INTEGRATION OF CERTAIN PRODUCTS CONTAINING JACOBI POLYNOMIALS

by

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1. Let

(1.1) $$R(\lambda, \mu, \nu, z) = \sum_{m=0}^{\infty} \frac{(-)^m (\lambda + m + 1)_m}{m!\, \Gamma(\mu + m + 1)\, \Gamma(\nu + m + 1)} z^m,$$

then the known formula [8, p. 151]

$$\left(\frac{1}{2} z\right)^{\mu+\nu} = \frac{\Gamma(\mu + 1)\, \Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} \sum_{n=0}^{\infty} \frac{(\mu + \nu + 2n)\, \Gamma(\mu + \nu + n)}{n!} J_{\mu+n}(z)\, J_{\nu+n}(z)$$

admits of the generalization

(1.2) $$\sum_{n=0}^{\infty} \frac{(\lambda + 2n)\, \Gamma(\lambda + n)}{n!} z^n R(\lambda + 2n, \mu + n, \nu + n, z) = \frac{\Gamma(\lambda + 1)}{\Gamma(\mu + 1)\, \Gamma(\nu + 1)},$$

since we have [8, p. 147]

(1.3) $$J_\mu(2z)\, J_\nu(2z) = z^{\mu+\nu} R(\mu + \nu, \mu, \nu, z^2).$$

A further generalization of (1.2) is due to AL-SALAM and CARLITZ who gave the formula [1, p. 914]

(1.4) $${}_pF_{q+1}\left[\begin{matrix} \alpha_1, \ldots \ldots, \alpha_p & ; \\ \mu + 1, \nu + 1, \beta_1, \ldots, \beta_{q-1} & ; \end{matrix} -4xy\right]$$

$$= \frac{\Gamma(\mu + 1)\, \Gamma(\nu + 1)}{\Gamma(\lambda + 1)} \sum_{n=0}^{\infty} \frac{(\lambda + 2n)\, \Gamma(\lambda + n)}{n!} x^n R(\lambda + 2n, \mu + n, \nu + n, x)$$

$$\cdot {}_{p+2}F_{q+1}\left[\begin{matrix} -n, \lambda + n, \alpha_1, \ldots \ldots, \alpha_p & ; \\ \frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1, \beta_{q-1} & ; \end{matrix}\; y\right].$$

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In a recent paper [6] the writer has exhibited (1.4) as a necessary consequence of the generalized expansion [7, p. 246 (3.1)]

$$\left(\frac{1}{2}z\right)^{\mu+\nu} {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p ; \\ \beta_1, \ldots, \beta_q ;\end{matrix} - a^2 z^2\right] {}_PF_Q\left[\begin{matrix}\varrho_1, \ldots, \varrho_P ; \\ \sigma_1, \ldots, \sigma_Q ;\end{matrix} - b^2 z^2\right]$$

$$(1.5) \quad = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{(\mu+\nu)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\,(\mu+\nu)_n}{n!} J_{\mu+n}(x)\, J_{\nu+n}(x)$$

$$\cdot\, F\left[\begin{matrix}-n, \mu+\nu+n, \mu+1, \nu+1 : \alpha_1, \ldots, \alpha_p ; \varrho_1, \ldots, \varrho_P ; \\ \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2) : \beta_1, \ldots, \beta_q ; \sigma_1, \ldots, \sigma_Q ;\end{matrix} a^2, b^2\right],$$

where the notation for the double hypergeometric function is due to Burchnall and Chaundy [3, p. 112] in preference, for the sake of brevity, to an earlier one introduced by Kampé de Fériet [2, p. 150].

For several additional properties of $R(\lambda, \mu, \nu, z)$ see [1] and [6].

In the present paper the integral

$$\int_0^c x^\sigma (c-x)^\eta R\left(\alpha, \beta, \gamma, \frac{x}{a}\right) R\left(\lambda, \mu, \nu, \frac{x}{b}\right) P_n^{(\xi,\eta)}(c-2x)\, dx,$$

where c is real, non-zero and finite, $Re(\sigma) > -1$ and $Re(\eta) > -1$, is evaluated in terms of a double hypergeometric series and its several interesting special cases are considered. A number of formulae hitherto known are shown to be generalized by the results of this paper.

