ON APPELL'S DOUBLE HYPERGEOMETRIC FUNCTIONS

by

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1. INTRODUCTION: Evaluating some of the infinite integrals involving confluent hypergeometric functions I came across a double series analogous to APPELL's double hypergeometric function of the form

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\ (b)_s\ (c)_{r+s}\ (d)_{r+s}}{(a')_r\ (b')_s\ (e)_{r+s}\ \underline{|r}\ \underline{|s}}\, x^r\, y^s \tag{1.1}$$

where as usual $(a)_r = \Gamma(a + r)/\Gamma(a)$. The series converges absolutely when $|x|, |y| < 1$.

The series (1.1) is a very special case of the KAMPE' de FERIET's hypergeometric function of two variables of higher order and in KAMPE' de FERIET's notation, the function

$$F\left[\begin{array}{c|cc|c} 2 & c, & d & \\ 1 & a, & b & x \\ 1 & e & & y \\ 1 & a', & b' & \end{array}\right] \tag{1.2}$$

In this paper, we shall study some of the properties of the series (1.1) and its relation with other known APPELL's double hypergeometric functions [3] We shall denote the series (1.1) symbolically as

$$F^{(6)}\left[\begin{matrix} c, & d: & a\ ; & b \\ & e: & a'; & b' \end{matrix}\ ;\ x,\ y\right] \tag{1.3}$$

When $d = e$, this series reduces to a well known series $F^{(2)}$, defined as

$$F^{(2)}\left[c\,;\, a,\, b\,;\, a',\, b'\,;\, x,\, y\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\,(b)_s\,(c)_{r+s}}{(a')_r\,(b')_s\,|\underline{r}\,|\underline{s}}\, x^r\, y^s \tag{1.4}$$

When we take $b = b'$ in the series (1.1), it reduces to

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\,(c)_{r+s}\,(d)_{r+s}}{(a')_r\,(e)_{r+s}\,|\underline{r}\,|\underline{s}}\, x^r\, y^s \tag{1.5}$$

which we shall symbolically denote as

$$F^{(5)}\left[\begin{matrix} c,\ d : a \\ e : a' \end{matrix}\,;\, x,\, y\right] \tag{1.6}$$

Again, when $a = a'$ in the series (1.5), it reduces to a known expansion

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_{r+s}\,(d)_{r+s}}{(e)_{r+s}\,|\underline{r}\,|\underline{s}}\, x^r\, y^s = {}_2F_1\left[c,\, d\,;\, e\,;\, x + y\right] \tag{1.7}$$

INTEGRAL REPRESENTATION OF $F^{(5)}$ and $F^{(6)}$:

For finding out the integral representation of $F^{(6)}$, we shall first find out the integral representation of $F^{(5)}$. Now, by definition we have

$$F^{(5)} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\,(c)_{r+s}\,(d)_{r+s}}{(a')_r\,(e)_{r+s}\,|\underline{r}\,|\underline{s}}\, x^r\, y^s$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\,(c)_r\,(d)_r}{(a')_r\,(e)_r\,|\underline{r}}\; {}_2F_1\left[c + r,\, d + r\,;\, e + r\,;\, y\right]\, x^r$$

Using EULER's integral

$${}_2F_1\left[a,\, b\,;\, c\,;\, z\right] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c - b)} \int_0^1 b^{b-1}\,(1 - t)^{c-b-1}\,(1 - tz)^{-a}\, dt \tag{2.1}$$

where $R(c) > R(b) > 0$ and $|\arg(1-z)| < \pi$, and on reversing the order of summation and integration, which is justified as the series is absolutely convergent for the conditions given above, we get

$$F^{(5)}\left[\begin{matrix} c, & d: & a \\ & e: & a' \end{matrix}; x, y\right] =$$

$$= \frac{\Gamma(e)}{\Gamma(d)\,\Gamma(e-d)} \int_0^1 t^{d-1}(1-t)^{e-d-1}(1-ty)^{-c}\, {}_2F_1\left[a, c; a'; \frac{tx}{1-ty}\right] \mathrm{dt} \tag{2.2}$$

