SOME EXTENSIONS OF THE MEHLER FORMULA

by

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1. The Hermite polynomial $H_n(a)$ may be defined by

$$\sum_{n=0}^{\infty} H_n(a) \frac{x^n}{n!} = e^{2ax - x^2}.$$

The bilinear expansión

$$(1.1) \quad \sum_{n=0}^{\infty} H_n(a) H_n(b) \frac{x^n}{n!} = (1 - 4x^2)^{-\frac{1}{2}} \exp \left\{ \frac{4abx - 4(a^2 + b^2) x^2}{1 - 4x^2} \right\}$$

is known as Mehler's formula [3].

In the present paper we shall obtain a number of extensions of (1.1). In the first place we show that

$$(1.2) \quad \sum_{m,n=0}^{\infty} H_{m+n}(a) H_m(b) H_n(c) \frac{x^m y^n}{m!\, n!}$$

$$= (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} \exp \left\{ \frac{-4a^2(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2} \right\}.$$

This can be extended further. For example we show that

$$(1.3) \quad \sum_{m,n,p=0}^{\infty} H_{m+n+p}(a) H_m(b) H_n(c) H_p(d) \frac{x^m y^n z^p}{m!\, n!\, p!}$$

$$= (1 - 4x^2 - 4y^2 - 4z^2)^{-\frac{1}{2}}$$

$$\cdot \exp \left\{ \frac{-4a^2(x^2 + y^2 + z^2) + 4a(bx + cy + dz) - 4(bx + cy + dz)^2}{1 - 4x^2 - 4y^2 - 4z^2} \right\}$$

and so on.

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In the next place we consider an expansion of a somewhat different nature, namely

$$(1.4)\qquad \sum_{m,n,p=0}^{\infty} H_{n+p}(a)\, H_{p+m}(b)\, H_{m+n}(c)\, \frac{x^m y^n z^p}{m!\, n!\, p!}$$

$$= d_1^{-\frac{1}{2}} \exp\left\{\Sigma a^2 - \frac{\Sigma a^2 - 4\Sigma a^2 x^2 - 4\Sigma abz + 8\Sigma abxy}{d_1}\right\},$$

where

$$d_1 = 1 - 4x^2 - 4y^2 - 4z^2 + 16xyz$$

and Σa^2, $\Sigma a^2 x^2$, Σabz, $\Sigma abxy$ are symmetric functions in the indicated variables.

The corresponding result for the sum

$$(1.5)\qquad \sum_{m,n,p,q=0}^{\infty} H_{m+n}(a)\, H_{n+p}(b)\, H_{p+q}(c)\, H_{q+m}(d)\, \frac{x^m y^n z^p t^q}{m!\, n!\, p!\, q!}$$

is contained in (5.1) below.

The identities (1.1) and (1.2) admit of the following extensión.

$$(1.6)\qquad \sum_{k=0}^{\infty} \frac{z^k}{k!} H_{k+m}(a)\, H_{k+n}(b)$$

$$= (1 - z^2)^{-\frac{1}{2}(m+n+1)} \exp\left[\frac{4abz - 4(a^2+b^2)z^2}{1-4z^2}\right]$$

$$\cdot \sum_{r=0}^{\min(m,n)} 2^{2r} r! \binom{m}{r}\binom{n}{r} z^r H_{m-r}\left(\frac{a-2bz}{\sqrt{1-z^2}}\right) H_{n-r}\left(\frac{b-2az}{\sqrt{1-z^2}}\right),$$

$$(1.7)\qquad \sum_{m,n=0}^{\infty} H_{m+n}(a)\, H_{m+r}(b)\, H_{n+s}(c)\, \frac{x^m y^n}{m!\, n!}$$

$$= S \cdot \sum_{j=0}^{r} \sum_{k=0}^{s} \binom{r}{j}\binom{s}{j} H_{r-j}(b)\, H_{s-k}(c)\, H_{j+k}\left(\frac{-2a + 4bx + 4cy}{\sqrt{1-4x^2-4y^2}}\right)$$

$$\cdot (2x)^j (2y)^k (1 - 4x^2 - 4y^2)^{-\frac{1}{2}(j+k)}$$

where S denotes the right member of (1.2).

