ON P-ANALYTIC TENSORS

by

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SUMMARY:

Covariant and contravariant p-analytic vectors in a Kähler space were studied by R. C. SRIVASTAVA in [1]*. The object of the present paper is to define and study covariant and contravariant p-analytic tensors in a Kähler space, which for $p = 1$ reduce to the covariant and contravariant analytic tensors studied by Yano and Bochner. Necessary and sufficient conditions for a tensor to be covariant (contravariant) p-analytic have been obtained and relationship between covariant p-analytic tensors and harmonic tensors of type p is investigated. It is also established that a contravariant (covariant) $(p + 1)$-analytic tensor is covariant (contravariant) p-analytic. Later a necessary and sufficient condition for a tensor to be both convariant (contravariant) p-and $(p + 1)$-analytic has been found.

(1) INTRODUCTION:

Let us consider a Kähler space M_{2n} with a positive definite metric g_{ji} and a mixed tensor $F_i^{\ h}$, which satisfy the following conditions

$$F_j^{\ i} F_i^{\ h} = - A_j^h, \tag{1.1}$$

$$F_j^{\ t} F_i^{\ s} g_{ts} = g_{ji} \tag{1.2}$$

and

$$F_i^{\ h}{}_{;j} = 0, \tag{1.3}$$

* Numbers in the square brackets refer to the references at the end of the paper.

Latin indices $i, j, k, \ldots$ take the values $1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}$ and Greek indices α, γ, μ, ν rum over the range $1, 2, \ldots, n$.

where semicolon (;) followed by an index denotes covariant differentiation with respect to Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ji}. If we put

$$F_{ji} = F_j{}^h g_{hi}, \tag{1.4}$$

then from (1.1) and (1.2), we can verify

$$F_{ji} = -F_{ij} \tag{1.5}$$

and from (1.3), we get

$$F_{ji;k} = 0. \tag{1.6}$$

For brevity we shall denote

$$T_{i_1 i_2 \ldots i_s}{}^{h_1 h_2 \ldots h_r} \text{ by } T^{h_1 h_2 \ldots h_r}_{i_1 i_2 \ldots i_s}; \qquad T^{i_1 i_2 \ldots i_s}{}_{h_1 h_2 \ldots h_r} \text{ by } T^{i_1 i_2 \ldots i_s}_{h_1 h_2 \ldots h_r} \tag{1.7}$$

and

$$T^{h_1 h_2 \ldots h_r}_{i_1 i_2 \ldots i_s;\, i_{s+1} i_{s+2} \ldots i_{s+p}} \text{ by } T^{h_1/h_r}_{i_1/i_s;\, i_{s+1}/i_{s+p}}. \tag{1.8}$$

A tensor $T^{h_1/h_r}_{i_1/i_s}$ is called pure in i_g and h_t if

$$*O^{c\,h_t}_{i_g\,b}\, T^{h_1 \ldots b \ldots h_r}_{i_1 \ldots c \ldots i_s} = 0, \tag{1.9}$$

where

$$*O^{cb}_{ji} = \frac{1}{2}\left(A^c_j A^b_i + F^c_j F^b_i\right) \tag{1.10}$$

and a tensor $T^{h_1/h_r}_{i_1/i_s}$ is said to be hybrid in i_g and h_t if

$$O^{c\,h_t}_{i_g\,b}\, T^{h_1 \ldots b \ldots h_r}_{i_1 \ldots c \ldots i_s} = 0, \tag{1.11}$$

where

$$O^{cb}_{ji} = \frac{1}{2}\left(A^c_j A^b_i - F^c_j F^b_i\right). \tag{1.12}$$

The operators O and $*O$ also satisfy te following relation

$$O + *O = A. \tag{1.13}$$

Also if,

$$H_{kj} = \frac{1}{2} K_{kjih} F^{ih}, \tag{1.14}$$

then we know that ([2], p. 72)

$$K_{ks} F_j^{\ s} = H_{kj} \ ; \qquad H_{ks} F_j^{\ s} = - K_{kj}. \tag{1.15}$$

(2) p-ANALYTIC TENSORS:

