MULTIPLE SUMS AND GENERATING FUNCTIONS

by

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1. INTRODUCTION. Put

$$H(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{i+j}{j} \binom{m-i+j}{j} \binom{i+n-j}{n-j} \binom{m+n-i-j}{n-j} .$$

PAUL BROCK prosed as a problem the identity

$$H(m, n) - H(m-1, n) - H(m, n-1) = \binom{m+n}{m}^2. \tag{1.1}$$

For references see [1].

In generalizing BROCK's identity, the writer [1] considered the generating functions

(1.2)

$$F(u_1, u_2, \dots, u_n) = \sum_{i_1, \dots, i_n = 0}^{\infty} \binom{i_1 + i_2}{i_2} \binom{i_2 + i_3}{i_3} \dots \binom{i_n + i_1}{i_1} u_1^{i1} u_2^{i2} \dots u_u^{in} ,$$

$$G(u_1, u_2, \dots, u_r) = \sum_{m_1, \dots m_r, = 0}^{\infty} H(m_1, m_2, \dots, m_r) u_1^{m1} u_2^{m2} \dots u_r^{mr} \tag{1.3}$$

with

$$H(m_1, m_2, \dots, m_r) = \sum \binom{i_1+i_2}{i_2} \binom{i_2+i_3}{i_1} \dots \binom{i_{2r}+i_1}{i_1} , \tag{1.4}$$

where the summation in over all nonnegative i_s such that

$$i_s + i_{r+s} = m_s \quad (s = 1, 2, \dots, r).$$

Supported in part by NSF grant GP-5174.

He showed that

$$(1.5) \qquad F(u_1, u_2, \ldots, u_n) = (P_n^2 - 4u_1 Q_n R_n)^{-\frac{1}{2}},$$

where

$$(1.6) \quad \begin{cases} P_n = f_{n-1}(u_2, u_3, \ldots, u_n) + u_1 f_{n-3}(u_3, u_4, \ldots, u_{n-1}), \\ Q_n = f_{n-2}(u_2, u_3, \ldots, u_{n-1}), \\ R_n = f_{n-2}(u_3, u_4, \ldots, u_n), \end{cases}$$

and $f_n(u_1, u_2, \ldots, u_n)$ is a polynominal defined recursively by

$$f_1(u_1) = 1 - u_1, \; f_2(u_1, u_2) = 1 - u_1 - u_2$$

and

$$(1.7) \; f_n(u_1, u_2, \ldots, u_n) = f_{n-1}(u_1, u_2, \ldots, u_{n-1}) - u_n f_{n-2}(u_1, u_2, \ldots, u_{n-2}).$$

As for (1.3) we have

$$G(u_1, u_2, \ldots, u_r) = U_r^{-1}. \; F(u_1, u_2, \ldots, u_r),$$

Where

$$(1.8) \; U_r = U_r(u_1, u_2, \ldots, u_r) = f_{r-1}(u_2, \ldots, u_r) - u_1 f_{r-3}(u_3, \ldots, u_{r-1}).$$

Thus for example

$$U_1 = 1, \; U_2 = 1 - u_1 - u_2, \; U_3 = 1 - u_1 - u_2 - u_3,$$

$$U_4 = 1 - u_1 - u_2 - u_3 - u_4 + u_1 u_3 + u_2 u_4.$$

The object of the present paper is first, to make the general results of [1] somewhat more explicit. This depends on a more explicit description of the polynominal U_n defined by (1.8). Incidentally the number of terms of U_n is equal to the LUCAS number L_n. We note that formula (1.5) can be rewritten in terms of U_m;

$$(1.9) \qquad F(u_1, u_2, \ldots, u_n) = (U_n^2 - 4u_1 u_2 \ldots u_n)^{-\frac{1}{2}}.$$

Indeed this formula is basic for the later applications.

As an application of the explicit form of U_n we can state explicitly a generalization of (1.1) involving the function $H(m_1, \ldots, m_r)$ defined by (1.4). The special cases $r = 3$ and 4 have already been obtained in [1]; for the general case see below.