The identities (1.2), (1.3), (1.4) (1.5) are proved first by means of infinite integrals. However we also give more elementary proofs. In particular, (1.4) and (1.5) are obtained making use of (1.6) and (1.7).

2. To prove (1.2) we follow a method used by Watson [3] in proving (1.1). We shall require the following formulas.

$$e^{-a^2} = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\ (-u^2 + 2iau)\, du, \tag{2.1}$$

$$H_n(a) = \pi^{-\frac{1}{2}} (-2i)^n e^{a^2} \int_{-\infty}^{\infty} u^n \exp\ (-u^2 + 2iau)\, du. \tag{2.2}$$

Then

$$\sum_{m,n=0}^{\infty} H_{m+n}(a)\, H_m(b)\, H_n(c) \frac{x^m y^n}{m!\, n!}$$

$$= \pi^{-\frac{1}{2}} e^{a^2} \sum_{m,n=0}^{\infty} H_m(b)\, H_n(c) \frac{(-2i)^{m+n} x^m y^n}{m!\, n!}$$

$$\cdot \int_{-\infty}^{\infty} u^{m+n} \exp\ (-u^2 + 2iau)\, du$$

$$= \pi^{-\frac{1}{2}} e^{a^2} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} H_m(b) \frac{(-2ixu)^m}{m!} \sum_{n=0}^{\infty} H_n(c) \frac{(-2iyu)^n}{n!}$$

$$\cdot \exp\ (-u^2 + 2iau)\, du$$

$$= \pi^{-\frac{1}{2}} e^{a^2} \int_{-\infty}^{\infty} \exp\ \{-u^2 + 2iau - 4ibxu + 4x^2u^2 - 4icyu + 4y^2u^2\}\, du$$

$$= \pi^{-\frac{1}{2}} e^{a^2} \int_{-\infty}^{\infty} \exp\ \{-(1 - 4x^2 - 4y^2)\, u^2 + 2i(a - 2bx - 2cy)\, u\}\, du$$

$$= \pi^{-\frac{1}{2}} (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} e^{a^2} \int_{-\infty}^{\infty} \exp\ \left\{-v^2 + 2iv \frac{a - 2bx - 2cy}{(1 - 4x^2 - 4y^2)^{-\frac{1}{2}}}\right\} dv$$

$$= (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} \exp\ \left\{a^2 - \frac{(a - 2bx - 2cy)^2}{1 - 4x^2 - 4y^2}\right\}$$

$$= (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} \exp\ \left\{\frac{-4a^2(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2}\right\}.$$

This completes the proof of (1.2).

In exactly the same way we can prove (1.3). The general formula is evidently

$$(2.3)\qquad \sum_{n_1,\ldots,n_k=0}^{\infty} H_{n_1+\ldots+n_k}(a)\, H_{n_1}(b_1)\ldots H_{n_k}(b_k)\,\frac{x_1^{n_1}\ldots x_k^{n_k}}{n_1!\ldots n_k!}$$

$$= (1-4\Sigma x_1^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2\Sigma x_1^2 + 4a\Sigma b_1 x_1 - 4(\Sigma b_1 x_1)^2}{1-4\Sigma x_1^2}\right\},$$

where

$$\Sigma x_1^2 = x_1^2 + x_2^2 + \ldots + x_k^2,$$

$$\Sigma b_1 x_1 = b_1 x_1 + b_2 x_2 + \ldots + b_k x_k.$$

3. We can prove (1.2) without the use of infinite integrals in the following way. Since

$$(3.1)\qquad H_m(a) = \sum_{2r\leq m} (-1)^r \frac{m!}{(m-2r)!\,r!}(2a)^{m-2r},$$

it follows that the left member of (1.2) is equal to

$$\sum_{m,n=0}^{\infty} H_{m+n}(a)\frac{x^m y^n}{m!\,n!}\sum_{2r\leq m}(-1)^r\frac{m!}{(m-2r)!\,r!}(2\mathrm{b})^{m-2r}$$

$$\cdot\sum_{2s\leq n}(-1)^s\frac{n!}{(n-2s)!}(2c)^{n-2s}$$

$$=\sum_{m,n=0}^{\infty}\frac{(2bx)^m(2cy)^n}{m!\,n!}\sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{x^{2r}y^{2s}}{r!\,s!}H_{m+n+2r+2s}(a)$$

$$=\sum_{n=0}^{\infty}\frac{(2bx+2cy)^n}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{(x^2+y^2)^k}{k!}H_{n+2k}(a).$$