Let us consider a self conjugate tensor eld $T^{h_1/h_r}_{i_1/i_s}$ $(1 \leqslant r, s)$, which is pure in all its indices, that is, which has the components of the form

$$T^{h_1/h_r}_{i_1/i_s} = (T^{\alpha_1/\alpha_r}_{\lambda_1/\lambda_s}, \ 0, \ 0, \ldots, 0, \ T^{\bar{\alpha}_1/\bar{\alpha}_r}_{\bar{\lambda}_1/\bar{\lambda}_s}). \tag{2.1}$$

The fact that $T^{h_1/h_r}_{i_1/i_s}$ is pure in all its indices is expressed by

$$* O^{gh}_{it} T^{h_1 \ldots t \ldots h_r}_{i_1 \ldots g \ldots i_s} = 0 \tag{2.2}$$

or

$$F_j^{\ a} T^{h_1 \ldots h \ldots h_r}_{i_1 \ldots a \ldots i_s} - F_b^{\ h} T^{h_1 \ldots b \ldots h_r}_{i_1 \ldots j \ldots i_s} = 0, \tag{2.3}$$

the indices a, b taking all the positions.

We define a p-analytic tensor in a Kähler space to be a self conjugate pure tensor $T^{h_1/h_r}_{i_1/i_s}$ such that $T^{h_1/h_r}_{i_1/i_s; i_{s+2}/i_{s+p} j}$ is pure in i_1 and j, that is, if

$$* O^{ba}_{ji_1} T^{h_1/h_r}_{ai_2/i_s; i_{s+2}/i_{s+p} b} = 0 \tag{2.4}$$

or

$$T^{h_1/h_r}_{i_1/i_s; i_{s+2}/i_{s+p} a} + F_a^{\ b} F_{i_1}^{\ l} T^{h_1/h_r}_{li_2/i_s; i_{s+2}/i_{s+p} b} = 0. \tag{2.5}$$

Multiplying by $F_j^{\ a}$ and summing over a yields

$$F_j^{\ a} T^{h_1/h_r}_{i_1/i_s; i_{s+2}/i_{s+p} a} - F_{i_1}^{\ l} T^{h_1/h_r}_{li_2/i_s; i_{s+2}/i_{s+p} j} = 0. \tag{2.6}$$

Differentiating covariantly (2.6) with respect to t and multiplying by F^{jt}, we get

$$F^{jt} F_j^{\ a} T^{h_1/h_r}_{i_1/i_s; i_{s+2}/i_{s+p} at} - F^{jt} F_{i_1}^{\ a} T^{h_1/h_r}_{ai_2/i_s; i_{s+2}/i_{s+p} jt} = 0$$

or

$$F^{jt} F_j^{\ a} T^{h_1/h_r}_{i_1/i_s; i_{s+2}/i_{s+p} at} - \frac{1}{2} F^{jt} F_{i_1}^{\ a} \left(T^{h_1/h_r}_{ai_2/i_s; i_{s+2}/i_{s+p} jt} - T^{h_1/h_r}_{ai_2/i_s; i_{s+2}/i_{s+p} tj}\right) = 0$$

or

$$F^{jt} F_j^{\ a} T^{h_1/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}\, at} + \frac{1}{2} F^{bj} F_{i_1}^{\ a} \Big(\sum_{t=1}^{r} K_{bjl}^{\ \ h_t} T^{h_1/h_{t-1}\, l h_{t+1}/h_r}_{a i_2/i_s;\, i_{s+2}/i_{s+p}}$$

$$- \sum_{t=2}^{s} K_{bji_t}^{\ \ l} T^{h_1/h_r}_{a i_2/i_{t-1}\, l i_{t+1}/i_s;\, i_{s+2}/i_{s+p}} - K_{bja}^{\ \ l} T^{h_1/h_r}_{l i_2/i_s;\, i_{s+2}/i_{s+p}}$$