However we can go considerably further. We show that the polynominal

$$U_{nk}(u_1, \ldots, u_n, \ldots, u_1, \ldots, u_n)$$

can be exhibited as a polynomial in $U_n(u_1, \ldots, u_n)$ and $u_1 u_2 \ldots u_n$. Using this result we are able to obtain results similar to (1.1) for the function

$$H_n^{(k)}(m_1, \ldots, m_n) = \sum \binom{j_1 + j_2}{j_2}\binom{j_2 + j_3}{j_3} \cdots \binom{j_{nk} + j_1}{j_1},$$

where the summation is over all nonnegative j_s such that

$$j_s + j_{s+n} + \ldots + j_{s+n(k-1)} = m_s \quad (s = 1, 2, \ldots, n).$$

The general result becomes quite complicated but can be obtained in any particular case without much difficulty. For the special case $n = 2$, $k = 3$, see (7.12) below. When $n = 1$, the general formula reduces to

$$\sum_{j2<k} (-1)^j \binom{k-j}{j} H_1^{(k)}(m-j) = \binom{2m}{m}.$$

2. We shall first recall some properties of $f_n(u_1, \ldots, u_n)$ defined by (1.7). As noted in [1], f_n is a continuant. It can be exhibited as a determinant of order $n + 1$:

$$f_n(u_1, \ldots, u_n) = \begin{vmatrix} 1 & u_1 & & & & \\ 1 & 1 & u_2 & & & \\ & 1 & 1 & u_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & 1 & u_n \\ & & & & 1 & 1 \end{vmatrix} \tag{2.1}$$

Indeed if we let D_1 denote the determinant in (2.1) we have

$$D_0 = 1,\ D_1 = 1 - u_1,\ D_2 = 1 - u_1 - u_2$$

and

$$D_n = D_{n-1} - u_n D_{n-2}. \tag{2.2}$$

Comparing (2.2) with (1.7) it is clear that $D_n = f_n$.

We find that

$$f_3 = 1 - u_1 - u_2 - u_3 + u_1 u_3$$

$$f_4 = 1 - u_1 - u_2 - u_3 - u_4 + u_1 u_3 + u_1 u_4 + u_2 u_4$$

$$f_5 = 1 - u_1 - u_2 - u_3 - u_4 - u_5 + u_1 u_3 - u_1 u_4 + u_1 u_5 -$$

$$- u_2 u_4 + u_2 u_5 + u_3 u_5 + u_1 u_3 u_5.$$

The number of terms in f_n is easily seen to be equal to the Fibonacci number F_{n+2} defined by $F_0 = 0$, $F_1 = 1$,

$$F_{n+2} = F_{n+1} + F_n \quad (n \geqq 0).$$

This is a variant of Sylvester's theorem on the number of terms in a continuant [2, p. 503].

The terms of f_n can be described as follows. Each term is of the form

$$(2.3) \qquad (-1)^k u_{i_1} u_{i_2} \dots u_{ik} \quad (k = 0, 1, 2, \dots),$$

where

$$i_{s+1} > i_s + 1 \quad (s = 1, 2, \dots, k-1)$$

and $i \leq i_1$, $i_k \leq n$. For fixed k, the number of terms (2.3) is equal o $\binom{n-k+1}{k}$. Indeed if the number of such terms is denoted by $N(n, k)$, we have

$$N(n, k) = N(n-1, k) + N(n-2, k-1),$$

from which it follows that

$$(2.4) \qquad N(n, k) = \binom{n-k+1}{k}.$$

Thus the total number of terms is f_n is equal to

$$\sum_{2k \leq n+1} \binom{n-k+1}{k} = F_{n+2}.$$

We note also that f_n satisfies

$$(2.5) \qquad f_n(u_1, \dots, u_n) = f_n(u_n, \dots, u_1).$$

Moreover we have the recurrence [1, p. 27]

$$(2.6) \qquad f_n(u_1, u_2, \ldots, u_n)$$
$$= f_r(u_{n-r+1}, \ldots, u_n)\, f_{n-r-1}(u_1, \ldots, u_{n-r+1})$$
$$- u_{n-r} f_{r-1}(u_{n-r+2}, \ldots, u_n) f_{n-r-2}(u_1, \ldots, u_{n-r+2}) \quad (n \geq r+2).$$

