SUMS OF ARITHMETIC FUNCTIONS

by

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1. By an arithmetic function will be meant a function from the positive integers to the complex numbers. An arithmetic function f is said to be *factorable* if $f(1) = 1$ and

$$f(ab) = f(a)f(b) \qquad ((a,\ b) = 1). \tag{1.1}$$

An arithmetic function g is said to be *additive* if

$$g(a+b) = g(a) + g(b) \qquad ((a,\ b) = 1); \tag{1.2}$$

thus necessarily $g(1) = 0$.

It is well known (see for example [2, pp. 101-113]) that if $r_s(n)$ denotes the number of representations of n as a sum of s squares,

$$n = x_1^2 + x_2^2 + \dots + x_s^2,$$

then $r_2(n)$, $r_4(n)$, $r_6(n)$, $r_8(n)$ are "essentially" factorable functions of n. In particular

$$\begin{cases} r_4(n) = 8\sigma(n) & (n \text{ odd}) \\ r_4(2_k n) = 24\sigma(n) & (n \text{ odd},\ k \geq 1), \end{cases}$$

where $\sigma(n)$ denotes the sum of the divisors of n. On the other hand [1, Ch. 9]

$$r_{24}(n) = \frac{16}{691}\,\sigma^*{}_{11}(n) + \frac{128}{691}\left\{ (-\,1)\ 259\ \tau\ (n) - 512\ \tau\ (\tfrac{1}{2}\ n) \right\},$$

where

$$\sigma^*{}_{11}(n) = \sum_{d \mid n} d^{11} \qquad (n \text{ odd})$$

$$\sigma^*{}_{11}(n) = \sum_{\substack{d \mid n \\ d \text{ even}}} d^{11} - \sum_{\substack{d \mid n \\ d \text{ odd}}} d^{11} \qquad (n \text{ even})$$

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and $\tau(n)$ is the RAMANUJAN function, which is known to be factorable.

In view of the above it is of interest to study equations of the type

$$h(a) = Af(a) + Bg(a), \tag{1.3}$$

where f, g are factorable and A, B are nonzero constants. It is convenient to first treat the case $B = -A$, so that $h(1) = 0$. This case is covered by Theorem 1 below. This theorem is then applied to the case $A + B \neq 0$; the final result is contained in Theorem 2.

We next discuss the equation

$$h(a) = Af(a)g(a), \tag{1.4}$$

where f, g are additive and A is a nonzero constant; the result is contained in Theorem 3. Finally we treat the mixed equations

$$h(a) - k(a) = f(a)g(a), \tag{1.5}$$

where h, k are factorable and f, g are additive and

$$h(a) = f(a) - g(a), \tag{1.6}$$

where h is additive and f, g are factorable.

We hope on a later occasion to consider equations of the type

$$h(a) = A_1 f_1(a) + \dots + A_k f_k(a),$$

where $f_1, \dots, f_k$ are factorable, for arbitrary $k \geq 2$.

2. We consider first the equation

$$h(a) = A(f(a) - g(a)), \tag{2.1}$$

where f, g are factorable and A is a nonzero constant. Clearly it is necessary that

$$h(1) = 0. \tag{2.2}$$

From (2.1) and the factorability of f and g we have

$$h(ab) = A[f(a)f(b) - g(a)g(b)] \qquad ((a, b) = 1). \tag{2.3}$$

We have also

$$h(b) = A(f(b) - g(b)). \tag{2.4}$$

Eliminating $g(a)$, $g(b)$ from (2.1), 2.3), (2.4) we get

$$A[h(b)f(a) + h(a)f(b)] = Ah(ab) + h(a)h(b) \quad ((a, b) = 1). \tag{2.5}$$

Similarly we have

$$A[h(c)f(a) + h(a)f(c)] = Ah(ac) + h(a)h(c) \qquad ((a, c) = 1),$$

$$A[h(c)f(b) + h(b)f(c)] = Ah(bc) + h(b)h(c) \qquad ((b, c) = 1).$$

From the last two equations we get

$$h(c)[h(b)f(a) - h(a)f(b)] = h(ac)h(b) - h(bc)h(a), \tag{2.6}$$

where

$$(a, b) = (a, c) = (b, c) = 1.$$

If $(a, d) = (b, d) = 1$ we have also

$$h(d)[h(b)f(a) - h(a)f(b)] = h(ad)h(b) - h(bd)h(a). \tag{2.7}$$

From (2.6) and (2.7) we get

$$[h(ac)h(b) - h(bc)h(a)]h(d) = [h(ad)h(b) - h(bd)h(a)]h(c), \tag{2.8}$$

where

$$(a, b) = (cd, ab) = 1. \tag{2.9}$$

Note that c and d need not be relatively prime. Also we may assume that

$$a > 1, \quad b > 1, \quad c > 1, \quad d > 1,$$

for if any of the parameters is equal to 1, it is easily verified that (2.8) reduces to an identity.