$$- \sum_{t=s+2}^{s+p} K_{bji_t}^{\ \ l} T^{h_1/h_r}_{a i_2/i_s;\, i_{s+2}/i_{t-1}\, l i_{t+1}/i_{s+p}} \Big) = 0$$

or

$$g^{ab} T^{h_1/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}\, ab} + \sum_{t=1}^{r} F_{i_1}^{\ a} H_l^{\ h_t} T^{h_1/h_{t-1}\, l h_{t+1}/h_r}_{a i_2/i_s;\, i_{s+2}/i_{s+p}}$$

$$- \sum_{t=2}^{s} F_{i_1}^{\ a} H_{i_t}^{\ l} T^{h_1/h_r}_{a i_2/i_{t-1}\, l i_{t+1}/i_s;\, i_{s+2}/i_{s+p}} - H_a^{\ l} F_{i_1}^{\ a} T^{h_1/h_r}_{l i_2/i_s;\, i_{s+2}/i_{s+p}}$$

$$+ \sum_{t=s+2}^{s+p} F^{bj} F_{i_1}^{\ a} K_{ji_t b}^{\ \ \ \ l} T^{h_1/h_r}_{a i_2/i_s;\, i_{s+2}/i_{t-1}\, l i_{t+1}/i_{s+p}} = 0. \tag{2.7}$$

By virtue of (2.3) and (1.15), above equation becomes

$$g^{ab} T^{h_1/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}\, ab} + \sum_{t=1}^{r} K_s^{\ h_t} T^{h_1/h_{t-1}\, a h_{t+1}/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}}$$

$$- \sum_{t=1}^{s} K_{i_t}^{\ a} T^{h_1/h_r}_{i_1/i_{t-1}\, a i_{t+1}/i_s;\, i_{s+2}/i_{s+p}}$$

$$+ \sum_{t=s+2}^{s+p} F^{bj} F_{i_1}^{\ a} K_{ji_t b}^{\ \ \ \ l} T^{h_1/h_r}_{a i_2/i_s;\, i_{s+2}/i_{t-1}\, l i_{t+1}/i_{s+a}} = 0, \tag{2.8}$$

which is necessary condition for $T^{h_1/h_r}_{i_1/i_s}$ to be p-analytic.
On the other hand, putting

$$S^{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}} = \left(F^{ja} T_a^{\ i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}} - F^{i_1 a} T_a^{\ i_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p} j} \right), \tag{2.9}$$

we have

$$\frac{1}{2} S^{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}} S_{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}$$

$$= \left(T^{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p} j} T_{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p} j} \right)$$

$$+ F^{ja} F^{li_1} \left(T_{ai_1}^{\ \ \ i_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p}} \right) \left(T_{li_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p} j} \right) \tag{2.10}$$

