AN INTEGRAL REPRESENTATION FOR THE PRODUCT OF TWO GENERALISED RICE'S POLYNOMIALS.

by

H. L. MANOCHA

SUMMARY: The product of two generalised Rice's polynomials is represented by an quinary integral.

1. INTRODUCTION

Some time back CARLITZ [1] gave an integral representation for the product $L_m^{(\alpha)}(x)\, L_n^{(\beta)}(y)$, where $L_n^{(\alpha)}(x)$ denotes the general LAGUERRE polynomial of degree n. In this paper we propose to derive integral representation for the product $H_m^{(\alpha,\beta)}(\xi, p;\ x)\ H_n^{(\gamma,\delta)}(\eta, q; y)$, where $H_n^{(\alpha,\beta)}(\xi, p;\ x)$ is a generalised Rice's polynomial defined by

$$(1.1)\quad H_m^{(\alpha,\beta)}(\xi, p; x) = \frac{(1+\alpha)_m}{m!}\,{}_3F_2\left[\begin{matrix}-m,\ m+\alpha+\beta+1,\ \xi;\ x\\ \alpha+1,\ p;\end{matrix}\right].$$

For $\alpha = 0 = \beta$, (1.1) reduces ty the original form, Rice [2],

$$(1.2)\qquad H_m(\xi, p; x) = {}_3F_2\left[\begin{matrix}-m,\ m+1,\ \xi;\ x\\ 1,\ p;\end{matrix}\right].$$

2. It follows from (1.1) that

$$H_m^{(\alpha,\beta)}(\xi, p; x)\ H_n^{(\gamma,\delta)}(\eta, p; y)$$

$$= \frac{(1+\alpha)_m\,(1+\gamma)_n}{m!\ n!}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r\,(-n)_s\,(m+\alpha+\beta+1)_r\,(n+\gamma+\delta+1)_o\,(\xi)_r\,(\eta)_s}{r!\ s!\ (1+\alpha)_r\ (1+\gamma)_s\ (p)_r\ (q)_s}\ x^r\ y^s$$

$$= \frac{\Gamma(\alpha+m+1)\,\Gamma(\gamma+n+1)\,\Gamma(p)\,\Gamma(q)}{\Gamma(m+\alpha+\beta+1)\,\Gamma(n+r+\delta+1)\,\Gamma(\xi)\,\Gamma(\eta)}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{\Gamma(m+n-r-s+1)}{\Gamma(m-r+1)\ \Gamma(n-s+1)}\,.$$

$$\cdot \frac{\Gamma(m+\alpha+\beta+r+1)\,\Gamma(n+\gamma+\delta+s+1)}{\Gamma(m+n+\alpha+\beta+\gamma+\delta+r+s+2)} \frac{\Gamma(\alpha+\gamma+r+s+1)}{\Gamma(\alpha+r+1)\,\Gamma(\gamma+s+1)} \frac{\Gamma(\xi+r)\,\Gamma(\eta+s)}{\Gamma(\xi+\eta+r+s)} \cdot$$

$$\cdot \frac{\Gamma(p+q+r+s-1)}{\Gamma(p+r)\,\Gamma(q+s)} \frac{\Gamma(m+n+\alpha+\beta+\gamma+\delta+r+s+2)\,\Gamma(\xi+\eta+\gamma+s)\,(-x)^r\,(-y)^s}{r\,!\,s\,!\,(m+n-r-s)\,!\,\Gamma(\alpha+\gamma+r+s+1)\,\Gamma(p+q+r+s-1)}.$$

Now it is known that [3]

$$(2.1)\quad \frac{\Gamma(\eta+v+1)}{\Gamma(\eta+1)\,\Gamma(v+1)} = \frac{2^{\mu+n}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-n)\theta i}\,\mathrm{Cos}^{\mu+n}\,\theta\, d\theta,\ (\mu+v)<-1$$

and

$$(2.2)\qquad \frac{\Gamma(\mu)\,\Gamma(v)}{\Gamma(\mu+v)} = \int_0^1 t^{\mu-1}\,(1-t)^{v-1}\,dt \qquad \mu>0,\ v>0,$$

so that

$$H_m^{(\alpha,\beta)}\,(\xi,\,p;\,x)\,H_n^{(\gamma,\delta)}\,(\mu,\,q;\,y)$$

$$= 2^{\alpha+\gamma+m+n+p+q-2}\,\frac{\Gamma(\alpha+m+1)\ \Gamma(\gamma+n\ 1)}{\pi^3\ (m+n)\,!\ \Gamma(\alpha+\gamma+1)}\ \frac{\Gamma(p)\ \Gamma(q)}{\Gamma(p+q-1)}\ \cdot$$

