ON THE SYSTEM OF SCALE CURVES OF A CARTOGRAM T

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1. INTRODUCTION. Let T represent a cartogram between two Riemannian spaces V_n and $\bar{V}_n$ each of dimension $n \geq 2$. That is, T represents a one to one point correspondence between V_n and $\bar{V}_n$ such that in any admissible coordinate system (x) corresponding points are given by the *same* set of curvilinear coordinates $(x^i) = (x^1, x^2, \dots, x^n)$. It is assumed that the quadratic differential forms $ds^2 = g_{ij}\, dx^i\, dx^j$ and $d\bar{s}^2 = \bar{g}_{ij}\, dx^i\, dx^j$ of V_n and $\bar{V}_n$ are both positive definite.

The *scale* $\varrho = e^{\omega} = d\bar{s}/ds > 0$, of such a cartogram T is given by the equation

$$\varrho^2 = e^{2\omega} = \left(\frac{d\bar{s}}{ds}\right)^2 = \bar{g}_{ij}\, \frac{dx^i}{ds}\, \frac{dx^j}{ds} > 0. \tag{1.1}$$

Relative to the geometrical operations of V_n, the variation $d\varrho/ds$ of the scale ϱ with respect to the arc length s of a curve C of V_n is given by the equation

$$\varrho\, \frac{d\varrho}{ds} = e^{2\omega}\, \frac{d\omega}{ds} = \frac{d\bar{s}}{ds}\, \frac{d^2\bar{s}}{ds^2} = \bar{g}_{ij}\, \frac{dx^i}{ds}\, \varkappa^j + \frac{1}{2}\, \bar{g}_{ij,k}\, \frac{dx^i}{ds}\, \frac{dx^j}{ds}\, \frac{dx^k}{ds}\,, \tag{1.2}$$

where K^i is the contravariant form of the geodesic curvature of C.

A *scale curve* C(1) of this cartogram T is a locus of a point P on the Riemannian space V_n along which the scale does not vary.

2. SOME GEOMETRICAL THEOREMS CONCERNING SCALE CURVES. From equation (1.2) it is readily seen that the system of scale curves

C of a cartogram T between two Riemannian spaces V_n and $\bar{V}_n$, for $n \geq 2$, obeys the single ordinary differential equation

$$\bar{g}_{ij}\,\varkappa^i \frac{dx^j}{ds} + \frac{1}{2}\,\bar{g}_{ij,k} \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{2.1}$$

Therefore a curve C in V_n is a scale curve C if and only if its contravariant tangent vector is orthogonal to the covariant vector

$$U_i = \bar{g}_{ij}\,\varkappa^j + \frac{1}{2}\,\bar{g}_{ij,k} \frac{dx^j}{ds}\frac{dx^k}{ds}\,. \tag{2.2}$$

The following result is deduced from the preceding discussion.

THEOREM 2.1 *If two Riemannian spaces V_n and $\bar{V}_n$, for $n \geq 2$, correspond by a cartogram T for which the scale is $\varrho = e^{\omega} = d\bar{s}/ds > 0$, and if at a point x of V_n the scalar geodesic curvature $\varkappa(2)$ of a scale curve C is $K = | K^i | > 0$, and if Θ with $0 \leq \Theta \leq \pi$ is the angle between the covariant vector $p_i = \bar{g}_{ij}\, dx^j/ds$ for which $p = | p_i | > 0$ and K^i then*

$$p\,\varkappa \cos \Theta + \frac{1}{2}\,\bar{g}_{ij,k} \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{2.3}$$

At any point x of V_n construct the tangent Euclidean space π_n generated by the set of contravariant vectors with initial points at x. In π_n the scale ellipsoid Σ_{n-1} of the cartogram T is defined by the equation

$$\bar{g}_{ij}\,(X^i - x^i)\,(X^j - x^j) = 1. \tag{2.4}$$

A cartogram T is *locally conformal* at a point x if and only if the scale ellipsoid Σ_{n-1} constructed in the tangent Euclidean space π_n at x is a sphere Σ_{n-1}. It is known that a necessary and sufficient condition for local conformality is that the characteristic equation of T, namely

$$| \bar{g}_{ij} - u\, g_{ij} | = 0, \tag{2.5}$$

possesses a root of multiplicity n at the point x.