2. We restrict c in the usual manner and make use of the formula [4, p. 284]

$$\int_0^1 t^\sigma (1-t)^\eta P_n^{(\xi,\eta)}(1-2t)\, \mathrm{d}t = \frac{\Gamma(\sigma+1)\,\Gamma(\eta+n+1)\,\Gamma(\xi-\sigma+n)}{n!\,\Gamma(\xi-\sigma)\,\Gamma(\eta+\sigma+n+2)},$$

$$Re(\sigma) > -1, \quad Re(\eta) > -1;$$

so that

$$\int_0^c x^\sigma (c-x)^\eta R\left(\alpha, \beta, \gamma, \frac{x}{a}\right) R\left(\lambda, \mu, \nu, \frac{x}{b}\right) P_n^{(\xi,\eta)}(c-2x)\, dx$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s} \left(\frac{1}{a}\right)^r \left(\frac{1}{b}\right)^s (\alpha + r + 1)_r\, (\lambda + s + 1)_s}{r\,!\, s\,!\, \Gamma(\beta + r + 1)\, \Gamma(\gamma + r + 1)\, \Gamma(\mu + s + 1)\, \Gamma(\nu + s + 1)}$$

$$\cdot \int_0^c x^{\sigma + r + s} (c - x)^\eta\, P_n^{(\xi,\eta)} (c - 2x)\, dx\,,$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s}\, c^{\eta+\sigma+1} (\alpha + r + 1)_r\, (\lambda + s + 1)_s}{r!\, s!\, \Gamma(\beta + r + 1)\, \Gamma(\gamma + r + 1)\, \Gamma(\mu + s + 1)\, \Gamma(\nu + s + 1)} \left(\frac{c}{a}\right)^r \left(\frac{c}{b}\right)^s$$

$$\cdot \frac{\Gamma(\sigma + r + s + 1)\, \Gamma(\eta + n + 1)\, \Gamma(\xi + n - \sigma - r - s)}{n\,!\, \Gamma(\xi - \sigma - r - s)\, \Gamma(\eta + \sigma + n + 2 + r + s)}\,,$$

and therefore

$$\int_0^c x^\sigma (c - x)^\eta\, R\left(\alpha, \beta, \gamma, \frac{x}{a}\right) R\left(\lambda, \mu, \nu, \frac{x}{b}\right) P_n^{(\xi,\eta)} (c - 2x)\, dx$$

$$(2.1) \qquad = \frac{c^{\eta+\sigma+1}\, \Gamma(\sigma + 1)\, (\xi - \sigma)_n\, \Gamma(\eta + n + 1)}{\eta!\, \Gamma(\beta + 1)\, \Gamma(\gamma + 1)\, \Gamma(\mu + 1)\, \Gamma(\nu + 1)\, \Gamma(\eta + \sigma + n + 2)}$$

$$.F \left[\begin{matrix} \sigma + 1,\ \sigma - \xi + 1 : \frac{1}{2} (\alpha + 1),\ \frac{1}{2} (\alpha + 2)\,;\ \frac{1}{2} (\lambda + 1),\ \frac{1}{2} (\lambda + 2)\,; \\ \eta+\sigma+n+2,\ \sigma-\xi+1-n : \alpha+1, \beta+1, \gamma+1\,; \lambda+1, \mu+1,\ \nu+1\,; \end{matrix}\ -\frac{4c}{a},\ -\frac{4c}{b} \right],$$

provided that $Re(\sigma) > -1$ and $Re(\eta) > -1$.

Alternatively, the last integral can be written in the form

$$\int_0^c x^{\sigma+1} (c^2 - x^2)^\eta\, R\left(\alpha, \beta, \gamma, \frac{x^2}{4a^2}\right) R\left(\lambda, \mu, \nu, \frac{x^2}{4b^2}\right) P_n^{(\xi,\eta)} (c^2 - 2x^2)\, dx$$

$$(2.2) \quad = \frac{c^{2\eta+\sigma+1}\, \Gamma(\frac{1}{2} \sigma + 1)\, (\xi - \frac{1}{2} \sigma)_n\, \Gamma(\eta + n + 1)}{2\, n\,!\, \Gamma(\beta + 1)\, \Gamma(\gamma + 1)\, \Gamma(\mu + 1)\, \Gamma(\nu + 1)\, \Gamma(\eta + \frac{1}{2} \sigma + n + 2)}$$

$$.\, F \left[\begin{matrix} \frac{1}{2} \sigma + 1, \frac{1}{2} \sigma - \xi + 1 : \frac{1}{2} (\alpha + 1),\ \frac{1}{2} (\alpha + 2)\,;\ \frac{1}{2} (\lambda + 1),\ \frac{1}{2} (\lambda + 2)\,; \\ \eta+\frac{1}{2}\sigma+n+2, \frac{1}{2}\sigma-\xi-n+1 : \alpha+1, \beta+1, \gamma+1\,; \lambda+1, \mu+1, \nu+1\,; \end{matrix}\ -\frac{c^2}{a^2},\ -\frac{c^2}{b^2} \right],$$

where $\qquad Re\,(\sigma) > -2$ and $Re\,(\eta) > -1$.

