^{(l)})$$

where

$$S(r, f^{(l)}) = o(T(r, f^{(l)}))\,;\ N(r, f^{(l+1)}) = \overline{N}(r, f^{(l)}) + N(r, f^{(l)})$$

Therefore

$$(1 + o(1))\, T(r, f^{(l)}) \leq \overline{N}\left(r, \frac{1}{f^{(l)} - x}\right) + \overline{N}(r, f^{(l)}) + N(r, 1/f^{(l)}).$$

Hence

$$\varliminf_{r \to \infty} \frac{\overline{N}(r, 1/(f^{(l)} - x))}{T(r, f^{(l)})} \geq 1.$$

Consequently

$$\overline{N}\left(r, \frac{1}{f^{(l)} - x}\right) \sim T(r, f^{(l)}) \sim A r^{\varrho}, \ (r \to \infty).$$

and this implies that

$$\bar{n}\left(r, \frac{1}{f^{(l)} - x}\right) \sim A\, \varrho\, r^{\varrho}, \ (r \to \infty).$$

The result is, therefore, proved.

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