GENERALIZED BESSEL-FOURIER FUNCTIONS

by

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1. THE ORTHOGONAL SEQUENCE.

The method of obtaining Bessel-Fourier sequences from solution $J_n(x)$ of Bessel's equation

$$(xy')' + \left(x - \frac{n^2}{x}\right) y = 0 \tag{1}$$

is well known [5, p. 576]. The purpose of this note is to show that the method is available for solutions of the more general differential equation

$$(xy')' + \left(cx^m + \frac{k}{x}\right) y = 0 \qquad (x > 0) \tag{2}$$

where $c\,(> 0)$, k, and m are constants. If the coefficient of y is designated $p(x)$, all solutions of (2) will be oscillatory on $[1, \infty)$, when

$$\int_1^\infty p(x)\,dx = +\infty$$

(see [3, Ch. 11]). When $c = 1$, $m = 1$, $k = -n^2$, equation (2) becomes the Bessel equation.

When $m \geq 0$, equation (2) has a regular singular point at $x = 0$. Its indicial equation is

$$\varrho^2 + k = 0 \qquad (m \geq 0). \tag{3}$$

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For c, k, m fixed let $y_1(x)$ be any nonnull solution of (2) on $(0, \infty)$. If λ and μ are any two constants ($\mu \lambda \neq 0$), the functions $z = y_1(\lambda x)$ and $w = y_1(\mu x)$ satisfy the differential equations

$$(4) \qquad \begin{aligned} x \lambda z'' + \lambda z' + \lambda^2 p(\lambda x) z = 0, \\ x \mu w'' + \mu w' + \mu^2 p(\mu x) w = 0, \end{aligned}$$

respectively. If we multiply the first equation in (4) by μw and the second $by - \lambda z$, and add, we have

$$x(wz'' - zw'') + (wz' - zw') + [\lambda p(\lambda x) - \mu p(\mu x)] wz = 0.$$

Accordingly, if a and b are any two points of $(0, \infty)$, with $a < b$,

$$(5) \quad \int_a^b [x(wz'' - zw'') + (wz' - zw')]\, dx = \int_a^b [\mu p(\mu x) - \lambda p(\lambda x)] wz\, dx\cdot$$

An integration by parts yields

$$(6) \quad [x(wz' - zw')]_a^b = c(\mu^{m+1} - \lambda^{m+1}) \int_a^b x^m y_1(\lambda x) y_1(\mu x)\, dx.$$

To avoid solutions oscillating infinitely often on (0,1) we take $k \leq 0$ and set

$$k = -n^2,$$

where n is a nonnegative real number. We assume, henceforth, that $m \geq 0$. We may then set $a = 0$ and rewrite (6) as (*)

$$(7) \qquad \begin{aligned} &|x[\lambda y_1(\mu x) y_1'(\lambda x) - \mu y_1(\lambda x) y_1'(\mu x)]|_0^b \\ &= c(\mu^{m+1} - \lambda^{m+1}) \int_0^b x^m y_1(\lambda x) y_1(\mu x)\, dx, \end{aligned}$$

if we choose $y_1(x)$ to be a principal solution of (2) associated with the point $x = 0$ (**) [2, 1]. If follows that

$$\lim_{x \to 0^+} xyy' = 0.$$

(*) This result can be obtained from a formula derived by Lommel [4, p. 522].

(**) The solution $y_1(x)$ is now identifiable as $c_1 J_{\frac{n}{\theta}} \left[\frac{\sqrt{c}}{\theta} x^\theta\right]$ $(n \geq 0)$, where $2\theta = m + 1$ and c_1 is any constant $\neq 0$.

We recall that $J_n(x)$ $(n \geq 0)$ is a principal solution of Bessel's equation associated with the point $x = 0$. From (7) we have

$$(8) \qquad c(\mu^{m+1} - \lambda^{m+1}) \int_0^b x^m y_1(\lambda x) y_1(\mu x)\, dx = b[\lambda y_1(\mu b) y_1'(\lambda b) - \mu y_1(\lambda b) y_1'(\mu b)].$$

Denote by α_i $(i = 1, 2, 3, \ldots)$ the positive roots of the equation

$$y_1(\alpha b) = 0,$$

in ascending order. Note that if α_r and α_s are any two such roots, $r \neq s$,

$$(9) \qquad \int_0^b x^m y_1(\alpha_r x) y_1(\alpha_s x)\, dx = 0 \qquad (\alpha_r < \alpha_s);$$

that is, the functions $x^{\frac{m}{2}} y_1(\alpha_i x)$ $(i = 1, 2, 3, \ldots)$ are orthogonal on the interval $[0, b]$.

