A REMARK ON THE STALKS IN THE L-RECIPROCAL IMAGE

por

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We have showed in a recent paper [4] the natural morphism between the stalks in the reciprocal image is an isomorphism. The object of this note is to extent this result for the case of l-reciprocal image, defined in [3].

We have taken — without essential modifications — the proof of [4], ameliorating and shorting the exposition.

We shall follow [1] and [2] in all the definitions and terminology not explicitly given. Through all the note the presheaves will take values on a category K. If E is a topological space, $\mathcal{R}(E)$ will be the family of the open sets.

1. PRELIMINARIES

1.1. — Let, $\psi: X \to Y$, be a continuous map and $\mathcal{B}$ a presheaf defined over Y. $\{\psi^1 \mathcal{B}, \omega\}$ is the left-reciprocal image of $\mathcal{B}$ if $\psi^1 \mathcal{B}$ is a presheaf over X, ω an element of $H(\mathcal{B}, \psi_* \psi^1 \mathcal{B})$ such that the map of Hom $(\psi^1 \mathcal{B}, \mathcal{F})$ in Hom $(\mathcal{B}, \psi_* \mathcal{F}$ defined by, $f \to (\psi_0 f) \circ \omega$ is $(1-1)$ for every presheaf, $\mathcal{F}$, defined over X [3].

1.2. — Let X be a topological space, $x \in X$, $f \in$ Hom (A, B). There exists a sheaf, $\mathcal{H}$, given by, $\mathcal{H}\, U = A$ if $x \in U \in \mathcal{R}(X)$, $\mathcal{H}\, U = B$ if $x \notin U \in \mathcal{R}(X)$, and $\mathcal{H} \varrho^U_{U'}$, the unit morphism if $x \in U'$ or $x \notin U$, and $H \varrho^U_{U'} = f$ in the other cases. It is immediately seen that $\mathcal{H}$ is a sheaf (s. [4], **1.3**). We shall note this sheaf with $\mathcal{H}(x, A, B, f)$.

1.3. — If $\psi: X \to Y$ is a continuous map, with the definition of **1.2** it is easy to prove that, $\psi_* H(x, A, B, f) = \mathcal{H}(\psi(x), A, B, f)$. It is also immediate the stalk of $\mathcal{H}(x, A, B, f)$ in x is the object A, being the unities the morphisms composing the inductive limit.

1.4. — The morphisms of $\mathcal{H}(x, A, B, f)$ in $\mathcal{H}(x', A', B', f')$ are the elements (a, b) belonging to $\operatorname{Hom}(A, A') \times \operatorname{Hom}(B, B')$ such that, $f' a = b f$.

1.5. — If $\mathcal{F}$ is a presheaf defined over X and $\{\mathcal{F}_x, \mathcal{F}\varrho_x^U\}_{x \in U \in \mathcal{R}(X)}$ is the stalk of $\mathcal{F}$ in x, the relations, $\mathcal{F}\varrho_\phi^U = \mathcal{F}\varrho_\phi^{U'} \mathcal{F}\varrho_{U'}^U$ for $x \in U \subset U' \in \mathcal{R}(X)$ and the concept of inductive limit involve the existence of a morphism, $\mathcal{F}\varrho_\phi^x : \mathcal{F}_x \to \mathcal{F}\phi$, uniquely determinated by the relations, $\mathcal{F}\varrho_\phi^U = F\varrho_\phi^x \mathcal{F}\varrho_x^U$ for $x \in U \in \mathcal{R}(X)$.

1.6. — We shall note with $\mathcal{H}(y, \mathcal{F}_x)$ the sheaf $\mathcal{H}(y, \mathcal{F}_x, \mathcal{F}\phi, \mathcal{F}\varrho_\phi^x)$ (s. **1.4**).

1.7. — The morphisms, $\mathcal{F}\varrho_\phi^U$ and $\mathcal{F}\varrho_x^U$ define a morphism of $\mathcal{F}$ in $\mathcal{H}(y, \mathcal{F}_x)$ (s. 1.4) that we shall note with $\alpha(y, \mathcal{F}_x)$.