Thus (1.2) reduces to

$$\sum_{n=0}^{\infty}\frac{(2bx+2cy)^n}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{(x^2+y^2)^k}{k!}H_{n+2k}(a)$$

$$=(1-4x^2-4y^2)^{-\frac{1}{2}}\exp\left\{\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{1-4x^2-4y^2}\right\},$$

which is equivalent to

$$(3.2)\qquad \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{w^{2k}}{k!}H_{n+2k}(a)$$

$$=(1-4w^2)^{-\frac{1}{2}}\exp\left\{\frac{-4a^2w^2+2az-z^2}{1-4w^2}\right\}.$$

The left hand side of (3.2) is equal to

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(a) \sum_{2k \leq n} (-1)^k \frac{n!}{(n-2k)!\,k!} z^{n-2k} w^{2k}$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} H_n(a) \sum_{2k \leq n} (-1)^k \frac{n!}{(n-2k)!\,k!} \left(\frac{z}{w}\right)^{n-2k}$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} H_n(a)\, H_n(z/2w),$$

by (3.1). Now applying (1.1), we get

$$(1-4w^2)^{-\frac{1}{2}} \exp\left\{\frac{2az - 4a^2w^2 - z^2}{1-4w^2}\right\},$$

so that we have proved (3.2).

We have therefore proved that (1.2) is implied by (1.1). This is in fact true of the general formula (2.3). To prove this assertion consider the left hand side of (2.3), which by (3.1) is equal to

$$\sum_{n_1,\ldots,n_k=0}^{\infty} H_{n_1+\cdots+n_k}(a) \frac{x_1^{n_1}\ldots x_k^{n_k}}{n_1!\ldots n_k!}$$

$$\cdot \sum_{r_1} (-1)^{r_1} \frac{n_1!}{(n_1-2r_1)!\,r_1!} (2b_1)^{n_1-2r_1} \ldots \sum_{r_k} (-1) \frac{n_k}{(n_k-2r_k)!\,r_k!} (2b_k)_{n_k-2r_k}$$

$$= \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(2b_1x_1)^{1n_1}\ldots(2b_kx_k)^{n_k}}{n_1!\ldots n_k!}$$

$$\cdot \sum_{r_1,\ldots,r_k=0}^{\infty} (-1)^{r_1+\cdots+r_k} \frac{x_1^{2r_1}\ldots x_k^{2r_k}}{r_1!\ldots r_k!} H_{n_1+\cdots+n_k+2(r_1+\cdots+s_k)}(a)$$

$$= \sum_{n=0}^{\infty} \frac{(2b_1x_1+\ldots+2b_kx_k)^n}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{(x_1^2+\ldots+x_k^2)^r}{r!} H_{n+2r}(a).$$

Hence (2.3) reduces to

$$\sum_{n=0}^{\infty} \frac{(2b_1x_1+\ldots+2b_kx_k)^n}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{(x_1^2+\ldots+x_k^2)^r}{r!} H_{n+2r}(a)$$

$$= (1-4\Sigma x_1^{2-\frac{1}{2}} \exp\left\{\frac{-4a^2\Sigma x_1^2 + 4a\Sigma b_1x_1 - 4(\Sigma b_1x_1)^2}{1-4\Sigma x_1^2}\right\},$$

which is equivalent to

$$(3.3)\qquad \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{w^{2r}}{r!} H_{n+2r}(a)$$
$$= (1 - 4w^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2 w^2 + 2az - z^2}{1 - 4w^2.}\right\}.$$

But this is the same as (3.2), which we have seen is implied by (1.1). Therefore (2.3) is implied by (1.1) for all k.