We note that (2.8) can also be written in the form

$$(2.10)\quad h(ac)h(b)h(d) + h(bd)h(a)h(c) = h(ad)h(b)h(c) + h(bc)h(a)h(d).$$

This is then a *necessary* condition that $h(a)$ satisfy (2.1).

In the next place, from (2.5) and (2.6) we get

$$(2.11)\quad 2Ah(b)h(c)f(a) = A[h(ab)h(c) + h(ac)h(b) - h(bc)h(a)] + \\ + h(a)h(b)h(c),$$

and

$$(2.12)\quad 2Ah(b)h(c)g(a) = A[h(ab)h(c) + h(ac)h(b) - h(bc)h(a)] - \\ - h(a)h(b)h(c),$$

where

$$(2.13)\qquad (a,\ b) = (a,\ c) = (b,\ c) = 1.$$

In (2.11) replace c by d, where $(ab,\ d) = 1$, so that

$$(2.14)\quad 2Ah(b)h(d)f(a) = A[h(ab)h(d) + h(ad)h(b) - h(bd)h(a)] + \\ + h(a)h(b)h(d).$$

Eliminating $f(a)$ from (2.11) and (2.12) we get

$$[h(ab)h(c) + h(ac)h(b) - h(bc)h(a)[h(d)$$
$$= [h(ab)h(d) + h(ad)h(b) - h(bd)h(a)]h(\mathrm{c}),$$

which reduces to

$$[h(ac)h(b) - h(bc)h(a)]h(d) = [h(ad)h(b) - h(bd)h(a)]h(c).$$

This is identical with (2.8). Thus when (2.8) is satisfied (2.11) and (2.14) are equivalent. Similarly if we replace b or c in (2.11) by any other numbers that satisfy (2.13).

3. We shall now obtain a second necessary contition. Returning to (2.1), replace a by b, c, d, where the four numbers are relatively prime in pairs. Then we have

$$h(a)h(b)h(c)h(d) = A^4(f(a) - g(a))(f(b) - g(b))(f(c) - g(c))(f(d) - g(d))$$

$$= A^4[f(abcd) + g(abcd)]$$

$$- A^4\Sigma[f(bcd)g(a) + f(a)g(bcd)]$$

$$+ A^4\Sigma[f(ab)g(cd) + f(cd)g(ab)].$$

On the other hand, (2.1) also implies

$$h(a)h(b) = A^2[f(ab) + g(ab)] - A^2[f(a)g(b) + f(b)g(a)].$$

Hence (3.1) gives

$$h(a)h(b)h(c)h(d) = A^4[f(abcd) + g(abcd)]$$

$$- A^2\Sigma\{A^2[f(abcd) + g(abcd)] - h(bcd)h(a)\}$$

$$+ A^2\Sigma\{A^2[f(abcd) + g(abcd)] - h(ab)h(cd)\}.$$

This reduces to

$$h(a)h(b)h(c)h(d) = A^2\{\Sigma h(a)h(bcd) - \Sigma h(ab)h(cd)\}. \tag{3.2}$$

We assume in what follows that there exist values of a, b, c, d relatively prime in pairs such that

$$h(a)h(b)h(c)h(d) \neq 0. \tag{3.3}$$

Then A^2 is uniquely determined by (3.2). Another way of putting it is that the quotient

$$\frac{\Sigma h(a)h(bcd) - \Sigma h(ab)h(cd)}{h(a)h(b)h(c)h(d)} \tag{3.4}$$

is constant (for all quadruples a, b, c, d, relatively prime in pairs, that satisfy (3.3)).

4. We now assume that (2.2), (2.10) and (3.3) are satisfied and that A^2 is determined by (3.2). Then the functions f, g are given by (2.11) and (2.12). It remains to show that f and g are factorable. If h is replaced by $-h$ (2.2), (2.10), (3.2), (3.3) remain unchanged while f and g are interchanged. It therefore suffices to show that f is factorable.