and

$$
\begin{aligned}
&[(T_{i_{\cdot}/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}{}^{j})\, (T^{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}) + \\
&\qquad\qquad + F^{ja} F^{il} (T_i{}^{i_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p}})\, (T_{li_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, a})]_{;j} \\
&= (g^{ab}\, T_{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, ab})\, (T^{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}) \\
&+ (T_{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, j})\, (T^{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, j}) \\
&+ F^{ja} F^{i_1 l} (T^{h_1/h_r}_{li_2/i_s;\, i_{s+2}/i_{s+p}\, aj})\, (T^{i_2/i_s;\, i_{s+2}/i_{s+p}}_{i_1 h_1/h_r}) \\
&+ F^{ja} F^{i_1 l} (T^{h_1/h_r}_{li_2/i_s;\, i_{s+2}/i_{s+p}\, a})\, (T^{i_2/i_s;\, i_{s+2}/i_{s+p}}_{i_1 h_1/h_r}{}_{j}) \\
&= (g^{ab}\, T^{h_1/h_r}_{ai_2/i_s;\, i_{s+2}/i_{s+p}\, ab})\, (T^{i_1/i_s;\, i_{s+2}/i_{s+p}}_{h_1/h_r}) \\
&- F^{jt} F_{i_1}{}^{a} (T^{h_1/h_r}_{ai_2/i_s;\, i_{s+2}/i_{s+p}\, jt})\, (T^{i_1/i_s;\, i_{s+2}/i_{s+p}}_{h_1/h_r}) \\
&+ (T^{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, j})\, (T_{i_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, j}) \\
&+ F^{ja} F^{li_1} (T_{i_1}{}^{i_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}{}_{a})\, (T_{li_2/i_s h_1/h_r;\, i_{s+2}/i_{s+p}\, j}) \\
&= [g^{ab}\, T^{h_1/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}\, ab} + \sum_{t=1}^{r} K_a{}^{h_t}\, T^{h_1/h_{t-1}\, a h_{t+1}/h_r}_{i_1/i_s;\, i_{s+2}/i_{s+p}} \\
&- \sum_{t=1}^{s} K_{i_t}{}^{a}\, T^{h_1/h_r}_{i_1/i_{t-1}\, a i_{t+1}/i_s;\, i_{s+2}/i_{s+p}} \\
&+ \sum_{t=s+2}^{s+p} F^{bj} F_{i_1}{}^{a} K_{ji_t b}{}^{l}\, T^{h_1/h_r}_{ai_2/i_s;\, i_{s+2}/i_{t-1}\, l i_{t+1}/i_{s+p}}]\, T^{i_1/i_s;\, i_{s+2}/i_{s+p}}_{h_1/h_r} \\
&+ \frac{1}{2} S^{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}\, S_{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}.
\end{aligned}
$$

Thus assuming the space to be compact, we apply Green's theorem and obtain

$$
\begin{aligned}
&\int_M [\{J.\ H.\ S.\ \text{of (2.8)}\}\, T^{i_1/i_s;\, i_{s+2}/i_{s+p}}_{h_1 h_r} \\
&\qquad\qquad + \frac{1}{2} S^{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}\, S_{ji_1/i_s h_1/h_r;\, i_{s+2}/i_{s+p}}]\, d\sigma = 0,
\end{aligned}
$$

which using (2.8), provides the proof of the following.

THEOREM 2.1

A necessary and sufficient condition for a pure tensor, say $T^{h_1/h_r}_{i_1/i_s}$ *in a compact Kähler space to be p-analytic is*

$$g^{ab}\,T^{h_1/h_r}_{i_1/i_s;\,i_{s+2}/i_{s+p}ab} + \sum_{t=1}^{r} K_a{}^{h_t}\,T^{h_1/h_{t-1}\,ah_{t+1}/h_r}_{i_1/i_s;\,i_{s+2}/i_{s+p}}$$

$$-\sum_{t=1}^{s} K_{i_t}{}^{a}\,T^{h_1/h_r}_{i_1/i_{t-1}\,ai_{t+1}/i_s;\,i_{s+2}/i_{s+p}} + \sum_{t=s+2}^{s+p} F^{bj}\,F_{i_1}{}^{a}\,K_{ji_tb}{}^{l}\,T^{h_1/h_r}_{ai_2/i_s;\,i_{s+2}/i_{t-1}\,li_{t+1}/i_{s+p}} = 0.$$

As corollarys to this theorem, we have following two theorems

THEOREM 2.2:

A necessary and sufficient condition for a pure covariant tensor T_{i_1/i_r} *in a compact Kähler space to be (covariant) p-analytic is that*

$$g^{st}\,T_{i_1/i_r;\,i_{r+2}/i_{r+p}\,st} - \sum_{s=1}^{r} K_{i_s}{}^{a}\,T_{i_1/i_{s-1}\,ai_{s-1}/i_r;\,i_{r+2}/i_{r+p}}$$

$$+ \sum_{s=r+2}^{r+p} F^{tj}\,F^{a}_{i_1}\,K_{ji_st}{}^{b}\,T_{ai_2/i_r;\,i_{r+2}/i_{s-1}\,bi_{s+1}/i_{r+p}} = 0. \qquad (2.12)$$