4. We turn now to the proof of (1.4). It is convenient to put

$$\overline{H}_n(a) = \sum_{2r \leq m} \frac{n!}{(n-2r)!\, r!} (2a)^{n-2r},$$

so that

$$\overline{H}_n(ia) = i^n H_n(a)$$

and

$$\sum_{n=0}^{\infty} \overline{H}(ia) \frac{x^n}{n!} = e^{2ax + x^2}.$$

Also (2.1) becomes

$$\overline{H}_n(a) = \frac{2^n e^{=a^2}}{\pi^{-\frac{1}{2}}} \int_{-\infty}^{\infty} u^n \exp(-u^2 - 2au)\, du.$$

It follows that

$$\sum_{m,n,p=0}^{\infty} \overline{H}_{n+p}(a)\, \overline{H}_{p+m}(b)\, \overline{H}_{m+n}(c) \frac{x^m y^n z^p}{m!\, n!\, p!}$$
$$= \frac{e^{-a^2-b^2-c^2}}{\pi^{3/2}} \sum_{m,n,p=0}^{\infty} \frac{(4x)^m (4y)^n (4z)^p}{m!\, n!\, p!}$$
$$\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^{n+p} v^{p+m} w^{m+n}$$
$$\cdot \exp(-u^2 - v^2 - w^2 - 2au - 2bv - 2cw)\, du\, dv\, dw$$
$$- \frac{e^{-a^2-b^2-c^2}}{\pi^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-u^2 - a^2 - w^2 - 2au - 2bv - 2cw$$
$$+ 4xvw + 4yuw + 4zuv)\, du\, dv\, dw.$$

Let $Q(u, v, w)$ denote the quadratic form

$$u^2 + v^2 + w^2 - 4xvw - 4yuw - 4zuv$$

with matrix

$$A = \begin{bmatrix} 1 & -2z & -2y \\ -2z & 1 & -2x \\ -2y & -2x & 1 \end{bmatrix}.$$

Also put

$$\xi = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \alpha = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then

$$Q(u, v, w) + 2au + 2bv + 2cw = \xi' A \xi + \xi' \alpha + \ddot{a}' \xi,$$

where the prime denotes the transpose.

Put $\xi + \eta + \beta$, where η, β are column vectors. Then

$$\begin{aligned} \xi' A \xi + \xi' \alpha + \alpha' \xi &= (\eta' + \beta') A (\eta + \beta) + (\eta' + \beta') \alpha + \alpha' (\eta + \beta) \\ &= \eta' A \eta - \eta' (A \beta + \alpha) + (\beta' A + \alpha') \eta + \beta' A \beta + \beta' \alpha + \alpha' \beta \end{aligned}$$

If $A \beta + \alpha = 0$ this reduces to

$$\eta' A \eta - \alpha' A^{-1} \alpha.$$

We have therefore

$$\begin{aligned} (4.1) \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-u^2 - v^2 - w^2 - 2au - 2bv - 2cw \\ & \qquad + 4xvw + 4yuw + 4zuv)\, du\, dv\, dw \\ & = \exp(\alpha' A^{-1} \alpha) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\eta' A \eta)\, du\, dv\, dw. \end{aligned}$$

It is known [1, p. 96] that

$$(4.2) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\alpha' A^{-1} \alpha)\, du\, dv\, dw = \pi^{\frac{3}{2}} \Delta^{-\frac{1}{2}},$$

where Δ is the determinan of A:

$$\Delta = 1 - 4x^2 - 4y^2 - 4z^2 - 16xyz. \tag{4.3}$$

To compute $\alpha' A^{-1} \alpha$ we take

$$A^{-1} = \Delta^{-1} \begin{bmatrix} 1 - 4x^2 & 2x - 4xy & 2y - 2xz \\ 2x - 4xy & 1 - 4y^2 & 2x - 4yz \\ 2y - 4xz & 2x - 4yz & 1 - 4z^2 \end{bmatrix},$$

$$\alpha' A^{-1} \alpha = \frac{1}{\Delta} \{\Sigma a^2 (1 - 4x^2) + 2 \Sigma ab (2z - 4xy)\} \tag{4.4}$$

$$= \frac{1}{\Delta} \{\Sigma a^2 - 4 \Sigma a^2 x^2 + 4 \Sigma abz - 8 \Sigma abxy\}.$$

Therefore, by (4.1), (4.2) and (4.4), we have

$$\sum_{m,n,p=0}^{\infty} \overline{H}_{n+p}(a) \overline{H}_{p+m}(b) \overline{H}_{m+n}(c) \frac{x^m y^n z^p}{m!\, n!\, p!} \tag{4.5}$$

$$= \Delta^{-\frac{1}{2}} \exp \{- \Sigma a^2 + \Delta^{-1} [\Sigma a^2 - 4 (\Sigma ax)^2 - 8 \Sigma abz]\}.$$