In place of (2.11) we take

$$(4.1) \quad 2Ah(c)h(d)f(a) = A[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)] + \\ + h(a)h(c)h(d),$$

which is indeed equivalent to (2.11). We have also

$$(4.2) \quad 2Ah(c)h(d)f(b) = A[h(bc)h(d) + h(bd)h(c) - h(b)h(cd)] + \\ + h(b)h(c)h(d),$$

$$(4.3) \quad 2Ah(c)h(d)f(ab) = A[h(abc)h(d) + h(abd)h(c) - h(ab)h(cd)] + \\ + h(ab)h(c)h(d).$$

It is assumed that a, b, c, d are relatively prime in pairs.

By (4.1), (4.2), (4.3) and (3.2) we get

$$4A^2h^2(c)h^2(d)[f(a)f(b) - f(ab)]$$

$$= \{A[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)] + h(a)h(c)h(d)\}$$
$$. \{A[h(bc)h(d) + h(bd)h(c) - h(b)h(cd)] + h(b)h(c)h(d)\}$$
$$- 2Ah(c)h(d)\{A[h(abc)h(d) + h(abd)h(c) - h(ab)h(cd)] + \\ + h(ab)h(c)h(d)\}$$
$$= A^2\{[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)]\ [h(bc)h(d) + h(bd)h(c) - \\ - h(b)h(cd)_1$$
$$- 2h(c)h(d)[h(abc)h(d) + h(abd)h(c) - h(ab)h(cd)]\}$$
$$+ Ah(c)h(d)\{[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)[h(b)$$
$$+ [h(bc)h(d) + h(bd)h(c) - h(b)h(cd)]h(a) - 2h(ab)h(c)h(d)\}$$
$$+ A^2h(c)h(d)\ \{\Sigma h(a)h(bcd)\text{-}\Sigma h(ab)h(cd)\}.$$

The coefficient of A is equal to

$$[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)]h(b) + [h(bc)h(d) + h(bd)h(c) - h(b)h(cd)]h(a) - 2h(ab)h(c)h(d)$$

$$= \{h(ac)h(b)h(d) + h(bd)h(a)h(c) - h(ab)h(c)h(d) - h(cd)h(a)h(b)\}$$

$$+ \{h(ad)h(b)h(c) + h(bc)h(a)h(d) - h(ab)h(c)h(d) - h(cd)h(a)h(b)\} = 0,$$

by (2.10).

As for the coefficient of A^2 we have

$$[h(ac)h(d) + h(ad)h(c) - h(a)h(cd)][h(bc)h(d) + h(bd)h(c) - h(b)h(cd)]$$

$$- 2h(c)h(d)[h(abc)h(d) + h(abd)h(c) - h(ab)h(cd)]$$

$$+ h(c)h(d) \ \{\Sigma h(a)h(bcd) - \Sigma h(ab)h(cd)\}$$

$$= h(c)h(d)\{h(a)h(bcd) + h(b)h(acd) - h(c)h(abd) - h(d)h(abc)\}$$

$$+ h(c)h(d)\{h(ab)h(cd) - h(ac)h(bd) - h(ad)h(bc)\}$$

$$+ h(c)h(d)\{h(ac)h(bd) + h(ad)h(bc)\} + h(ac)h(bc)h^2(d) + h(ad)h(bd)h^2(c)$$

$$- h(a)h(cd)[h(bc)h(d) + h(bd)h(c)]$$

$$- h(b)h(cd)[h(ac)h(d) + h(ad)h(c)] + h(a)h(b)h^2(cd)$$

$$= h(d)\{h(acd)h(b)h(c) + h(ac)h(bc)h(d) - h(abc)h(c)h(d) - h(ac)h(cd)h(b)\}$$

$$+ h(c)\{h(bcd)h(a)h(d) + h(ad)h(bd)h(c) - h(abd)h(d)(c) - h(bd)h(cd)h(a)\}$$

$$+ h(cd)\{h(ab)h(c)h(d) + h(cd)h(a)h(b) - h(ad)h(b)h(c) - h(bc)h(a)h(d)\}.$$

Each of the quantities in braces vanishes by (2.10). This is obvious for the third; for the first we use the quadruple b, d, c, ac while for the second we use a, c, d, bd.