When $n = 0$ the special case $c = 1$ of (2.1) corresponds to the known formula [6, p. 142 (6.1)]

$$(2.3)\qquad \int_0^1 R(\alpha, \beta, \gamma, \tfrac{1}{4} xt)\, R(\lambda, \mu, \nu, \tfrac{1}{4} yt)\, t^{\sigma-1} (1-t)^{\varrho-1}\, dt$$

$$= \frac{\Gamma(\sigma)\, \Gamma(\varrho)\, \{\Gamma(\sigma+\varrho)\}^{-1}}{\Gamma(\beta+1)\, \Gamma(\gamma+1)\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$$

$$F\left[\begin{matrix} \sigma : \tfrac{1}{2}(\alpha+1), \tfrac{1}{2}(\alpha+2)\,;\, \tfrac{1}{2}(\lambda+1), \tfrac{1}{2}(\lambda+2)\,; \\ \sigma+\varrho : \alpha+1, \beta+1, \gamma+1\,;\, \lambda+1, \mu+1, \nu+1\,; \end{matrix} -x, -y\right],$$

which is valid whenever both $Re(\sigma)$ and $Re(\varrho)$ are positive.

Next, put $\alpha = \beta + \gamma$, $\lambda = \mu + \nu$ and change the notation slightly. By virtue of (1.3), the formula (2.2) will then express an integral involving the product

$$J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\nu\left(\frac{x}{b}\right) J_\varrho\left(\frac{x}{b}\right) P_n^{(\xi,\eta)}(c^2 - 2x^2)$$

in terms of a hypergeometric series, and we have

$$\int_0^c x^{\sigma+1} (c^2-x^2)^\eta\, J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\nu\left(\frac{x}{b}\right) J_\varrho\left(\frac{x}{b}\right) P_n^{(\xi,\eta)}(c^2-2x^2)\, dx$$

$$(2.4)\qquad = \frac{2^{\sigma-\delta-1}\, c^{2\eta+\delta+1}\, \Gamma(\tfrac{1}{2}\delta+1)\, (\xi-\tfrac{1}{2}\delta)_n\, \Gamma(\eta+n+1)}{a^{\lambda+\mu}\, b^{\nu+\varrho}\, \Gamma(\lambda+1)\, \Gamma(\mu+1)\, \Gamma(\nu+1)\, \Gamma(\varrho+1)\, \Gamma(\eta+\tfrac{1}{2}\delta+n+2)}$$

$$\cdot F\left[\begin{matrix} \tfrac{1}{2}\delta+1, \tfrac{1}{2}\delta-\xi+1 : \tfrac{1}{2}(\lambda+\mu+1), \tfrac{1}{2}(\lambda+\mu+2)\,;\, \tfrac{1}{2}(\nu+\varrho+1), \tfrac{1}{2}(\nu+\varrho+2)\,; \\ \eta+\tfrac{1}{2}\delta+n+2, \tfrac{1}{2}\delta-\xi-n+1 : \lambda+1, \mu+1, \lambda+\mu+1\,;\, \nu+1, \varrho+1, \nu+\varrho+1\,; \end{matrix} -\frac{c^2}{a^2}, -\frac{c^2}{b^2}\right],$$

where $\delta = \lambda + \mu + \nu + \varrho + \sigma$, $Re(\delta) > -2$ and $Re(\eta) > -1$.