We remark that

(2.7)

$$(f_n(u_1, u_2, \ldots, u_n))^{-1} = \sum_{i_1,\ldots,i_n=0}^{\infty} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\cdots\binom{i_{n-1}+i_n}{i_n} u_1^{i_1} u_2^{i_2} \ldots u_n^{i_n}.$$

Turning next to U_n, it follows from (1.8) and (2.3) that the general term is of the form

$$(2.8) \qquad (-1)^k u_{i_1} u_{i_2} \ldots u_{i_k} \quad (k = 0, 1, 2, \ldots)$$

where

$$i_{s+1} > i_s + 1 \quad (s = 1, 2, \ldots, k-1)$$

and in addition the combination $i_1 = 1$, $i_k = n$ is ruled out. For example we have

$$U_5 = 1 - u_1 - u_2 - u_3 - u_4 - u_5 + u_1 u_3 + u_1 u_4 + u_2 u_4$$
$$+ u_2 u_5 + u_3 u_5,$$
$$U_6 = 1 - u_1 - u_2 - u_3 - u_4 - u_5 - u_6 + u_1 u_3 + u_1 u_4 + u_1 u_5$$
$$+ u_2 u_4 + u_2 u_5 + u_2 u_6 + u_3 u_5 + u_3 u_6 + u_4 u_6 - u_1 u_3 u_5 - u_2 u_4 u_6.$$

For fixed k, the number of terms (2.6) in U_n is equal to

$$(2.9) \qquad \binom{n-1}{k} + \binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k}.$$

This can be proved directly or by making use of (1.8). It follows from (2.9) or (1.8) that the total number of terms in U_n is equal to the Lucas number L_n defined by $L_1 = 1$, $L_2 = 3$,

$$L_{n+2} = L_{n+1} + L_n \quad (n \geq 1).$$

It follows also from (2.6) that U_n satisfies

(2.10) $$U_n(u_1, u_2, \ldots u_n) = U_n(u_n, u_{n-1}, \ldots, u_1)$$

as well as

(2.11) $$U_n(u_1, u_2, \ldots, u_n) = U_n(u_2, u_3, \ldots, u_n, u_1).$$

As we have seen, f_n sarisfies (2.8); however f_n does not satisfy (2.9).

We may state

THEOREM 1. *The polynominal* $U_n = U_n(u_1, \ldots, u_n)$ *defined by* (1.8) *has the explicit expression*

(2.12) $$U_n = 1 - \sum_i u_i + \sum u_{i_1} u_{i_2} - \ldots + (-1)^k \sum u_{i_1} u_{i_2} \ldots u_{i_k} + \ldots,$$

where in the general term

$$i_s + 1 < i_{s+1} \quad (s = 1, 2, \ldots, k-1),$$

but the combination $i_1 = 1$, $i_k = k$ *is not allowed. moreover* U_n *satisfies* (2.10) *and* (2.11).

Another relation involving f_n and U_n is implied by the explicit representations, namely

(2.13)

$$f_n(u_1, \ldots, u_n) = U_n(u_1, \ldots, u_n) + u_1 u_n f_{n-4}(u_3, \ldots, u_{n-2}) \; (n \geq 4).$$

Repeated application of this formula yields

(2.14) $$f_n(u_1, \ldots, u_n) = U_n(u_1, \ldots, u_n) + u_1 u_n U_{n-4}(u_3, \ldots, u_{n-2}) + u_1 u_3 u_{n-2} u_n U_{n-8}(u_5, \ldots, u_{n-4}) + \ldots,$$

where in general the final term contains an f. For example,

$$f_4(u_1, \ldots, u_4) = U_4(u_1, \ldots, u_4) + u_1 u_4,$$
$$f_5(u_1, \ldots, u_5) = U^5(u_1, \ldots, u_5) + u_1 u_5 f_1(u_3),$$
$$f_6(u_1, \ldots, u_6) = U_6(u_1, \ldots, u_6) + u_1 u_6 f_2(u_3, u_4).$$