THEOREM 2.3:

A necessary and sufficient condition for a pure contravariant tensor T^{i_1/i_r} *in a compact Kähler space to be (contravariant) p-analytic is that*

$$g^{st}\,T^{i_1/i_r;\,i_{r+2}/i_{r+p}}{}_{st} + \sum_{s=1}^{r} K_a{}^{i_s}\,T^{i_1/i_{s-1}\,a\,i_{s+1}/i_r;\,i_{r+2}/i_{r+p}}$$

$$- \sum_{s=r+2}^{r+p} F^{tj}\,F_a{}^{i_1}\,K_{btj}{}^{i_s}\,T^{ai_2/i_r;i_{r+2}/i_{s-1}\,bi_{s+1}/i_{r+p}} = 0. \qquad (2.13)$$

In particular, if $p = 1$, (2.11) becomes

$$g^{ab}\,T^{h_1/h_r}_{i_1/i_s;\,ab} + K_a{}^{h_1}\,T^{ah_2/h_r}_{i_1/i_s} + \ldots + K_a{}^{h_r}\,T^{h_1/h_{r-1}\,a}_{i_1/i_s}$$

$$- K_{i_1}{}^{a}\,T^{h_1/h_r}_{ai_2/i_s} - \ldots - K_{i_s}{}^{a}\,T^{h_1/h_r}_{i_1/i_{s-1}\,a} = 0,$$

which is a necessary and sufficient condition for a pure tensor $T^{h_1/h_r}_{i_1/i_s}$ in a compact Kähler space to be analytic. (Yano, p. 101, theorem 9.1)

(3) Relation between p-analytic and p-harmonic tensors:

A necessary and sufficient condition for a skew symmetric tensor field T_{i_1/i_r} to be harmonic of type p is that [3]

$$S\{T_{i_1/i_r;\,i_{r+2}/i_{r+p}}\} = 0, \tag{3.1}$$

where

$$S\{T_{i_1/i_r;\,i_{r+2}/i_{r+p}}\} = g^{st}\,T_{i_1/i_r;\,i_{r+2}/i_{r+p}st}$$
$$-\sum_{s=1}^{r} K_{i_s}{}^{a}\,T_{i_1/i_{s-1}ai_{s+1}/i_r;\,i_{r+2}/i_{r+p}}$$
$$-\sum_{\substack{s,t=1\\ s<t}}^{r} K_{i_t i_s}{}^{ba}\,T_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_r;i_{r+2}/i_{r+p}}$$
$$+\sum_{s=1}^{r}\sum_{t=r+2}^{r+p} K^{b}{}_{i_t i_s}{}^{a}\,T_{i_1/i_{s-1}ai_{s+1}/i_r;\,i_{r+2}/i_{t-1}bi_{t+1}/i_{r+p}} = 0. \tag{3.2}$$

But, if T_{i_1/i_r} is pure in all its indices, then $K_{kj}{}^{ih}$ being hybrid in i and h, equation (3.2) reduces to

$$S\{T_{i_1/i_r;\,i_{r+2}/i_{r+p}}\} = g^{st}\,T_{i_1/i_r;\,i_{r+2}/i_{r+p}st}$$
$$-\sum_{s=1}^{r} K_{i_s}{}^{a}\,T_{i_1/i_{s-1}ai_{s+1}/i_r;\,i_{r+2}/i_{r+p}}$$
$$+\sum_{s=1}^{r}\sum_{t=r+2}^{r+p} K^{b}{}_{i_t i_s}{}^{a}\,T_{i_1/i_{s-1}ai_{s+1}/i_r;\,i_{r+2}/i_{t-1}bi_{t+1}/i_{r+p}} = 0. \tag{3.3}$$