$$\cdot\frac{\Gamma(m+n+\alpha+\beta+\gamma+\delta+2)}{\Gamma(m+\alpha+\beta+1)\ \Gamma(n+\gamma+\delta+1)}\ \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)\ \Gamma(\eta)}$$

$$\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^{m+\alpha+\beta}\ v^{\xi-1}\ (1-u)^{\eta+\gamma+\delta)}\ (1-v)^{\eta-1}$$

$$e^{(\alpha-\gamma)\theta i+(p-q)\theta i+(m-n)\psi i}\ \mathrm{Cos}^{\alpha+\gamma}\ \theta\ \mathrm{Cos}^{p+q-2}\ \phi\ \mathrm{Cos}^{m+n}\ \psi.$$

$${}_3F_2\ [-m-n,\ m+n+\alpha+\beta+\gamma+\delta+2,\ \xi+\eta;\ \alpha+\gamma+1,\ p+q-1;\ 2\ (uvxe^{(\theta+\phi-\psi)i}$$

$$+\ (1-u)\ (1-v)\ y\ e^{-(\theta+\phi-\psi)i})\ \mathrm{Cos}\ \theta\ \mathrm{Cos}\ \phi\ \mathrm{Sec}\ \psi]\ d\theta d\phi d\psi du dv,$$

which by virtue of (1.1), yields

$$(2.3)\qquad H_m^{(\alpha,\beta)}\ (\xi,\,p;\,x)\ H_n\ \ (\eta,\ q;\ y)$$

$$= 2^{\alpha+\gamma+m+n+p+q-2}\ \frac{\Gamma(\alpha+m+1)\,\Gamma(\gamma+n+1)}{\pi^3\,(\alpha+\gamma+m+n+1)}\ \frac{\Gamma(m+n+\alpha+\beta+\gamma+\delta+2)}{\Gamma(m+\alpha+\beta+1)\ \Gamma(n+\gamma+\delta+1)}$$

$$\cdot \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)\,\Gamma(\eta)} \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q-1)} \int_0^1 \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^{m+\alpha+\beta}\, v^{\xi-1}\, (1-u)^{\eta+\gamma+\delta}\,.$$

$$.\,(1-v)^{\eta-1}\, e^{(\alpha-\gamma)\theta i+(p-q)\phi i+(m-n)\psi i}\, \mathrm{Cos}^{\alpha+\gamma}\,\theta\, \mathrm{Cos}^{p+q-2}\,\phi\, \mathrm{Cos}^{m+n}\,\psi\,.$$

$$H_{m+n}^{(\alpha+\gamma,\beta+\delta+1)}\,[\xi+\eta,\, p+q-1;\, 2\,(uvxe^{(\theta+\phi-\psi)i}+(1-v)\,ye^{(-\theta+\phi-\psi)i})$$

$$\mathrm{Cos}\,\theta\,\mathrm{Cos}\,\phi\,\sec\psi]\; d\theta d\phi d\psi\, dudv\,.$$

For $\alpha = \beta = 0 = =$, (2.3) reduces to

$$(2.4)\quad H_m(\xi, p; x)\, H_n(\eta, q; y) = (m+n+1)\, \frac{2^{m+n+p+q-2}}{\pi^3}\, \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)\,\Gamma(\eta)}\, \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q-1)}$$

$$\int_0^1 \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^m\,(1-u)^n\, v^{\xi-1}\,(1-v)^{\eta-1}\, e^{(p-q)\phi i+(mi+(m-n)\psi)i}\, +$$

$$\mathrm{Cos}^{p+q-2}\,\mathrm{Cos}^{m+n}\,\psi\, H_{m+n}^{(0,1)}\,[\xi+\eta,\, p+q-1:\, 2\,(uvxe^{(\theta+\phi-\psi)i}\, +$$

$$(1-u)\,(1-v)\,y\,e^{-(\theta+\phi-\psi)i})\,\mathrm{Cos}\,\theta\,\mathrm{Cos}\,\phi\,\mathrm{Sec}\,\psi]\; d\theta d\phi d\psi dudv.$$

I am grateful to Dr. B.L. SHARMA for his guidance during the preparation of the paper.

REFERENCES

[1] CARLITZ, L. *An integral representation for the product of two Laguerre polynomials*, Bell. Un Mat Ital., (3) Vol. 17 (1961), pp. 25-28.

[2] RICE, S.O. *Some properties of* ${}_3F_2$ *(-n, n% 1,; 1, p; v)*. Duke Math. Journal, 6, 1940, pp. 108-119.

[3] WHITTAKER, E.T. AND WATSON, G. *A course of modern analysis*. 4th ed. Cambridge (1962) pp. 263,253.

DEPARTMENT OF APPLIED SCIENCES
PUNJAB ENGINEERING COLLEGE,
CHANDIGARH (INDIA).