3. We shall now show that

(3.1) $$P_n^2 - 4u_1 Q_n R_n = U_n^2 - 4u_1 u_2 \ldots u_n,$$

where P_n, Q_n, R_n are defined by (1.6) and U_n by (1.8). The proof depends upon the identity

(3.2) $$f_{n-1}(u_1, u_2, \ldots, u_{n-1}) f_{n-1}(u_2, u_3, \ldots, u_n)$$
$$-f_n(u_1, u_2, \ldots, u_n) f_{n-2}(u_2, u_3, \ldots, u_{n-1}) = u_1 u_2 \ldots u_3.$$

For $n = 2$, (3.2) reduces to

$$(1-u_1)(1-u_2) - (1-u_1-u_2) = u_1 u_2,$$

which is correct. Assuming that (3.2) holds up and including the value n, we have, by (1.7),

$$f_n(u_1, \ldots, u_n) f_n(u_2, \ldots, u_{n+1}) - f_{n+1}(u_1, \ldots, u_{n+1}) f_{n-1}(u_2, \ldots, u_n)$$
$$= \{f_{n-1}(u_2, \ldots, u_n) - u_{n+1} f_{n-2}(u_2, \ldots, u_{n-1})\} f_n(u_1, \ldots, u_n)$$
$$= \{f_n(u_1, \ldots, u_n) - u_{n+1} f_{n-1}(u_1, \ldots, u_{n-1})\} f_{n-1}(u_2, \ldots, u_n)$$
$$= u_{n+1}\{ f_{n-1}(u_1, \ldots, u_{n-1}) f_{n-1}(u_2, \ldots, u_n) - f_n(u_1, \ldots, u_n)$$
$$= f_{n-2}(u_2, \ldots, u_{n-1})\} = u_1 u_2 \ldots u_n u_{n+1}.$$

In the next place we have, by (1.8) and (3.2),

$$U_n^2 - 4u_1 u_2 \ldots u_n \{f_{n-1}(u_2, \ldots, u_n) - u_1 f_{n-3}(u_3, \ldots, u_{n-1})\}^2 -$$
$$- 4u_1 \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{n-2}(u_3, \ldots, u_n) - f_{1n-1}(u_2, \ldots, u_n)$$
$$f_{n-3}(u_3, \ldots, u_{n-1})\} = \{f_{n-1}(u_2, \ldots, u_n) + u_1 f_{n-3}(u_3, \ldots, u_{n-1})\}^2 -$$
$$- 4u_1 f_{n-2}(u_2, \ldots, u_{n-1}) f_{n-2}(u_3, \ldots, u_n) = P_n^2 - 4u_1 Q_n R_n.$$

This evidently proves (3.1).

An interesting feature of (3.1) is that the right member is obviously unaltered by cyclic substitutions of $u_1, u_2, \ldots, u_n$ but it is not apparent that the left member has this property.

Combining (3.1) with (1.5) we have at once for the generating function defined by (1.2)

(3.3) $$F(u_1, u_2, \ldots, u_n) = (U_n^2 - 4u_1 u_2 \ldots u_n)^{-\frac{1}{2}}.$$

It is evident from (1.2) that $F(u_1, u_2, \ldots, u_n)$ is unaltered by cyclic substitutions of $u_1, u_2, \ldots, u_n$.

4. We now consider $U_{2n}(u_1, u_2, \ldots, u_{2n})$, where

(4.1) $$u_{n+j} = u_n \quad (j = 1, 2, \ldots, n).$$

For brevity we put

$$(4.2)\quad U_{n,2} = U_{n,2}(u_1, \ldots, u_n) = U_{2n}(u_1, \ldots, u_n, u_1, \ldots, u_n).$$

We shall show that

$$(4.3)\qquad U_{n,2} = U_n^2 - 2u_1 u_2 \ldots u_n,$$

where $U_n = U_n(u_1, \ldots, u_n)$.