Again, using the EULER's integral (2.1), in the relation (2.2), we get

$$F^{(5)}\left[\begin{matrix} c, & d: & a \\ & e: & a' \end{matrix}; x, y\right]$$

$$= \frac{\Gamma(e)\,\Gamma(a')}{\Gamma(c)\,\Gamma(d)\,\Gamma(e-d)\,\Gamma(a'-c)} \int_0^1\int_0^1 t^{d-1}\,(1-t)^{e-d-1}\,(1-ty)^{-c}\,u^{c-1}$$

$$\times\,(1-u)^{a'-c-1}\,(1-ty-u\,tx)^{-a}\,\mathrm{dudt} \tag{2.3}$$

where $R(c) > R(b) > 0$.

Now,

$$F^{(6)}\left[\begin{matrix} c, & d: & a\ ; & b \\ & e: & a'\ ; & b' \end{matrix}; x, y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_r\,(b)_s\,(c)_{r+s}\,(d)_{r+s}}{(a')_r\,(b')_s\,(e)_{r+s}\,\underline{|r}\,\underline{|s}}\, x^r\, y^s$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\,(c)_r\,(d)_r}{(a')_r\,(e)_r\,\underline{|r}}\, x^r.\; {}_3F_2\left[b, c+r, d+r; b', e+r; y\right]$$

Using the result [1., p. 200]

$${}_{p+1}F_{q+1}\left[\begin{matrix} v, & a_1, \ldots, a_p \\ \mu+v, & b_1, \ldots, b_q \end{matrix}; \alpha y\right] = \frac{\Gamma(\mu+v)}{\Gamma(\mu)\,\Gamma(v)}\, y^{-\mu-v+1}$$

$$\times \int_0^1 x^{v-1}(y-x)^{\mu-1}\, {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; \alpha x\right] dx$$

we get

$$F^{(6)}\left[\begin{matrix} c, & d: & a; & b \\ & e: & a'; & b' \end{matrix}; x, y\right] = \frac{\Gamma(b')}{\Gamma(b'-b)\,\Gamma(b)} \sum_{r=0}^{\infty} \frac{(a)_r\,(c)_r\,(d)_r\,x^r}{(a')_r\,(e)_r\,\lfloor r}$$

$$\int_0^1 t^{b-1}\,(1-t)^{b'-1}\,{}_2F_1\left[c+r, d+r; e+r; t\,y\right] \mathrm{dt} = I$$

Expanding hypergeometric function ${}_2F_1$ in infinite series and reversing the order of summation and integration, we get

$$I = \frac{\Gamma(b')}{\Gamma(b'-b)\,\Gamma(b)} \int_0^1 t^{b-1}\,(1-t)^{b'-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\,(c)_{r+s}\,(d)_{r+s}}{(a')_r\,(e)_{r+s}\,\lfloor r\,\lfloor s} x^r\,(y\,t)^s\,\mathrm{dt}$$

Interpreting the involved double series with the help of (1.6), we get

$$I = \frac{\Gamma(b')}{\Gamma(b'-b)\,\Gamma(b)} \int_0^1 t^{b-1}\,(1-t)^{b'-1}\,F^{(5)}\left[\begin{matrix} c, & d: & a \\ & e: & a' \end{matrix}; x, yt\right] \mathrm{dt}$$