Replacing a, b, c by ia, ib, ic and x, y, z by $-x, -y, -z$, (1.5) becomes

$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a) H_{p+m}(b) H_{m+n}(c) \frac{x^m y^n z^p}{m!\, n!\, p!} \tag{4.6}$$

$$= \Delta_1^{-\frac{1}{2}} \exp \{\Sigma a^2 - \Delta_1^{-1} [\Sigma a^2 - 4 (\Sigma ax)^2 + 8 \Sigma abz]\},$$

where

$$\Delta_1 = 1 - 4x^2 - 4y^2 - 4z^2 + 16xyz. \tag{4.7}$$

5. By a similar analysis we can show that

$$\sum_{m,n,p,q=0}^{\infty} \overline{H}_{m+n}(a) \overline{H}_{n+p}(b) \overline{H}_{p+q}(c) \overline{H}_{q+m}(d) \frac{x^m y^n z^p w^q}{m!\, n!\, p!\, q!} \tag{5.1}$$

$$= \Delta^{-\frac{1}{2}} \exp \{- \Sigma a^2 + d^{-1} G\},$$

where $\Delta = \det A$,

$$(5.2) \qquad A = \begin{bmatrix} 1 & -2y & \cdot & -2w \\ -2y & 1 & -2z & \cdot \\ \cdot & -2z & 1 & -2x \\ -2x & \cdot & -2w & 1 \end{bmatrix},$$

$$(5.3) \qquad G = \alpha' A^{-1}, \qquad \alpha' = [a, b, c, d].$$

Generally one can evaluate the cyclic sum

$$(5.4) \qquad \sum_{n_1, \ldots, n_k = 0}^{\infty} \overline{H}_{n_1+n_2}(a_1) \overline{H}_{n_2+n_3}(a_2) \ldots \overline{H}_{n_k+n_1}(a_k) \frac{x_1^{n_1} \ldots x_k^{n_k}}{n_1! \ldots n_k!}.$$

Note that, for $w = 0$, the left member of (5.1) becomes

$$(5.5) \qquad \sum_{m, n, p=0}^{\infty} \overline{H}_{m+n}(a) \overline{H}_{n+p}(b) \overline{H}_p(c) \overline{H}_m(d) \frac{x^m y^n z^p}{m!\, n!\, p!}.$$

6. We shall now give a different proof of (1.4) that does not require infinite integrals. We shall require the following formula.

$$(6.1) \qquad \sum_{k=0}^{\infty} \frac{z^k}{k!} H_{k+m}(a) H_{k+n}(b)$$

$$= (1 - 4z^2)^{-\frac{1}{2}(m+n+1)} \exp \left[\frac{4abz - 4(a^2+b^2)z^2}{1 - 4z^2} \right]$$

$$\cdot \sum_{\min(m,n)}^{\infty} 2^{2r} r! \binom{m}{r} \binom{n}{r} z^r H_{m-r} \left(\frac{a - 2bz}{\sqrt{1-4z^2}} \right) H_{n-r} \left(\frac{b - 2az}{\sqrt{1-4z^2}} \right).$$

This evidently reduces to (1.1) when $m = n = 0$.

Let $F_{m,n}$ denote the left member of (6.1). Since [2, p. 197]

$$(6.2) \qquad \sum_{m=0}^{\infty} \frac{x^m}{m!} H_{m+k}(a) = \exp(2ax - x^2) H_k(a - x),$$

we have

$$\sum_{m,n=0}^{\infty} F_{m,n} \frac{x^m y^n}{m!\, n!} = \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!\, n!} \sum_{k=0}^{\infty} H_{k+m}(a) H_{k+n}(b)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{m=0}^{\infty} \frac{x^m}{m!} H_{m+k}(a) \sum_{n=0}^{\infty} \frac{y^n}{n!} H_{n+k}(b)$$

$$= \exp\,(2ax + 2by - x^2 - y^2) \sum_{k=0}^{\infty} \frac{z^k}{k!} H_k(a-x)\, H_k(b-y)$$

$$= (1 - 4z^2)^{-\frac{1}{2}} \exp \Big\{ 2ax + 2by - x^2 - y^2$$

$$+ \frac{4(a-x)(b-y)z - 4(a-x)^2 z^2 - 4(b-y)^2 z^2}{1 - 4z^2} \Big\},$$

so that

$$\sum_{m,n=0}^{\infty} F_{m,n} \frac{x^m y^n}{m!\, n!} = (1 - 4z^2)^{-\frac{1}{2}} \tag{6.3}$$

$$\cdot \exp \left\{ \frac{4abz - 4(a^2+b^2)z^2 + 2ax + 2by - x^2 - y^2 - 4bxz - 4ayz + 4xyz}{1 - 4z^2} \right\}.$$