We have proved

$$f(a)f(b) = f(ab) \qquad ((a,\ b) = 1).$$

This completes the proof of

THEOREM 1. *Let* $h(a)$ *be an arithmetic function such that*

(i) $$h(1) = 0,$$

(ii) $$h(ac)h(b)h(d) + h(bd)h(a)h(c) = h(ad)h(b)h(c) + h(bc)h(a)h(d),$$

where

$$(a, b) = (cd, ab) = 1,$$

(iii) $$h(a)h(b)h(c)h(d) = A^2\{\Sigma h(a)h(bcd) - \Sigma h(ab)h(cd)\},$$

where A *is constant and* a, b, c, d *are relatively prime in pairs; moreover*

(iv) $$h(a)h(b)h(c)h(d) \neq 0$$

for at least one quadruple a, b, c, d *relatively prime in pairs. Then*

$$h(a) = A[f(a) - g(a)],$$

where f, g *are factorable functions.*

5. We now consider the equation

(5.1) $$h(a) = Af(a) + Bg(a),$$

where f, g are factorable and A, B are constants such that

(5.2) $$AB(A + B) \neq 0.$$

Note that

(5.3) $$h(1) = A + B \neq 0.$$

If $(a, b) = 1$ it follows from (5.1) that

$$h(b) = Af(b) + Bg(b),\ h(ab) = Af(a)f(b) + Bg(a)g(b),$$

so that

$$Bh(ab) = ABf(a)f(b) + [h(a) - Af(a)][h(b) - Af(b)],$$

or

$$A(A + B)f(a)f(b) - Ah(b)f(a) - Ah(a)f(b) = Bh(ab) - h(a)h(b).$$

This can be rewritten in the form

$$(5.4)\quad A[A+B)f(a)-h(a)][A+B)f(b)-h(b)]=(A+B)Bh(ab)-Bh(a)h(b).$$

Since

$$(A+B)f(a)-h(a)=B[f(a)-g(a)],$$

(5.4) becomes

$$(5.5)\quad AB\,\phi(a)\,\phi(b)=h(1)h(ab)-h(a)h(b),$$

where

$$(5.6)\quad \phi(a)=f(a)-g(a).$$

If $(ab, c)=1$ then in addition to (5.5) we have

$$(5.7)\quad \begin{cases} AB\,\phi(a)\,\phi(c)=h(1)h(ac)-h(a)h(c),\\ AB\,\phi(b)\,\phi(c)=h(1)h(bc)-h(b)h(c).\end{cases}$$

For brevity put

$$(5.8)\quad \Delta(a, b)=h(1)h(ab)-h(a)h(b).$$

Then (5.5), (5.7) become

$$(5.9)\quad AB\,\phi(a)\,\phi(b)=\Delta(a, b),\ AB\,\phi(a)\,\phi(c)=\Delta(a, c),\ AB\,\phi(b)\,\phi(c)=\Delta(b, c),$$

where

$$(a, b)=(a, c)=(b, c)=1.$$

From (5.9) we get

$$(5.10)\quad (AB)^3\,\phi^2(a)\,\phi^2(b)\,\phi^2(c)=\Delta(b, c)\,\Delta(c, a)\,\Delta(a, b).$$

Now it is easily verified that

$$\begin{aligned}(A+B)^2h(abc)-h(a)h(b)h(c)-\Sigma h(a)\,\Delta(b, c)\\ =-AB(A-B)\,\Pi\,[f(a)-g(a)]\\ =-AB(A-B)\,\phi(a)\,\phi(b)\,\phi(c).\end{aligned}$$

Hence (5.10) gives

$$\{h^2(1)h(abc) - h(1)\Sigma h(a)h(bc) + 2h(a)h(b)h(c)\}^2 = \lambda \Delta(b, c)\Delta(c, a)\Delta(a,b) \tag{5.11}$$

where

$$\lambda = \frac{(A - B)^2}{AB} = \frac{h^2(1)}{AB} - 4. \tag{5.12}$$

A relation simpler that (5.11) can be obtained. For a, b, c, d relatively prime in pairs we have

$$h(1)h(abcd) - \Sigma h(a)h(bcd) + \Sigma h(ab)h(cd)$$

$$= (A+B)[Af(abcd)+Bg(abcd)] - \Sigma[Af(a)+Bg(a)][Af(bcd)+Bg(bcd)]$$
$$+ \Sigma[Af(ab) + Bg(ab)][Af(cd) + Bg(cd)]$$
$$= AB[f(abcd) + g(abcd)] - AB\Sigma[f(a)g(bcd) + g(a)f(bcd)]$$
$$+ AB\Sigma[f(ab)g(cd) + g(ab)f(cd)].$$

On the other hand

$$\phi(a)\,\phi(b)\,\phi(c)\,\phi(d) = \Pi\,(f(a) - g(a))$$
$$= f(abcd) + g(abcd) - \Sigma\,[f(a)g(bcd) + g(a)f(bcd)]$$
$$+ \Sigma[f(ab)g(cd) + g(ab)f(cd)]$$

and therefore

$$AB\,\{h(1)h(abcd) - \Sigma h(a)h(bcd) + \Sigma h(ab)h(cd)\} = \Delta(a, c)\Delta(b, d), \tag{5.13}$$

where a, b, c, d, are relatively prime in pairs.