3. When $a = b$, the double series in (2.2) is equal to

$$\sum_{k=0}^{\infty} \frac{(\tfrac{1}{2}\delta+1)_k\, (\tfrac{1}{2}\delta-\xi+1)_k\, (\tfrac{1}{2}\lambda+\tfrac{1}{2})_k\, (\tfrac{1}{2}\lambda+1)_k \left(-\dfrac{c^2}{a^2}\right)^k}{k!\, (\eta+\tfrac{1}{2}\sigma+n+2)_k\, (\tfrac{1}{2}\sigma-\xi-n+1)_k\, (\lambda+1)_k\, (\mu+1)_k\, (\nu+1)_k}$$

$$\cdot \sum_{r=0}^{k} \frac{(-k)_r\, (\tfrac{1}{2}\alpha+\tfrac{1}{2})_r\, (\tfrac{1}{2}\alpha+1)_r\, (-\lambda-k)_r\, (-\mu-k)_r\, (-\nu-k)}{r!\, (\alpha+1)_r\, (\beta+1)_r\, (\gamma+1)_r\, (\tfrac{1}{2}-\tfrac{1}{2}\lambda-k)_r\, (-\tfrac{1}{2}\lambda-k)_r},$$

and this simplifies as a ${}_4F_5$ if we further set

$$\beta = \gamma + \tfrac{1}{2} = \tfrac{1}{2}\alpha \text{ and } \mu = \nu + \tfrac{1}{2} = \tfrac{1}{2}\lambda.$$

Therefore, a special case of (2.2) is

$$\int_0^c x^{\lambda+1} (c^2 - x^2)^\eta \, R\left(\mu, \tfrac{1}{2}\mu, \tfrac{1}{2}\mu - \tfrac{1}{2}, \frac{x^2}{16a^2}\right)$$

$$\cdot\, R\left(\nu, \tfrac{1}{2}\nu, \tfrac{1}{2}\nu - \tfrac{1}{2}, \frac{x^2}{16a^2}\right) P_n^{(\xi,\eta)}(c^2 - 2x^2)\, dx$$

$$(3.1) \qquad = \frac{2^{\mu+\nu-1} c^{2\eta+\lambda+1} \Gamma(\tfrac{1}{2}\lambda+1)\,(\xi - \tfrac{1}{2}\lambda)_n\, \Gamma(\eta+n+1)}{\pi\, \Gamma(\mu+1)\, \Gamma(\nu+1)\, \Gamma(\eta+\tfrac{1}{2}\lambda+n+2)}$$

$$\cdot\, {}_4F_5\left[\begin{matrix} \tfrac{1}{2}\lambda+1, \tfrac{1}{2}\lambda-\xi+1, \tfrac{1}{2}(\mu+\nu+1), \tfrac{1}{2}(\mu+\nu+2); \\ \eta+\tfrac{1}{2}\lambda+n+2, \tfrac{1}{2}\lambda-\xi-n+1, \mu+1, \nu+1, \mu+\nu+1; \end{matrix} -\frac{c^2}{a^2}\right],$$

valid when $Re(\lambda) > -2$ and $Re(\eta) > -1$.

The last formula when re-written in view of the relation [1, p 911].

$$(3.2) \qquad R(2\nu, \nu, \nu - \tfrac{1}{2}, z^2) = \frac{z^{-2\nu}}{\pi^{1/2}} J_{2\nu}(4z)$$

gives us

$$\int_0^c x^{\lambda+1} (c^2 - x^2)^\eta J_\mu\left(\frac{x}{a}\right) J_\nu\left(\frac{x}{a}\right) P_n^{(\xi,\eta)}(c^2 - 2x^2)\, dx$$

$$(3.3) \qquad = \frac{c^{2\eta+\delta+1} \Gamma(\tfrac{1}{2}\delta+1)\,(\xi-\tfrac{1}{2}\delta)_n\, \Gamma(\eta+n+1)}{2\,(2a)^{\mu+\nu}\, \Gamma(\mu+1)\, \Gamma(\nu+1)\, \Gamma(\eta+\tfrac{1}{2}\delta+n+2)}$$

$$\cdot\, {}_4F_5\left[\begin{matrix} \tfrac{1}{2}\delta+1, \tfrac{1}{2}\delta-\xi+1, \tfrac{1}{2}(\mu+\nu+1), \tfrac{1}{2}(\mu+\nu+2); \\ \eta+\tfrac{1}{2}\delta+n+2, \tfrac{1}{2}\delta-\xi-n+1, \mu+1, \nu+1, \mu+\nu+1; \end{matrix} -\frac{c^2}{a^2}\right],$$

where $Re(\delta) > -2$, $\delta = \lambda + \mu + \nu$ and $Re(\eta) > -1$.