Now let $G_{m,n}$ denote the right member of (6.1). Then

$$\sum_{m,n=0}^{\infty} G_{m,n} \frac{x^m y^n}{m!\, n!} = (1 - 4z^2)^{-\frac{1}{2}} \exp \left\{ \frac{4abz - 4(a^2+b^2)z^2}{1 - 4z^2} \right\}$$

$$\cdot \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!\, n!} = (1 - 4z^2)^{-\frac{1}{2}(m+n)} \sum_{r=0}^{\min(m,n)} 2^{2r} r! \binom{m}{r}\binom{n}{r} z^r$$

$$\cdot H_{m-r}\left(\frac{a - 2bz}{\sqrt{1 - 4z^2}}\right) H_{n-r}\left(\frac{b - 2az}{\sqrt{1 - 4z^2}}\right).$$

The triple sum is equal to

$$\sum_{r=0}^{\infty} \frac{2^{2r}(xyz)^r}{r!} (1 - 4z^2)^{-r} \sum_{m=0}^{\infty} H_m\left(\frac{a - 2bz}{\sqrt{1 - 4z^2}}\right) \frac{x^m}{m!} (1 - 4z^2)^{-\frac{1}{2}m}$$

$$\cdot \sum_{n=0}^{\infty} H_n\left(\frac{b - 2az}{\sqrt{1 - 4z^2}}\right) \frac{y^n}{n!} (1 - 4z^2)^{-\frac{1}{2}n}$$

$$= \exp \left\{ \frac{4xyz}{1 - 4z^2} + \frac{2(a - 2bz)x - x^2}{1 - 4z^2} + \frac{2(b - 2az)y - y^2}{1 - z^2} \right\}.$$

Thus

$$\sum_{m,n=0}^{\infty} G_{m,n} \frac{x^m y^n}{m!\, n!} = (1 - 4z^2)^{-\frac{1}{2}} \exp \left\{ \frac{4abz - 4(a^2+b^2)z^2}{1 - 4z^2} \right\} \tag{6.4}$$

$$\cdot \exp \left\{ \frac{2ax + 2by - x^2 - y^2 - 4bxz - 4ayz + 4xyz}{1 - 4z^2} \right\}.$$

Comparing (6.4) with (6.3), we get (6.1).

The special case $n = 0$ of (6.0) should be noted:

(6.5)
$$\sum_{k=0}^{\infty} \frac{z^k}{k!} H_{k+m}(a) H_k(b)$$
$$= (1 - 4z^2)^{-\frac{1}{2}(m+1)} \exp\left[\frac{4abz - 4(a^2+b^2)z^2}{1-4z^2}\right] H_m\left(\frac{a-2bz}{\sqrt{1-4z^2}}\right).$$

In particular, for $b = 0$, (6.4) reduces to

(6.6)
$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!} H_{2k+m}(a)$$
$$= (1 - 4z^2)^{-\frac{1}{2}(m+1)} \exp\left[\frac{-4a^2z^2}{1-4z^2}\right] H_m\left(\frac{a}{\sqrt{1-4z^2}}\right).$$