Returning to (5.9) it is evident that

$$AB\,\phi^2(a) = \frac{\Delta(a, b)\Delta(a, c)}{\Delta(b, c)}, \tag{5.14}$$

where $(a, b) = (ab, c) = 1$. Hence if $(ab, d) = 1$ we have also

$$AB\,\phi^2(a) = \frac{\Delta(a,b)\ \Delta(a, d)}{\Delta(b, d)}.$$

It follows that

(5.15) $$\Delta(a, c)\Delta(b, d) = \Delta(a, d)\Delta(b, c)$$

provided

(5.16) $$(a, b) = (ab, cd) = 1.$$

Note that $(c, d) = 1$ is not assumed.

6. We now assume that $h(a)$ satisfies (5.13) and (5.15) and that

(6.1) $$\Delta(a, c)\Delta(b, d) \neq 0$$

for some quadruple a, b, c, d relatively prime in pairs. Then AB is determined by (5.13); this together with $A + B = h(1)$ gives A and B. The function $\phi(a)$ is given by)5·14). We now apply Theorem 1 to ϕ. It will accordingly suffice to show that

(6.2) $$\phi(ac)\,\phi(b)\,\phi(d) + \phi(db)\,\phi(a)\,\phi(c)$$
$$= \phi(ad)\,\phi(b)\,\phi(c) + \phi(bc)\,\phi(a)\,\phi(d)$$

where

(6.3) $$(a, b) = (ab, cd) = 1$$

and

(6.4) $$\Sigma\,\phi(a)\,\phi(bcd) - \Sigma\,\phi(ab)\,\phi(cd) = \phi(a)\,\phi(b)\,\phi(c)\,\phi(d) \neq 0,$$

where a, b, c, d are relatively prime in pairs.

We show first that (6.1) implies (6.4). Since

$$AB\,\phi(a)\,\phi(c) = \Delta(a, c), \quad AB\,\phi(a)\,\phi(d) = \Delta(a, d),$$
$$AB\,\phi(ac)\,\phi(b) = \Delta(ac, b), \quad AB\,\phi(bd)\,\phi(a) = \Delta(bd, a),$$
$$AB\,\phi(ad)\,\phi(b) = \Delta(ad, b), \quad AB\,\phi(bc)\,\phi(a) = \Delta(bc, a),$$

(6.2) becomes

(6.5) $$[\Delta(ac, b) - \Delta(bc, a)]\,\Delta(a, d) = [\Delta(ad, b) - \Delta(bd, a)]\Delta(a,c)$$

But

$$h(1)[\Delta(ac, b) - \Delta(bc, a)] = h(1)[h(bc)h(a) - h(ac)h(b)]$$
$$= h(a)\Delta(b, c) - h(b)\,\Delta\,(a, c),$$
$$h(1)[\Delta(ad, b) - \Delta(bd, a)] = h(a)\Delta(b, d) - h(b)\Delta(a, d),$$

so that (6.5) reduces to

$$[h(a)\Delta(b, c) - h(b)\Delta(a, c)]\Delta(a, d) = h(a)\,[\Delta(b, d) - h(b)\Delta(a, d)]\Delta(a, c).$$

This is an obvious consequence of (5.15).

In the next place we may replace (6.4) by

$$AB\{\Sigma\Delta(a, bcd) - \Sigma\Delta(ab, cd)\} = \Delta(a, c)\Delta(b, d)$$

which in turn may be replaced by

$$AB\{h(1)h(abcd) - \Sigma h(a)h(bcd) + \Sigma h(ab)h(cd)\} = \Delta(a, c)\Delta(b, d).$$

This is evidently identical with (5.13).

This completes the proof of

THEOREM 2. *Let* h(1) = A + B ≠ 0 *and let* a, b, c, d *be four numbers that are relatively prime in pairs. Assume that*

(i) $$\Delta(a, c)\Delta(b, d) \neq 0$$

for at least one such quadruple and that

(ii) $$AB\{h(1)h(abcd) - \Sigma h(a)h(bcd) + \Sigma h(ab)h(cd)\} = \Delta(a, c)\Delta(b, d),$$

where AB *is independent of* a, b, c, d. *Moreover assume that*

(iii) $$\Delta(a, c)\Delta(b, d) = \Delta(a, d)\Delta(b, c)$$

for

$$(a, b) = (ab, cd) = 1.$$

Then

$$h(a) = Af(a) + Bg(a),$$

where f and g are factorable.