and a necessary and sufficient condition for a pure skew symmetric covariant tensor T_{i_1/i_r} in a compact Kähler space to be (covariant) p-analytic is that

$$g^{st}\,T_{i_1/i_r;\,i_{r+2}/i_{r+p}st} - \sum_{s=1}^{r} T_{i_1/i_{s-1}ai_{s+1}/i_r;\,i_{r+2}/i_{r+p}}\,K_{i_s}{}^{a}$$
$$+\sum_{s=r+2}^{r+p} F^{tj}\,F_{i_1}{}^{a}\,K_{jti_s}{}^{b}\,T_{ai_2/i_r;\,i_{r+2}/i_{s-1}bi_{s+1}/i_{r+p}} = 0. \tag{3.4}$$

(3.3) and (3.4) provide the proof of the following

THEOREM 3.1 :

A necessary and sufficient condition for a pure skew symmetric covariant p-analytic tensor in a compact Kähler space to be p-harmonic is

$$\sum_{s=r+2}^{r+p} F^{tj} F_{i_1}{}^{a} K_{jti_s}{}^{b} T_{ai_2/i_r;\, i_{r+2}/i_{s-1} b i_{s+1}/i_{r+p}} - \sum_{s=1}^{r} \sum_{t=r+2}^{r+p} T_{i_1/i_{s-1} a i_{s+1}/i_r;\, i_{r+2}/i_{t-1} b i_{t+1}/i_{r+p}} K^{b}{}_{i_t i_s}{}^{a} = 0. \tag{3.5}$$

Next, a necesary and sufficient condition for a skew symmetric tensor T^{i_1/i_r} in a compact Kähler space to be p-Killing is [3]

$$G\{T^{i_1/i_r;\, i_{r+2}/i_{r+p}}\} = 0, \tag{3.6}$$

where

$$G\{T^{i_1/i_r;\, i_{r+2}/i_{r+p}}\} = g^{st}\, T^{i_1/i_r;\, i_{r+2}/i_{r+p}}{}_{st} + \frac{1}{r}\sum_{s=1}^{r} T^{i_1/i_{s-1} a i_{s+1}/i_r;\, i_{r+2}/i_{r+p}} K_a{}^{i_s} + \frac{1}{r}\sum_{\substack{s,t=1\\ s<t}}^{r} T^{i_1/i_{s-1} a i_{s+1}/i_{t-1} b i_{t+1}/i_r;\, i_{r+2}/i_{r+p}} K^{i_t i_s}{}_{ba} - \frac{1}{r}\sum_{s=1}^{r}\sum_{t=r+2}^{r+p} T^{i_1/i_{s-1} a i_{s+1}/i_r;\, i_{r+2}/i_{t-1} b i_{t+1}/i_{r+p}} K_b{}^{i_t i_s}{}_a = 0 \tag{3.7}$$

and

$$T_{i_1}{}^{i_2/i_r;\, i_{r+2}/i_{r+p} i_1} = 0. \tag{3.8}$$

But, if T^{i_1/i_r} is pure in all its indices, then $K_{kj}{}^{ih}$ being hybrid in i and h, equation (3.7) becomes

$$g^{st}\, T^{i_1/i_r;\, i_{r+2}/i_{r+p}}{}_{st} + \frac{1}{r}\sum_{s=1}^{r} K_a{}^{i_s}\, T^{i_1/i_{s-1} a i_{s+1}/i_r;\, i_{r+2}/i_{r+p}} - \frac{1}{r}\sum_{s=1}^{r}\sum_{t=r+2}^{r+p} T^{i_1/i_{s-1} a i_{s+1}/i_r;\, i_{r+2}/i_{t-1} b i_{t+1}/i_{r+p}} K_b{}^{i_t i_s}{}_a = 0 \tag{3.9}$$

and a necessary and sufficient condition for a pure skew symmetric contravariant tensor T^{i_1/i_r} in a compact Kähler space to be (contravariant) p-analytic is equation (2.13). Equations (3.8), (3.9) and (2.13) provide the proof of the following two theorems.