THEOREM 2.2. *If a cartogram T between two Riemannian spaces V_n and $\bar{V}_n$, for $n \geq 2$, is not locally conformal at a point x then, at x,* *there is at least one unit contravariant tangent vector dx^i/ds such that there exists a curve C_0 passing through x in the direction of dx^i/ds, for which the contravariant vector geodesic curvature vector $\overset{i}{\varkappa_0}$ at x, obeys the differential condition* (2.1) *and the set of n relations $p^i = r dx^i/ds$ is satisfied for some scalar $r > 0$. The scalar geodesic curvature of C_0 satisfies the equation*

$$p \varkappa_0 + \frac{1}{2} \bar{g}_{ij,k} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \tag{2.6}$$

For, if dx^i/ds is a characteristic vector corresponding to a characteristic root u_0 of (2.5) then $p_i = \bar{g}_{ij}\, dx^j/ds = u_0 g_{ij}\, dx^j/ds$. Thus $p^i = g^{ij} p_j = u_0 g^{ij} g_{ja}\, dx^a/ds = u_0\, dx^i/ds$. Hence $r = u_0$ and it is seen that p^i is parallel to dx^i/ds. Therefore, since dx^i/ds is orthogonal to $\varkappa^i$ it follows that the angle Θ of Theorem 2.1 must be zero yielding (2.6).

THEOREM 2.3. *(An analogue of Meusnier's Theorem)* (3) *If, under the conditions of Theorem* 2.1, *$K > 0$ is the scalar geodesic curvature of a scale curve C tangent to the curve C_0 at the point x and if Θ with $0 \leq \Theta \leq \pi$ is the angle between their two corresponding vector geodesic curvatures $\varkappa^i$ and $\overset{i}{\varkappa_0}$, then*

$$\varkappa \cos \Theta = \varkappa_0 \tag{2.7}$$

This follows immediately from a comparison of equations (2.3) and (2.6).

3. SOME CONFORMAL PROPERTIES OF A NON-CONFORMAL CARTOGRAM T. If T is a cartogram between two Riemannian spaces V_n and $\overline{V}_n$, for $n \geq 2$, then at a fixed point x the characteristic directions corresponding to the roots of the characteristic equation (2.5) decompose the tangent Euclidean space π_n into a union of disjoint mutually orthogonal Euclidean subspaces π_{p_k} with $1 \leq p_k \leq n$ and $p_1 + p_2 + \dots + p_n = n$. The dimension of each π_{p_k} is p_k and equals the multiplicity of the corresponding charateristic root of (2.5). These Euclidean subspaces, for $m \geq 2$, intersect only at the point x. The cartogram T is locally conformal at x if and only if $m = 1$ and $p_1 = n$.

THEOREM 3.1. *For a cartogram T between two Riemannian spaces V_n and $\overline{V}_n$, for $n \geq 2$, consider the set of all scale curves C of V_n such that each one passes through a fixed point of V_n and the scalar geodesic* *curvature of each one at this point is $\varkappa > 0$. A scale curve of this cartogram T obeys the differential condition*

$$g_{ij,k} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \tag{3.1}$$

if and only if its unit contravariant tangent vector dx^i/ds is in exactly one of the characteristic Euclidean subspaces of T at the point x.

The differential condition (3.1) is valid for *every* point x and every unit contravariant tangent vector dx^i/ds of V_n if and only if the map T is a conformal cartogram T between V_n and $\bar{V}_n$, for which the logarithmic scale is a point function $\omega = \omega(x)$.

For this situation the differential equation of the system of all scale curves C of a conformal cartogram T between V_n and $\bar{V}_n$ whose logarithmic scale is $\omega = \omega(x)$ is

$$\frac{\partial \omega}{\partial x^k} \frac{dx^k}{ds} = 0. \tag{3.2}$$

If the two Riemannian spaces correspond by a *non-homothetic* conformal cartogram T then $\omega = \omega(x)$ is non-constant point function. Under these circumstances the system of all scale curves C of V_n is composed of all curves on the simple family of ∞^1 subspaces Σ_{n-1} defined by the equation

$$\omega = \omega(x) = \omega(x^1, x^2, \ldots, x^n) = \text{constant}. \tag{3.3}$$

Finally, a map T between two Riemannian spaces V_n and $\bar{V}_n$, each of dimension $n \geq 2$, is a homothetic cartogram T if and only if *every* curve C of V_n is a scale curve C of the cartogram T. (4)

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REFERENCES

1. KASNER, E. and DECICCO, J., «*Scale curves in general cartography*». Proceedings of the National Academy of Sciences, Vol. 30, No. 8, pp. 211-215, 1944.

2. EISENHART, L. P., «*Riemannian Geometry*». Princeton University Press, 1949.

3. WEATHERBURN, C. E., «*Differential Geometry of Three Dimensions*». Vol. I, Cambridge University Press, 1961.

4. DECICCO, J. and ANDERSON, R. V., «*Some theorems on isothermal families in Riemannian space V_n*». Ricerche di Matematica, Vol. 16, pp. 192-201. Naples, Italy, 1967.