7. We have made no use of (5.11) in proving Theorem 2. It may be of interest to point out that when (5.15) holds, (5.11) and (5.13) are equivalent. We shall now prove this assertion.

For brevity put

$$\Delta(a, b, c) = h^2(1)h(abc) - h(1)\Sigma h(a)h(bc) + 2h(a)h(b)h(c),$$

$$\Delta(a, b, c, d) = h(1)h(abcd) - \Sigma h(a)h(bcd) + \Sigma h(ab)h(cd).$$

Thus (5.11) and (5.13) become

$$\Delta^2(a, b, c) = \lambda\Delta(b, c)\Delta(c, a)\Delta(a, b) \tag{7.1}$$

and

$$AB\Delta(a, b, c, d) = \Delta(a, c)\Delta(b, d), \tag{7.2}$$

respectively.

Since

$$\Delta(a, b, c) = AB(A \rightarrow B)\ \phi\ (a)\ \phi\ (b)\ \phi\ (c),$$

$$\Delta(a, b, c, d) = AB\ \phi\ (a)\ \phi\ (b)\ \phi\ (c),$$

it follows that

$$\Delta^2(a, b, c)\ \phi^2\ (d) = (A - B)^2\Delta^2(a, b, c, d).$$

Then

$$\frac{\Delta^2(a, b, c)}{\Delta\,(b, c)\Delta\,(c, a)\Delta\,(a, b)} \quad \frac{\Delta\,(b, c)\ \phi^2\ (d)}{\Delta(b, d)\Delta(c, d)} \tag{7.3}$$

$$= (A - B)^2 \frac{\Delta^2(a, b, c, d)}{\Delta(a, b)\Delta(a, c)\Delta(b, d)\Delta(c, d)}$$

$$= (A - B)^2 \left\{\frac{\Delta\,(a, b, c, d)}{(a, c)\Delta(b, d)}\right\}^2,$$

by (5.15). Since

$$AB\ \phi^2\ (d) = \frac{\Delta(b, d)\ \Delta\ (c, d)}{\Delta(b, c)},$$

(7.3) becomes

$$\frac{\Delta^2(a, b, c)}{\Delta(b, c)\Delta(c, a)\Delta(a, b)} = A^2B^2\lambda\left\{\frac{(a, b, c, d)}{\Delta(a, c)\Delta(b, d)}\right\}^2 \tag{7.4}$$

and the asserted equivalence follows at once.

We remark that λ may vanish. This evidently occurs if and only if $A = B$. The condition (7.1) now reduces to simply

$$\Delta(a, b, c) = 0. \tag{7.5}$$

8. We now briefly discuss some problems involving additive functions. To begin with we consider the equation

$$h(a) = Af(a)g(a), \tag{8.1}$$

where f, g are additive and A is a nonzero constant. The condition

$$h(1) = 0 \tag{8.2}$$

is obviously necessary. Also there is no loss in generality in assuming that $A = 1$. We may accordingly take

$$h(a) = f(a)g(a),\ h(b) = f(b)g(b),\ h(ab) = [f(a) + f(b)][g(a) + g(b)],$$

where $(a, b) = 1$. Then

$$h(ab) - h(a) - h(b) = f(a)g(b) + f(b)g(a). \tag{8.3}$$

If $(ab, c) = 1$ then by (8.3)

$$\begin{aligned} h(abc) - h(a) - h(bc) &= f(a)[g(b) + g(c)] + [f(b) + f(c)]g(a) \\ &= \{h(ab) - h(a) - h(b)\} + \{h(ac) - h(a) - h(c)\}. \end{aligned}$$

It follows that

$$h(abc) - \Sigma h(ab) + \Sigma h(a) = 0, \tag{8.4}$$

where

$$(b,\ c) = (c,\ a) = (a,\ b) = 1. \tag{8.5}$$

Thus we have the two necessary conditions (8.2) and (8.4).

In addition we shall also assume that there exist three integers a, b, c, that satisfy (8.5) and such that

$$h(a)h(b)h(c) \neq 0. \tag{8.6}$$

We shall require the following

LEMMA. *If a, b, c satisfy* (8.5) *and* (8.6) *and* $(abc, d) = 1$, *then we can find complex numbers α, β, γ, δ such that the determinants of order two in the array*

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ h(a)/\alpha & h(b)/\beta & h(c)/\gamma & h(c)/\delta \end{bmatrix}$$

are all different from zero.