THEOREM 3.2:

In a compact Kähler space a contravariant p-analytic tensor T^{i_1/i_r} is p-Killing, if it satisfies

$$
\begin{aligned}
&(r-1)\sum_{s=1}^{r} K_a{}^{i_s}\, T^{i_1/i_{s-1}\, a i_{s+1}/i_r;\, i_{r+2}/i_{r+p}} \\
&+\sum_{s=1}^{r}\sum_{t=r+2}^{r+p} K_b{}^{i_t i_s}{}_a\, T^{i_1/i_{s-1}\, a i_{s+1}/i_r;\, i_{r+2}/i_{t-1}\, b i_{t+1}/i_{r+p}} \\
&-r\sum_{s=r+2}^{r+p} F^{tj}\, F_a{}^{i_1}\, K_{btj}{}^{i_s}\, T^{a i_2/i_r;\, i_{r+2}/i_{s-1}\, b i_{s+1}/i_{r+p}} = 0
\end{aligned}
\tag{3.10}
$$

and

$$T_{i_1}{}^{i_2/i_r;\; i_{r+2}/i_{r+p}\, i_1} = 0.$$

THEOREM 3.3:

A p-Killing tensor T_{i_1/i_r} in a compact Kähler space is contravariant p-analytic, if it satisfies (3.10).

(4) Suppose tensor T_{i_1/i_s} is a covariant $(p+1)$-analytic, then from section (2), it satisfies

$$F_j{}^a\, T^{i_1/i_s}{}_{;\, i_{s+2}/i_{s+p+1}\, a} - F^{i_1 a}\, T_a{}^{i_2/i_s}{}_{;\, i_{s+2}/i_{s+p+1}\, j} = 0. \tag{4.1}$$

Multiplying by $F^{j i_{s+p+1}}$ and summing over j and i_{s+p+1}, we get

$$g^{ab}\, T^{i_1/i_s;\, i_{s+1}/i_{s+p}}{}_{ab} - F^{jb}\, F^{i_1 a}\, T_a{}^{i_2/i_s;\, i_{s+2}/i_{s+p}}{}_{bj} = 0$$

or

$$g^{ab}\, T^{i_1/i_s;\, i_{s+2}/i_{s+p}}{}_{ab} - F^{jb}\, F_a{}^{i_1}\, T^{a i_2/i_s;\, i_{s+2}/i_{s+p}}{}_{jb} = 0. \tag{4.2}$$

After simplifying the above equation, we get

$$
\begin{aligned}
g^{ab}\, T^{i_1/i_s;\, i_{s+2}/i_{s+p}}{}_{ab} &+ \sum_{r=1}^{s} K_a{}^{i_r}\, T^{i_1/i_{r-1}\, a i_{r+1}/i_s;\, i_{s+2}/i_{s+p}} \\
&+ \sum_{t=s+2}^{s+p} F^{bj}\, F_a{}^{i_1}\, K_{bcj}{}^{i_t}\, T^{a i_2/i_s;\, i_{s+2}/i_{t-1}\, c i_{t+1}/i_{s+p}} = 0,
\end{aligned}
\tag{4.3}
$$

which is necessary and sufficient condition for the tensor to be (contravariant) p-analytic. Hence we have

THEOREM 4.1 :

A covariant $(p+1)$-analytic tensor is contravariant p-analytic

Next, let T^{i_1/i_s} be a contravariant $(p+1)$-analytic tensor, then again from section (2), we get

$$F_j^{\ a}\, T_{i_1/i_s;\, i_{s+2}/i_{s+p+1}\, a} - F_{i_1 a}\, T^{a}_{\ i_2/i_s;\, i_{s+2}/i_{s+p+1}\, j} = 0. \tag{4.4}$$