7. Making use of (6.1) we have

$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a) H_{p+m}(b) H_{m+n}(c) \frac{x^m y^n z^p}{m!\, n!\, p!}$$
$$= \sum_{m,n=0}^{\infty} H_{m+n}(c) \frac{x^m y^n}{m!\, n!} \sum_{p=0}^{\infty} H_{n+p}(a) H_{m+p}(b) \frac{z^p}{p!}$$
$$= \sum_{m,n=0}^{\infty} H_{m+n}(c) \frac{x^m y^n}{m!\, n!} \cdot (1-4z^2)^{-\frac{1}{2}(m+n+1)} \exp\left[\frac{4abz - 4(a^2+b^2)z^2}{1-4z^2}\right]$$
$$\cdot \sum_{r=0}^{\min(m,n)} 2^{2r} r! \binom{m}{r}\binom{n}{r} z^r H_{n-r}\left(\frac{a-2bz}{\sqrt{1-4z^2}}\right) H_{m-r}\left(\frac{b-2az}{\sqrt{1-4z^2}}\right)$$
$$= (1-4z^2)^{-\frac{1}{2}} \exp\left[\frac{4abz - 4(a^2+b^2)z^2}{1-4z^2}\right]$$
$$\cdot \sum_{m,n,r=0}^{\infty} H_m\left(\frac{b-2az}{\sqrt{1-4z^2}}\right) H_n\left(\frac{a-2bz}{\sqrt{1-4z^2}}\right) H_{m+n+2r}(c)$$
$$\cdot \frac{x^m y^n (4xyz)^r}{m!\, n!\, r!} (1-4z^2)^{-\frac{1}{2}(m+n+2r)}.$$

The triple sum is equal to

$$\sum_{m,n=0}^{\infty} H_m\left(\frac{b-2az}{\sqrt{1-4z^2}}\right) H_n\left(\frac{a-2bz}{\sqrt{1-4z^2}}\right) \frac{x^m y^n}{m!\, n!} (1-4z^2)^{-\frac{1}{2}(m+n)}$$
$$\cdot \sum_{r=0}^{\infty} H_{m+n+2r}(c) \frac{(4xyz)^r}{r!} (1-4z^2)^{-r}.$$

By (6.6) the inner sum is equal to

$$\left(1+\frac{16xyz}{1-4z^2}\right)^{-\frac{1}{2}(m+n+1)}$$

$$\cdot\exp\left(\frac{16c^2xyz}{1-4z^2+16xyz}\right)H_{m+n}\left(\frac{c\sqrt{1-4z^2}}{\sqrt{1-4z^2+16xyz}}\right).$$

We have therefore

$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a)\,H_{p+m}(b)\,H_{m+n}(c)\,\frac{x^m y^n z^p}{m!\,n!\,p!} \tag{7.1}$$

$$=(1-4z^2+16xyz)^{-\frac{1}{2}}\exp\left\{\frac{4abz-4(a^2+b^2)z^2}{1-4z^2}+\frac{16c^2xyz}{1-4z^2+16xyz}\right\}$$

$$\cdot\sum_{m,n=0}^{\infty} H_m\left(\frac{b-2az}{\sqrt{1-4z^2}}\right)H_n\left(\frac{a-2bz}{\sqrt{1-4z^2}}\right)H_{m+n}\left(\frac{c\sqrt{1-4z^2}}{\sqrt{1-4z^2+16xyz}}\right)$$

$$\cdot\frac{x^m y^n}{m!\,n!}(1-4z^2+16xyz)^{-\frac{1}{2}(m+n)}.$$

Now applying (1.2), a little manipulation leads to (1.4).

8. We have, by (6.2) and (1.2),

$$\sum_{r,s=0}^{\infty}\frac{u^r v^s}{r!\,s!}\sum_{m,n=0}^{\infty} H_{m+n}(a)\,H_{m+r}(b)\,H_{n+s}(c)\,\frac{x^m y^n}{m!\,n!}$$

$$=\sum_{m,n=0}^{\infty} H_{m+n}(a)\,\frac{x^m y^n}{m!\,n!}\sum_{r,s=0}^{\infty} H_{m+r}(b)\,H_{n+s}(c)\,\frac{u^r v^s}{r!\,s!}$$

$$=\exp(2bu+2cv-u^2-v^2)\sum_{m,n=0}^{\infty} H_{m+n}(a)\,H_m(b-u)\,H_n(c-v)\,\frac{x^m y^n}{m!\,n!}$$

$$=\exp(2bu+2cv-u^2-v^2)\cdot(1-4x^2-4y^2)^{-\frac{1}{2}}$$

$$\cdot\exp\left\{\frac{-4a^2(x^2+y^2)+4a[(b-n)x+(c-v)y]-4[(b-u)x+(c-v)y]^2}{1-4x^2-4y^2}\right\}$$