The proof of the lemma is immediate.

Now assume that the function $h(a)$ satisfies (8, 2), (8.4) and (8.6). Let a, b denote any two of the four numbers a, b, c, d in the lemma and let u, v denote any two positive integers such that the numbers a, b, u, v are relatively prime in pairs. Then by (8.3) we have

$$\begin{cases} f(a)g(u) + g(a)f(v) = h(au) - h(a) - h(u), \\ f(a)g(v) + g(a)f(v) = h(av) - h(a) - h(v), \\ f(a)g(uv) + g(a)f(uv) = h(auv) - h(a) - h(uv). \end{cases} \tag{8.7}$$

It follows from (8.7) that

$$f(a)[g(uv) - g(u) - g(v)] + g(a)[f(uv) - f(u) - f(v)]$$

$$= [h(auv) - h(a) - h(uv)] - [h(au) - h(a) - h(u)] - [(av) - h(a) - h(v)]$$

$$= h(auv) - h(au) - h(av) - h(uv) + h(a) + h(u) + h(v).$$

Hence by (8,4) we have

$$f(a)[g(uv) - g(u) - g(v)] + g(a)[f(uv) - f(u) - f(v)] = 0. \tag{8.8}$$

Similarly we have

$$f(b)[g(uv) - g(u) - g(v)] + g(b)[f(uv) - f(u) - f(v)] = 0. \tag{8.9}$$

We now define

$$f(a) = \alpha,\ g(a) = h(a)/\alpha,\ f(b) = \beta,\ g(b) = h(b)/\beta,\ \dots$$

Then by the lemma

$$\begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix} \neq 0,$$

so that (8.8) and (8.9) give

$$f(uv) = f(u) + f(v),\ g(uv) = g(u) + g(v). \tag{8.10}$$

Clearly u, v may represent any pair of relatively prime integers.

We have therefore proved

THEOREM 3. *Let* h(1) = 0 *and*

$$h(abc) - \Sigma h(ab) + \Sigma h(a) = 0 \tag{8.11}$$

for all triples a, b, c, *such that*

$$(b,\ c) = (c,\ a) = (a,\ b) = 1. \tag{8.12}$$

Also assume that there exists a triple a_1, b_1, c_1, *satisfying* (8.11) *and such that*

$$h(a_1)h(b_1)h(c_1) \neq 0.$$

Then we have

$$h(a) = f(a)g(a), \tag{8.13}$$

where f *and* g *are additive functions.*

As we have seen, (8.11) is a necessary condition for (8.13).

9. As a special case of (8.13) we may consider the equation

$$h(a) - u(a) = f(a)g(a), \tag{9.1}$$

where f, g are additive, h is factorable and u is defined by

$$u(1) = 1,\ u(a) = 0\ (a > 1). \tag{9.2}$$

Applying the necessary condition (8.11) we get

$$h(abc) - \Sigma h(ab) + \Sigma h(a) = 0, \tag{9.3}$$

where a, b, c, are relatively prime in pairs and $a > 1$, $b > 1$, $c > 1$.

Since h is factorable (9.3) becomes

$$h(abc) - \Sigma h(ab) + \Sigma h(a) = 0$$

or

$$[h(a) - 1][h(b) - 1][h(c) - 1] = 1. \tag{9.4}$$

Thus a necessary condition for (9.1) is

$$h(a) \neq 1 \qquad (a = 1). \tag{9.5}$$

Let $(ab, d) = 1$. Then by (9.4)

$$[h(a) \to 1][h(b) \to 1][h(d) - 1] = 1,$$

so that

$$h(c) = h(d). \tag{9.6}$$

It follows from (9.6) that

$$h(a) = C \neq 1 \quad (a > 1). \tag{9.7}$$

Thus (9.1) becomes

$$f(a)g(a) = C \quad (a > 1). \tag{9.8}$$

Again applying the condition (8.11) we get $C = 0$.