And since [8, p. 150]

$$J_\mu(z)\, J_\nu(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\, d\theta,$$

(3.3) assumes the form

$$ {}_4F_5\left[\begin{matrix} \tfrac{1}{2}\delta+1,\ \tfrac{1}{2}\delta-\xi+1,\ \tfrac{1}{2}(\mu+\nu+1),\ \tfrac{1}{2}(\mu+\nu+2)\,; \\ \eta+\tfrac{1}{2}\delta+n+2,\ \tfrac{1}{2}\delta-\xi-n+1,\ \mu+1,\ \nu+1,\ \mu+\nu+1\,; \end{matrix} \ -\frac{c^2}{a^2}\right] $$

$$ (3.4) \qquad = \frac{2^{\mu+\nu+2}\,\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma(\eta+\tfrac{1}{2}\delta+n+2)}{c^{2\eta+\delta+1}\,\Gamma(\tfrac{1}{2}\delta+1)\,(\xi-\tfrac{1}{2}\delta)_n\,\Gamma(\eta+n+1)} $$

$$ \cdot\frac{a^{\mu+\nu}}{\pi}\int_0^c\int_0^{\frac{1}{2}\pi} x^{\lambda+1}(c^2-x^2)^{\eta}\,J_{\mu+\nu}\left(\frac{2x}{a}\cos\theta\right)P_n^{(\xi,\eta)}(c^2-2x^2)\cos(\mu-\nu)\theta\,dx\,d\theta, $$

valid when $Re(\delta) > -2$, $\delta = \lambda + \mu + \nu$ and $Re(\eta) > -1$.

If in the last formula we set $n = 0$ and change the notation slightly, we shall obtain the known result [6, p. 144 (6.7)].

$$ {}_3F_4\left[\begin{matrix} \sigma,\ \tfrac{1}{2}(\mu+\nu+1),\ \tfrac{1}{2}(\mu+\nu+2)\,; \\ \sigma+\varrho,\ \mu+1,\ \nu+1,\ \mu+\nu+1\,; \end{matrix}\ -4x^2\right]\frac{\Gamma(\sigma)\,\Gamma(\varrho)}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma(\sigma+\varrho)} $$

$$ (3.5) \quad = \frac{4}{\pi x^{\mu+\nu}}\int_0^1\int_0^{\frac{1}{2}\pi} t^{2\sigma-\mu-\nu-1}(1-t^2)^{\varrho-1}J_{\mu+\nu}(4\,xt\cos\theta)\cos(\mu-\nu)\theta\,dt\,d\theta, $$

which holds when $Re(\sigma) > 0$, $Re(\mu+\nu) > -1$ and $Re(\varrho) > 0$.

Also in the limit when $b \to \infty$, (2.2) yields

$$ \int_0^c x^{\sigma+1}(c^2-x^2)^{\eta}\,R\left(\alpha,\,\beta,\,\gamma,\,\frac{x^2}{4a^2}\right)P_n^{(\xi,\eta)}(c^2-2x^2)\,dx $$

$$ (3.6) \qquad = \frac{c^{2\eta+\sigma+1}\,\Gamma(\tfrac{1}{2}\sigma+1)\,(\xi-\tfrac{1}{2}\sigma)_n\,\Gamma(\eta+n+1)}{2\,n!\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\eta+\tfrac{1}{2}\sigma+\eta+2)} $$

$$ \cdot\,{}_4F_5\left[\begin{matrix} \tfrac{1}{2}\sigma+1,\ \tfrac{1}{2}\sigma-\xi+1,\ \tfrac{1}{2}(\alpha+1),\ \tfrac{1}{2}(\alpha+2)\,; \\ \eta+\tfrac{1}{2}\sigma+n+2,\ \tfrac{1}{2}\sigma-\xi-n+1,\ \alpha+1,\ \beta+1,\ \gamma+1\,; \end{matrix}\ -\frac{c^2}{a^2}\right] $$

under the constraints already stated.

It is not difficult to give a direct proof of (3.6) following the method illustrated in § 2.

In (3.6) put $\alpha = \beta + \gamma$, employ (1.3) and make a slight change in the notation. We shall again arrive at the formula (3.3).

4. Finally we remarck that formulae involving GEGENBAUER polynomials (or the ultraspherical polynomials) defined by means of

$$(1 - 2\,xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C^{\nu}_n\,(x)\,t^n$$

are special cases of the results of the preceding sections, since

$$C_n^{\alpha+\frac{1}{2}}(x) = \frac{(1+2\,\alpha)_n}{(1+\alpha)_n}\,P_n^{(\alpha,\alpha)}(x)\;.$$

For integrals involving LEGENDRE polynomials we further recall the well-known relation

$$P_n\,(x) = C_n^{\frac{1}{2}}\,(x) = P_n^{(0,0)}\,(x)\,.$$

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