Now, using the relation (2,3), we get the required integral representation of the series $F^{(6)}$ as

$$F^{(6)}\left[\begin{matrix} c, & d; & a; & b \\ & e: & a'; & b' \end{matrix}; x_1 y\right] = \frac{\Gamma(b')\,\Gamma(e)\,\Gamma(a')}{\Gamma(c)\,\Gamma(d)\,\Gamma(e-d)\,\Gamma(a'-c)\,\Gamma(b'-b)\,\Gamma(b)}$$

$$\times \int_0^1 \int_0^1 \int_0^1 t^{b-1}\,(1-t)^{b'-1}\,v^{d-1}\,(1-v)^{e-d-1}\,(1-vyt)^{a-c}\,u^{c-1}$$

$$(1-u)^{a'-c-1}\,(1-v\,y\,t-u\,v\,x)^{-a}\,d\,u\,d\,v\,\mathrm{dt} \tag{2.4}$$

3. We have the following two elementary expansions as

$${}_2F_1\left[a, b; c; x\right]\,{}_2F_1\left[a', b'; c'; y\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_r\,(a')_s\,(b)_r\,(b')_s}{\lfloor s\,\lfloor r\,(c)_r\,(c')_s} x^r\,y^s \tag{3.1}$$

and

$${}_2F_1\left[a, b; c; x+y\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s}\,(b)_{r+s}}{(c)_{r+s}\,\lfloor r\,.\,\lfloor s} x^r\,y^s \tag{3.2}$$

CHAUNDI & BURCHNALL [2., p. 250] have introduced the inverse pair of symbolic operators

$$\nabla (h) \equiv \frac{\Gamma(h)\ \Gamma(\delta + \delta' + h)}{\Gamma(\delta + h)\ \Gamma(\delta' + h)}, \ \triangle (h) \equiv \frac{\Gamma(\delta + h)\ \Gamma(\delta' + h)}{\Gamma(h)\ \Gamma(\delta + \delta' + h)}$$

where $\delta, \delta' \equiv x\ \delta/\delta x,\ y\ \delta/\delta y,$

then

$$\nabla (h)\ (h)_r\ (h)_s\ x^r\ y^s = (h)_{r+s}\ x^r\ y^s$$

Now using these definitions, the series can also be expressed in terms of product of simple hypergeometric function as

$$F^{(6)} \left[\begin{matrix} c, d: a; b \\ e: a'; b' \end{matrix} ; x, y \right] = \nabla (c) \nabla (d) \triangle (e)\ {}_3F_2 \left[a, c, d;\ a', c;\ x \right]$$

$${}_3F_2 \left[b, c, d;\ b', e;\ y \right] \qquad (3.3)$$

In the relation (3.3) if take $d = e$ we get the known result [2., p. 253]

$$F^{(2)}\ [c: a, b; a', b'; x, y] = \nabla (c)\ {}_2F_1\ [c, a; a'; x]\ {}_2F_1\ [c, b; b'; y] \qquad (3.4)$$

Again, using the same definition and the result (3.1) we get the result

$$F^{(6)} \left[\begin{matrix} a, b' : b, a' \\ c : b', a \end{matrix} ; x, y \right] = \nabla (a)\ \nabla (b') \triangle (c)\ {}_2F_1 \left[a, b; c; x \right]$$

$${}_2F_1 \left[a', b';\ c;\ y \right] \qquad (3.5)$$

Now, using the relation [2., p. 270]

$${}_{p+1}F_p^{(2)} \left[\begin{matrix} a : b_1, \ldots, b_p; b'_1, \ldots, b'_p \\ c_1, \ldots, c_p; c'_1, \ldots, c'_p \end{matrix} ; x, y \right] =$$

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s}\ (b_1)_r \ldots (b_p)_r\ (b'_1)_s \ldots (b'_p)_s}{\underline{|r}\ \underline{|s}\ (c_1)_r \ldots (c_p)_r\ (c'_1)_s \ldots (c'_p)_s} x^r\ y^s$$

$$= \nabla (a)\ {}_{p+1}F_p \left[\begin{matrix} a, b_1, \ldots, b_p \\ c_1, \ldots, c_p \end{matrix} ; x \right] {}_{p+1}F_p \left[\begin{matrix} a, b'_1, \ldots, b'_p \\ c'_1, \ldots, c'_p \end{matrix} ; y \right] \qquad (3.6)$$