$$=\exp(2bu+2cv-u^2-v^2)$$

$$\cdot(1-4x^2-4y^2)^{-\frac{1}{2}}\exp\left\{\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{1-4x^2-4y^2}\right\}$$

$$\cdot\exp\left\{\frac{2(2a+4bx+cy)\,ux+vy)-4(ux+vy)^2}{1-4x^2-4y^2}\right\}$$

$$= (1-4x^2-4y^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{1-4x^2-4y^2}\right\}$$

$$\cdot \sum_{r,s=0}^{\infty} H_r(b)\, H_s(c) \frac{u^r v^s}{r!\, s!} \sum_{j,k=0}^{\infty} H_{j+k}\left(\frac{-2a+4bx+4cy}{\sqrt{1-4x^2-4y^2}}\right) \frac{(2ux)^j (2vy)^k}{j!\, k!}$$

$$(1-4x^2-4y^2)^{-\frac{1}{2}(j+k)}.$$

Comparison of coefficients of $u^r v^s$ gives

$$(8.1) \qquad \sum_{m,n=0}^{\infty} H_{m+n}(a)\, H_{m+r}(b)\, H_{n+s}(c) \frac{x^m y^n}{m!\, n!}$$

$$= (1-4x^2-4y^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{1-4x^2-4y^2}\right\}$$

$$\cdot \sum_{j=0}^{r} \sum_{k=0}^{s} \binom{r}{j}\binom{s}{k} H_{r-j}(b)\, H_{s-k}(c)\, H_{j+k}\left(\frac{-2a+4bx+4cy}{\sqrt{1-4x^2-4y^2}}\right)$$

$$\cdot (2x)^j (2y)^k (1-4x^2-4y^2)^{-\frac{1}{2}(j+k)}.$$

For $r = s = 0$, it is evident that (8.1) reduces to (1.2).
Similarly one can evaluate the sum

$$(8.2) \qquad \sum_{m,n,p=0}^{\infty} H_{m+n+p}(a)\, H_{m+r}(b)\, H_{n+s}(c)\, H_{p+t}(d) \frac{x^m y^n z^p}{m!\, n!\, p!}.$$

For brevity let S denote the right member of (1.2). Then it follows from (8.1) that

$$\sum_{m,n,r,s=0}^{\infty} H_{m+n}(a)\, H_{m+r}(b)\, H_{n+s}(c)\, H_{r+s}(d) \frac{x^m y^n u^r v^s}{m!\, n!\, r!\, s!}$$

$$= S \cdot \sum_{j,k=0}^{\infty} \frac{(2xu)^j (2yv)^k}{j!\, k!} (1-4x^2-4y^2)^{-\frac{1}{2}(j+k)} H_{j+k}\left(\frac{-2a+4bx+4cy}{\sqrt{1-4x^2-4y^2}}\right)$$

$$\cdot \sum_{j,k=0}^{\infty} H_r(b)\, H_s(c)\, H_{r+s+j+k}(d) \frac{u^r v^s}{r!\, s!}$$

$$= S \cdot \sum_{r,s,n=0}^{\infty} H_{r+s+n}(d)\, H_r(b)\, H_s(c)\, H_n \frac{-2a+4bx+4cy}{\sqrt{1-4x^2-4y^2}}$$

$$\cdot \frac{u^r v^s (2xu+2yv)^n}{r!\, s!\, n!} (1-4x^2-4y^2)^{-\frac{1}{2}}.$$

We now apply (1.3). After some manipulation and change of notation, we get (5.1). It will suffice to verify the external factor $d^{-\frac{1}{2}}$. By (1.3) this appears as

$$(1 - 4x^2 - 4y^2)(1 - 4u^2 - 4v^2) - (4xu + 4yv)^2.$$

On the other hand, by (5.2),

$$\Delta = \det A = \begin{vmatrix} 1 & 2y & \cdot & 2x \\ 2y & 1 & 2v & \cdot \\ \cdot & 2v & 1 & 2u \\ 2x & \cdot & 2u & 1 \end{vmatrix}$$

$$= (1 - 4y^2)(1 - 4u^2) - (2v)(2v) - (2x)(2x) - (2yv)(4xu) - (4xy)(4uv) + (4xv)^2$$

$$= 1 - 4x^2 - 4y^2 - 4u^2 - 4v^2 + 16x^2v^2 + 16y^2u^2 - 32xyuv.$$

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