1.8. — If $\eta \in \operatorname{Hom}$ $(\mathcal{F}, \mathcal{B}$, being $\mathcal{F}$ and $\mathcal{B}$ presheaves over X, $x \in X$ and $\{\mathcal{F}_x, \mathcal{F}\varrho_x^U\}$, $\{\mathcal{B}_x, \mathcal{B}\varrho_x^U\}$ are the respective stalks of $\mathcal{F}$ and $\mathcal{B}$ in x, the morphisms η_x and $\eta\,\phi$ define a morphism, $\alpha(y, \eta_x)$ of $\mathcal{H}(y, \mathcal{F}_x)$ in $\mathcal{H}(y, \mathcal{B}_x)$, by reason that the relations, for $x \in U \in \mathcal{R}(X)$, $\mathcal{B}\varrho_\phi^x \eta_x \mathcal{F}\varrho_x^U = \mathcal{B}\varrho_\phi^x \mathcal{B}\varrho_x^U \eta\, U = \mathcal{B}\varrho_\phi^U \eta\, U = \eta\,\phi\,\mathcal{F}\varrho_\phi^U = \eta\,\phi\,\mathcal{F}\varrho_\phi^x \mathcal{F}\varrho_x^U$ and the concept of inductive limit imply $\mathcal{B}\varrho_\phi^x \eta_x = \eta\,\phi\,\mathcal{F}\varrho_\phi^x$. We remark that, $\psi_* \alpha(y, \eta_x) = \alpha(\psi(y), \eta_x)$ for any continuous map, ψ, as follows immediately from definitions.

1.9. — Let X, Y, be topological spaces, $\psi : X \to Y$ a continuous map, $\mathcal{B}$ a presheaf defined over Y, $\{\psi^1 \mathcal{B}, \omega\}$ the 1-reciprocal image of $\mathcal{B}$. If $\psi_x : \psi_* \psi^1 \mathcal{B}_{\psi(x)} \to \psi^1 \mathcal{B}_x$ is the natural morphism between the stalks, defined by the relations, $\psi_x \cdot \psi_* \psi^1 \mathcal{B}\varrho_{\psi(x)}^V = \psi^1 \mathcal{B}\varrho_x^{\psi^{-1}V}$ for $\psi(x) \in V \in \mathcal{R}(Y)$, and we have in mind the relations for $\psi(x) \in V \in \mathcal{R}(Y)$, $\psi_* \psi^1 \mathcal{B}\varrho_\phi^{\psi(x)} \cdot \psi_* \psi^1 \mathcal{B}\varrho_{\psi(x)}^V = \psi_* \psi^1 \mathcal{B}\varrho_\phi^V = \psi^1 \mathcal{B}\varrho_\phi^{\psi^{-1}V} = \psi^1 \mathcal{B}\varrho_\phi^x \cdot \psi^1 \mathcal{B}\varrho_x^{\psi^{-1}V} = \psi^1 \mathcal{B}\varrho_\phi^x \cdot \psi_x \cdot \psi_* \psi^1 \mathcal{B}\varrho_{\psi(x)}^V$ and the concept of inductive limit, we find $\psi_* \psi^1 \mathcal{B}\varrho_\phi^{\psi(x)} = \psi^1 \mathcal{B}\varrho_\phi^x \cdot \psi_x$, and then, ψ_x and the unit morphism define a morphism from $\mathcal{H}(y, \psi_* \psi^1 \mathcal{B}_{\psi(x)})$ to $\mathcal{H}(y, \psi^1 \mathcal{B}_x)$ that we shall denote $\alpha(y, \psi_x)$. It is easyly shown that $\psi_* \alpha(y, \psi_x) = \alpha(\psi(y), \psi_x)$.

2. The natural morphism between the stalks

2.1. — From now on, X, Y will be topological spaces, $\psi : X \to Y$ a continuous map, $\mathcal{B}$ a presheaf defined over Y, $\{\psi^1 \mathcal{B}, \omega\}$ the 1-reciprocal image of $\mathcal{B}$, $\{\mathcal{B}_{\psi(x)}, \mathcal{B}\varrho_{\psi(x)}^V\}_{\psi(x) \in V \in \mathcal{R}(Y)}$,

$$\{\psi_* \psi^1 \mathcal{B}_{\psi(x)}, \psi_* \psi^1 \mathcal{B}\varrho_{\psi(x)}^V\}_{\psi(x) \in V \in \mathcal{R}(Y)},$$

the stalks of $\mathcal{B}$ and $\psi_* \psi^1 \mathcal{B}$ in $\psi(x)$, respectively, and

$$\{\psi^1 \mathcal{B}_x, \psi^1 \mathcal{B} \varrho_x^U\}_{x \in U \in \mathcal{R}(X)}$$

the stalk of $\psi^1 \mathcal{B}$ in x. We have to remark we only suppose the existence of this inductive limits: no other assumptions are made on the category K.

2.2. — Accordingly with **1.7**, $\alpha(\psi(x), \mathcal{B}_{\psi(x)}) : \mathcal{B} \to \mathcal{H}(\psi(x), \mathcal{B}_{\psi(x)}) = \psi_* \mathcal{H}(x, \mathcal{B}_{\psi(x)})$ and taking into account the definition of 1-reciprocal image, there is a morphism, $\beta : \psi^1 \mathcal{B} \to \mathcal{H}(x, \mathcal{B}_{\psi(x)})$, such that, $\alpha(\psi(x), \mathcal{B}_{\psi(x)}) = \psi_* \beta \circ \omega$. The morphism β origins a morphism, $\beta_x : \psi^1 \mathcal{B}_x \to [\mathcal{H}(x, \mathcal{B}_{\psi(x)})]_x = \mathcal{B}_{\psi(x)}$, uniquely determinated by the relations, $\beta_x \psi^l \mathcal{B} \varrho_x^U = \beta U$, for $x \in U$.