In the relation (3.3), we get

$$\nabla(d)\,\triangle(e)\,F^{(6)}\begin{bmatrix} c,\,d : a';\, b' \\ e : a';\, b' \end{bmatrix}; \varkappa, y\Bigg] = {}_3F_2{}^{(2)}\begin{bmatrix} c : a,\, d;\, b,\, d \\ d,\, e;\, b',\, e \end{bmatrix}; \varkappa. y\Bigg] \quad (3.7)$$

We can also relate the series $F^{(6)}$ with other known APELL'S Hypergeometric series [3] in the following way:

$$\text{a)}\quad F^{(6)}\left[\begin{matrix} \alpha_1, \delta : \beta_1 ; \beta' \\ \gamma : \delta_1 ; \delta \end{matrix}; \varkappa, y\right] = \nabla(\delta)\; F^{(1)}\left[\alpha' ; \beta_1, \beta' ; \gamma ; \varkappa, y\right] \quad (3.8)$$

$$\text{b)}\quad F^{(6)}\left[\begin{matrix} \alpha_1, \beta' : \beta_1 ; \gamma \\ \gamma : \beta_1' ; \gamma' \end{matrix}; \varkappa_1\, y\right] = \nabla(\beta')\triangle(\gamma)F^{(2)}\left[\alpha ; \beta_1, \beta' ; \gamma_1, \gamma' ; \varkappa, y\right] \quad (3.9)$$

$$\text{c)}\quad F^{(6)}\left[\begin{matrix} \beta_1, \beta' : \alpha_1 ; \alpha' \\ \gamma : \beta'_1 ; \beta \end{matrix}; \varkappa_1\, y\right] = \nabla(\beta)\nabla(\beta')F^{(3)}\left[\alpha_1, \alpha' ; \beta, \beta' ; \gamma ; \varkappa, y\right] \quad (3.10)$$

and

$$\text{d)}\quad F^{(6)}\left[\begin{matrix} \alpha,\ \beta' :\ \delta,\ \delta \\ \delta :\ \gamma,\ \gamma' \end{matrix}; \varkappa, y\right] = \triangle(\delta)\, F^{(4)}\left[\alpha_1,\ \beta ;\ \gamma,\ \gamma' ;\ \varkappa,\ y\right] \quad (3.11)$$

CHAUNDI & BURCHNALL [2. p. 250] have given the following results:

$$\nabla(b)\; F^{(1)}\,[a ; b, b ; c ; \varkappa, y] = {}_2F_1\,[a, b ; c ; \varkappa + y], \quad (3,12)$$

$$F^{(1)}\,[a ; b, b' ; c_1 ; \varkappa, y] = \nabla(a)\triangle(c)\; {}_2F_1\,[a, b ; c ; \varkappa]\; {}_2F_1\,[a, b' ; c ; y] \quad (3.13)$$

$$\nabla(b)\triangle(c)\; F^{(2)}\,[a ;\ b_1\, b ;\ c,\ c ;\ \varkappa,\ y] = {}_2F_1\,[a_1,\ b ;\ c ;\ \varkappa + y] \quad (3.14)$$

$$F^{(3)}\,[a_1, a' ; b_1, b' ; c ; x, y] = \triangle(c)\; {}_2F_1\,[a_1\, b ; c ; \varkappa]\; {}_2F_1\,[a'_1\, b' ;\ c ;\ y] \quad (3.15)$$

and

$$\triangle(\gamma)\; F^{(4)}\,[\alpha_1,\ \beta ;\ \gamma_1,\ \gamma ;\ \varkappa_1\, y] = {}_2F_1\,[\alpha_1,\ \beta ;\ \gamma ;\ \varkappa + y] \quad (3.16)$$

Now using the result (3.12) in the relation (3.8) after putting $\beta' = \beta$ we get

$$F^{(6)}\left[\begin{matrix} \alpha,\ \delta :\ \beta ;\ \beta \\ \gamma :\ \delta ;\ \delta \end{matrix}; \varkappa, \gamma\right] = \nabla(\delta)\triangle(\beta)\; {}_2F_1\left[\alpha_1, \beta ; \gamma ; \varkappa + y\right] \quad (3.17)$$