Multiplying equation (4.4) by $F^{j i_{s+p+1}}$ and summing over j and i_{s+p+1}, we get

$$g^{ab}\, T_{i_1/i_s\, i_{s+2}/i_{s+p}\, ab} + F^{jb} F_{i_1}^{\ a}\, T_{a i_2/i_s;\, i_{s+2}/i_{s+p}\, jb} = 0. \tag{4.5}$$

Solving equation (4.5), we get

$$g^{ab}\, T_{i_1/i_s;\, i_{s+2}/i_{s+p}\, ab} - \sum_{r=1}^{s} K_{i_r}^{\ a}\, T_{i_1/i_{r-1}\, a i_{r+1}/i_s;\, i_{s+2}/i_{s+p}}$$
$$+ \sum_{r=s+2}^{s+p} F^{bj} F_{i_1}^{\ a} K_{j b i_r}^{\ \ \ c}\, T_{a i_2/i_s;\, i_{s+2}/i_{r-1}\, c i_{r+1}/i_{s+p}} = 0, \tag{4.5}$$

which is a necessary and sufficient condition for the tensor T_{i_1/i_s} to be covariant p-analytic. Thus we can state

THEOREM 4.2 :

A contravariant $(p+1)$-analytic tensor is covariant p-analytic.

Theorems (4.1) and (4.2) together give the following two theorems.

THEOREM 4.3 :

A covariant p-analytic tensor is contravariant $\{p-(2r+1)\}$-analytic and covariant $(p-2r)$-analytic.

THEOREM 4.4 :

A contravariant p-analytic tensor is covariant $\{p-(2r+1)\}$-analytic and contravariant $(p-2r)$-analytic.

Next, assume that T^{i_1/i_s} is a contravariant $(p+1)$-analytic and p-analytic tensor. From theorem (4.2), it is also covariant p-analytic tensor and therefore

$$O^{ab}_{i_1 j}\, T_{a i_2/i_s;\, i_{s+2}/i_{s+p} b} = 0 \tag{4.7}$$

and

$$*\, O^{ab}_{i_1 j}\, T_{a i_2/i_s;\, i_{s+2}/i_{s+p} b} = 0. \tag{4.8}$$

Adding (4.7) and (4.8), we get

$$(*\, O^{ab}_{i_1 j} + O^{ab}_{i_1 j})\, T_{a i_2/i_s;\, i_{s+2}/i_{s+p} b} = 0$$

i. e.

$$A^a_{i_1} A^b_j\, T_{a i_2/i_s;\, i_{s+2}/i_{s+p} b} = 0$$

i. e.

$$T_{i_1/i_s;\, i_{s+2}/i_{s+p} j} = 0 \tag{4.9}$$

and consequently from ([4], p. 170)

$$T_{i_1/i_s; j} = 0. \tag{4.10}$$

Conversely, if

$$T_{i_1/i_s; j} = 0$$

T^{i_1/i_s} is contravariant p-and $(p+1)$-analytic. This proves that

THEOREM 4.5:

A necessary and sufficient condition for a tensor to be contravariant p-and $(p+1)$-analytic is that

$$T_{i_1/i_s; j} = 0. \tag{4.11}$$

Using similar technique, we can establish the following

THEOREM 4.6:

A necessary and sufficient condition for a tensor to be covariant p-and $(p+1)$-analytic is that

$$T_{i_1/i_s; j} = 0. \tag{4.12}$$

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REFERENCES

[1] SRIVASTAVA, R., C., *On p-analytic vectors* (under publication)

[2] YANO, K., *Differential geometry on complex and almost complex spaces.* Pergamon Press (1965).

[3] SRIVASTAVA, R., C., *On generalizations of harmonic and Killing tensors,* Annales Polonici Mathematici XXI (1968), p. 139-153.

[4] YANO, K., & BOCHNER, S., *Curvature and Betti number,* Annales of Math. Studies 32 (1953).