2.3. — For $\psi(x) \in V \in \mathcal{R}(Y)$, we have, $\beta_x \cdot \psi_x \cdot \omega_{\psi(x)} \cdot \mathcal{B}\varrho_{\psi(x)}^V = \beta_x \cdot \psi_x \cdot \psi_* \psi^1 \mathcal{B} \varrho_{\psi(x)}^V \cdot \omega V = \beta_x \cdot \psi^1 \mathcal{B}\varrho_x^{\psi^{-1}V} \cdot \omega V = \beta \psi^{-1} V \cdot \omega V = (\psi_* \beta \circ \omega) V = \alpha(\psi(x), \mathcal{B}_{\psi(x)}) V = \mathcal{B} \varrho V_{\psi(x)}^V$, and accordingly with the concept of inductive limite, $\beta_x \cdot \psi_x \cdot \omega_{\psi(x)}$ is the unit morphism.

2.4. — Let we take the morphism, $\gamma = \alpha(\psi(x), \psi_x) \circ \alpha(\psi(x), \omega_{\psi(x)}) \circ \circ \alpha(\psi(x), \beta_x) \circ \psi_* \alpha(x, \psi^1 \mathcal{B}_x) \circ \omega : \mathcal{B} \to \mathcal{H}(\psi(x), \psi^1 \mathcal{B}_x)$, that can be expressed in the form, $\gamma = \psi_* [\alpha(x, \psi_x) \circ \alpha(x, \omega_{\psi(x)}) \circ \alpha(x, \beta_x) \circ \circ \alpha(x, \psi^1 \mathcal{B}_x)] \circ \omega$. If $\psi(x) \in V \in \mathcal{R}(Y)$, $\gamma V = \psi_x \cdot \omega_{\psi(x)} \cdot \beta_x \cdot \cdot \psi^1 \mathcal{B} \varrho_x^{\psi^{-1} V} \cdot \omega V = \psi_x \cdot \omega_{\psi(x)} \cdot \beta \psi^{-1} V \cdot \omega V = \psi_x \cdot \omega_{\psi(x)} \cdot \mathcal{B} \varrho_{\psi(x)}^V = \psi_x \cdot \psi_* \psi^1 \mathcal{B} \varrho_{\psi(x)}^V \cdot \omega V = \psi^1 \mathcal{B} \varrho_x^{\psi^{-1} V} \cdot \omega V = [\psi_* \alpha(x, \psi^1 \mathcal{B}_x) \circ \omega] V$. If $\psi(x) \notin V \in \mathcal{R}(Y)$, $\gamma V = \omega \phi \cdot \beta \phi \cdot \psi^1 \mathcal{B} \varrho_\phi^{\psi^{-1} V} \cdot \omega V = \omega \phi \cdot \beta \phi \cdot \omega \phi \cdot \cdot \mathcal{B} \varrho_\phi^V = \omega \phi \cdot (\psi_* \beta \circ \omega) \phi \cdot \mathcal{B} \varrho_\phi^V = \omega \phi \cdot \mathcal{B} \varrho_\phi^V = \psi_* \psi^1 \mathcal{B} \varrho_\phi^V \cdot \omega V = \{\psi_* \alpha(x, \psi^1 \mathcal{B}_x) \circ \omega] V$. Hence, $\psi_* \alpha(x, \psi^1 \mathcal{B}_x) \circ \omega = \psi_* [\alpha(x, \psi_x) \circ \alpha(x, \omega_{\psi(x)}) \circ \alpha(x, \beta_x) \circ \alpha(x, \psi^1 \mathcal{B}_x)] \circ \omega$,

and taking into account the definition of 1-reciprocal image, we find, $\alpha(x, \psi^1 \mathcal{B}_x) = \alpha(x, \psi_x) \circ \alpha(x, \omega_{\psi(x)}) \circ \alpha(x, \beta_x) \circ \alpha(x, \psi^1 \mathcal{B}_x)$, and so, for $x \in U \in \mathcal{R}(X)$, we have, $\psi^1 \mathcal{B} \varrho_x^U = \psi_x \cdot \omega_{\psi(x)} \cdot \beta_x \cdot \psi^1 \mathcal{B} \varrho_x^U$, and by the definition of inductive limit, $\psi_x \cdot \omega_{\psi(x)} \cdot \beta_x$ has to be an anity.

2.5. — From **2.3** and **2.4** follows that the morphism $\psi_x \cdot \omega_{\psi(x)}$ — the natural morphism between the stalks — is an isomorphism.

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