One can show that the integral

$$(10) \qquad \int_0^b x^m y_1^2(\lambda x)\, dx = \frac{1}{c(m+1)\lambda^m}\left[\lambda b^2 y_1'^2(\lambda b) + \left(c\lambda^m b^{m+1} - \frac{n^2}{\lambda}\right) y_1^2(\lambda b)\right]$$

in the usual way by dividing both members of (8) by $\mu - \lambda$ and taking the limit as $\mu \to \lambda$ with the aid of l'Hospital's rule. The following method is just as simple.

It is easy to verify the identity

$$(11) \qquad [(\lambda x)^2 y_1'^2(\lambda x) + \lambda x p(\lambda x) y_1^2(\lambda x)]' = \lambda[x p(\lambda x)]' y_1^2(\lambda x)$$

if we recall that

$$\lambda x y_1''(\lambda x) = -y_1'(\lambda x) - p(\lambda x) y(\lambda x).$$

Noting that

$$[x p(\lambda x)]' = c\lambda^m (m+1) x^m$$

we can integrate both members of (11) obtaining (10). Again, since $y_1(x)$ is a principal solution we have been able to employ the fact that

$$\lim_{x \to 0^+} (xy')^2 = 0, \qquad \lim_{x \to 0^+} xy^2 = 0. \text{ (*)}$$

When $\lambda = \alpha_r$ equation (10) becomes

$$\int_0^b x^m y_1^2(\alpha_r x)\, dx = \frac{b^2 y_1'^2(\alpha_r b)}{c(m+1)\alpha_r^{m-1}}. \tag{12}$$

In the case of Bessel functions it is customary to take $b = 1$.

If $f(x)$ is, say, continuous on the interval $[0, b]$, one can obtain a formal expansion of $f(x)$ as a series

$$f(x) \sim \sum_1^\infty a_i x^{\frac{m}{2}} y_1(\alpha_i x),$$

in the usual way. It is not difficult to verify that

$$a_i = \frac{c(m+1)\alpha_i^{m-1}}{b^2 y_1'^2(\alpha_i b)} \int_0^b x^{\frac{m}{2}} f(x) y_1(\alpha_i x)\, dx \qquad (i = 1, 2, 3, \ldots).$$

It is clear that $y_1'(\alpha_i b) \neq 0$.

2. Power series expansion of the principal solution.

We now suppose that m is an integer ≥ 0. In this case, according to the classical theory of the regular singular point, there exists a power series expansion of the principal solution of the form

$$y_1(x) = x^n \sum_0^\infty a_i x^i \qquad (a_0 \neq 0),$$

where $k = -n^2$, and $n \geq 0$. When $m \geq 1$, the usual calculation yields

$$a_1 = a_2 = \ldots = a_m = 0,$$

$$a_{i+m+1} = \frac{-c a_i}{(i+m+1)(i+m+2n+1)} \qquad (i = 0, 1, 2, \ldots).$$

(*) To see this easily, consider two cases $n > 0$ and $n = 0$.

These become the coefficients of $J_n(x)$ when $c = m = 1$. When $m = 0$, we have

$$a_i = \frac{-c\,a_{i-1}}{i\,(i + 2n)} \qquad (i = 1,\ 2,\ 3,\ \ldots),$$

with a_0 arbitrary but $\neq 0$ as in the first case. The test-ratio test applied to the series $\sum_0^\infty a_i x^i$ yields the fact that the series converges for all x [as it must, of course, from the form of (2)].

The case $m = n = 0$. When $m = n = 0$, it is readily verified that

$$a_i = (-1)^i \frac{c^i a_0}{(i!)^2} \qquad (i = 1,\ 2,\ \ldots)$$

with a_0 arbitrary but $\neq 0$. In this case, equations (9) and (12) become, respectively,

$$\int_0^b y_1(\alpha_r x)\, y_1(\alpha_s x)\, dx = 0 \qquad (\alpha_r < \alpha_s),$$

$$\int_0^b y_1^2(\alpha_r x)\, dx = \frac{b^2 \alpha_r y_1'^2(\alpha_r b)}{c},$$

and the functions $y_1(\alpha_j x)$ $(j = 1,\ 2,\ 3,\ \ldots)$ are orthogonal on the interval $[0,\ b]$.

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