Hence (9.1) reduces to

$$f(a)g(a) = 0 \quad (a > 1). \tag{9.9}$$

Theorem 3 now does not apply. However the general solution of (9.9) is easily obtained. Clearly (9.9) implies

$$f(a)g(b) + g(a)f(b) = 0 \ \ ((a, b) = 1, a > 1, b > 1). \tag{9.10}$$

If we assume

$$f(a) \neq 0, f(b) = 0, g(a) = 0, g(b) \neq 0,$$

(9.10) is contradicted. Hence one of the functions f, g vanishes identically. We conclude that the general solution of (9.1) es given by

$$(9.11)\qquad \begin{cases} h = u,\ f = 0,\ g \text{ arbitrary}, \\ h = u,\ g = 0,\ f \text{ arbitrary}. \end{cases}$$

In view of this result it may be of interest to discuss the more general equation

$$(9.12)\qquad h(a) - k(a) = f(a)g(a),$$

where h, k are factorable and f, g are additive. Applying the condition (8.11) we obtain

$$(9.13)\qquad h(abc) - \Sigma h(ab) + \Sigma h(a) = k(abc) - \Sigma h(ab) + \Sigma k(a),$$

where

$$(b, c) = (c, a) = (a, b) = 1.$$

Since h and k are factorable (9.13) reduces to

$$(9.14)\qquad [h(a) - 1][h(b) - 1][h(c) - 1] = [k(a) - 1][k(b) - 1][k(c) - 1r.$$

We assume that there exist two numbers $a > 1$, $b > 1$, $(a, b) = 1$ such that

$$(9.15)\qquad [h(a) - 1]]h(b) - 1][k(a) - 1][k(b) - 1] \neq 0.$$

Let $(ab, d) = 1$. Then by (9.14) we have

$$(9.16)\qquad [h(a) - 1][k(b) - 1][h(d) - 1] = [k(a) - 1][k(b) - 1][k(d) - 1].$$

Therefore, by (9.15),

$$[h(c) - 1][k(d) - 1] = [h(d) \rightarrow 1][k(c) - 1],$$

or

$$h(c) - k(c) = h(d) - k(d).$$

This evidently implies

$$(9.17)\qquad h(c) - k(c) = A \quad ((c, ab) = 1),$$

where A is constant.

Combining (9.17) with (9.12) we get

$$f(d)g(d) = A \quad ((d,\ ab) = 1,\ d > 1). \tag{9.18}$$

Applying (8.11) once more we get $A = 0$.

We now replace (9.15) by the stronger condition

$$[h(a) - 1][h(b) - 1][h(c) - 1][k(a) - 1][k(b) - 1][k(c) - 1] \neq 0 \tag{9.19}$$

for some triple a, b, c, such that

$$(b, c) = (c, a) = (a, b) = 1,\ a > 1,\ b > 1,\ c > 1.$$

It then follows from (9.18) that

$$f(d)g(d) = A$$

for all $d > 1$ and as before $A = 0$.

We have therefore

THEOREM 4. *Consider the equation*

$$h(a) - k(a) = f(a)g(b), \tag{9.20}$$

where h, k *are factorable and* f, g *are additive. Assume that* h, k, *satisfy* (9.19). *Then*

$$h = k,\ fg = 0. \tag{9.21}$$

Finally we remark that the equation

$$h(a) = f(a) - g(a), \tag{9.22}$$

whre h is additive and f, g are factorable implies

$$[f(a) - 1][f(b) - 1] = [g(a) - 1][g(b) - 1] \quad ((a, b) = 1). \tag{9.23}$$

If we assume that

$$[f(a) - 1][f(b) - 1][(g(a) - 1][g(b) - 1] \neq 0 \tag{9.24}$$

for some pair $a > 1$, $b > 1$, $(a,\ b) = 1$, then it follows from (9.23) that

$$1 - g(c) = A(1 - f(c)), \tag{9.25}$$

for all c, where A is constant. Since (9.25) implies

$$f(c) - g(c) = (1 - A)(f(c) - 1),$$

(9.22) becomes

$$h(c) = (1 - A)(f(c) - 1).$$

By the additivity of h this gives

$$(f(c) - 1)(f(d) - 1) = 0 \quad ((c, d) = 1). \tag{9.26}$$

This evidently contradicts (9.24).

Hence (9.22) implies

$$[f(a) - 1][f(b) - 1][g(a) - 1][g(b) - 1] = 0 \quad ((a, b) = 1)$$

and so by (9.23)

$$(f(a) - 1)(f(b) - 1) = 0 \quad ((a, b) = 1). \tag{9.27}$$

Conversely if (9.23) and (9.27) are satisfied then h, as defined by (9.22), is additive.

We may state

THEOREM 5. *The equation*

$$h(a) = f(a) - g(a),$$

where h *is additive and* f, g *are factorable, is satisfied if and only if* f *and* g *satisfy*

$$(f(a) - 1)(f(b) - 1) = (g(a) - 1)(g(b) - 1) = 0$$

for all (a, b) = 1.

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