Using relation (3.13) in the result (3.8) we get

$$F^{(6)}\begin{bmatrix}\alpha,\ \delta:\ \beta;\ \beta' \\ \gamma:\ \delta;\ \delta'\end{bmatrix}; x, y\Big] = \nabla(\delta)\nabla(\alpha)\triangle(\gamma)\ {}_2F_1[\alpha, \beta; \gamma; x]\ {}_2F_1[\alpha, \beta'; \gamma; y] \tag{3,18}$$

Using relation (3.4) in (.9) & the relation (3.15) in (3.10) we get

$$F^{(6)}\left[\begin{matrix}\alpha, \beta': \beta; \gamma \\ \gamma; \beta'; \gamma'\end{matrix}; x, y\right] = \nabla(\beta')\nabla(\alpha)\triangle(\gamma)\ {}_2F_1[\alpha, \beta; \gamma; x]\ {}_2F_1[\alpha, \beta'; \gamma'; y] \tag{3.19}$$

and

$$F^{(6)}\left[\begin{matrix}\beta, \beta': \alpha; \alpha' \\ \gamma: \beta'; \beta\end{matrix}; x, y\right] = \nabla(\beta)\nabla(\beta')\triangle(\gamma)\ {}_2F_1[\alpha, \beta; \gamma; x]\ {}_2F_1[\alpha', \beta'; \gamma; y] \tag{3.20}$$

Using relation (3.16) in (3.11) after putting $\gamma' = \gamma$ we get

$$F^{(6)}\left[\begin{matrix}\alpha, \beta: \delta; \delta' \\ \delta: \gamma; \gamma\end{matrix}; x, y\right] = \Delta(\delta)\,\Delta(\gamma)\ {}_2F_1\left[\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; x+y\right] \tag{3.21}$$

Using the result (3.14) in the result (3.9) after putting $\beta' = \beta$ & $\gamma' = \gamma$ we get known result (3.2)

4. Two expansions

Using expansion (2. P. 252)

$$F^{(2)}[a; b, b'; c, c'; x, y] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (b')_r}{\underline{|r}\ (c)_r (c')_r} x^r y^r\ {}_2F_1[a+r, b+r; c+r; x]$$

$$ {}_2F_1[a+r, b'+r; c'+r; y] \tag{4.1}$$

in the result (3.9) we get

$$\nabla(\gamma)\triangle(\beta')\ F^{(6)}\left[\begin{matrix}\alpha, \beta': \beta; \gamma \\ \gamma: \beta'; \gamma'\end{matrix}; x, y\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r (\beta')_r}{\underline{|r}\ (\gamma)_r (\gamma')_r} x^r y^r\ {}_2F_1[\alpha+r; \beta+r; \gamma+r; x]\ {}_2F_1[\alpha+r, \beta'+ + r; \gamma'+r; y] \tag{4.2}$$

Writing the result (3.18) in the form

$$\nabla(\gamma)\,\triangle(\delta)\,F^{(6)}\begin{bmatrix}\alpha_1\ \delta:\ \beta;\ \beta' \\ \gamma:\ \delta;\ \delta\end{bmatrix}; \varkappa, y\Big] = F^{(2)}\,[\alpha;\ \beta,\ \beta';\ \gamma,\ \gamma';\ \varkappa,\ y]$$

and expending $F^{(2)}$ with the help of result (4.1) we get

$$\nabla(\gamma)\,\triangle(\delta)\,F^{(6)}\left[\begin{matrix}\alpha,\ \delta:\ \beta;\ \beta' \\ \gamma:\ \delta;\ \delta\end{matrix}; \varkappa, y\right]$$

$$= \sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r\,(\beta')_r}{\underline{|r}\,(\gamma)_r\,(\gamma)_r}\,x^r\,y^r\;{}_2F_1\,[\alpha+r,\ \beta+r;\ \gamma+r;\ \varkappa]\;{}_2F_1\,[\alpha+r,\ \beta'+r;\ \gamma+r;\ y]$$

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