By (1.8) and (4.2)

$$U_{n,2} = f_{2n-1}(u_2, \ldots, u_n, u_1, \ldots, u_n) - u_1 f_{2n-3}(u_3, \ldots, u_n, u_1, \ldots, u_{n-1}).$$

But by (2.6) and (4.1)

$$f_{2n-1}(u_2, \ldots, u_n, u_1, \ldots, u_n)$$
$$= f^2{}_{n-1}(u_2, \ldots, u_n) - u_1 f_{n-2}(u_3, \ldots, u_n) f_{n-2}(u_2, \ldots, u_{n-1})$$

and

$$f_{2n-3}(u_3, \ldots, u_n, u_1, \ldots, u_{n-1})$$
$$= f_{n-2}(u_2, \ldots, u_{n-1}) f_{n-2}(u_3, \ldots, u_n) - u_1 f^2{}_{n-3}(u_3, \ldots, u_{n-1}),$$

so that

$$\begin{aligned} U_{n,2} &= f^2{}_{n-1}(u_2, \ldots, u_n) - 2u_1 f_{n-2}(u_2, \ldots, u_{n-2}) f_{n-2}(u_3, \ldots, u_n) \\ &\quad + u^2{}_1 f^2{}_{n-3}(u_3, \ldots, u_{n-1}) \\ &= \{f_{n-1}(u_2, \ldots, u_n) - u_1 f_{n-3}(u_3, \ldots, u_{n-1})\} \\ &\quad - 2u_1 \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{n-2}(u_3, \ldots, u_n) \\ &\quad - f_{n-1}(u_2, \ldots, u_n) f_{n-3}(u_3, \ldots, u_{n-1})\} \\ &= U_n^2(u_1, \ldots, u_n) - 2u_1 u_2 \ldots u_n \end{aligned}$$

by (1.8) and (3.2).

It follows from (4.3) that

$$\begin{aligned} & U^2{}_{n2}(n_1, \ldots, u_n, u_1, \ldots, u_n) - 4u^2{}_1 u^2{}_2 \ldots u^2{}_n \\ &= \{U^2{}_n(u_1, \ldots, u_n) - 2u_1 u_2 \ldots u_n\}^2 - 4u^2{}_1 u^2{}_2 \ldots u^2{}_n \\ &= U^2{}_n(u_1, \ldots, u_n)\{U^2{}_n(u_1, \ldots, u_n) - 4u_1 u_2 \ldots u_n\}, \end{aligned}$$

and therefore

$$(4.4) \qquad \{U^2{}_{2n}(u_1, \ldots, u_n, u_1, \ldots u_n) - 4u^2{}_1\, u^2{}_2 \ldots u^2{}_n\}^{\frac{1}{2}}$$
$$U_n(u_1, \ldots, u_n)\ \{U^2{}_n(u_1, \ldots, u_n) - 4u_1\, u_2 \ldots u_n\}^{-\frac{1}{2}}.$$

Combining (4.4) with (1.3), (1.4) and (1.9) we get

$$(4.5) \qquad G(u_1, u_2, \ldots, u_n) = U_n{}^{-1}(U^2{}_n - 4u_1\, u_2 \ldots u_n)^{-\frac{1}{2}}$$
$$= U_n{}^{-1}\ F(u_1, u_2, \ldots, u_n)$$

in agreement with (7.8) of [1].

5. We shall now show that (4.3) can be generalized considerably. We assume in what follows that

$$u_j = u_{j+n} = \ldots = u_{j+n(k-1)} \quad (j = 1, 2, \ldots, n).$$

Put

$$(5.1) \qquad U_{n,k} = U_{n,k}(u_1, \ldots, u_n)$$
$$= U_{nk}(u_1, \ldots, u_n, \ldots, u_1, \ldots, u_n).$$

We show first that

$$(5.2) \qquad U_{n,k+1} = U_n\, U_{n,k} - u_1\, u_2 \ldots u_n\, U_{n,k-1} \quad (k \geq 2).$$

The proof of (5.2) depends upon the following two identities:

$$(5.3) \qquad f_{n-2}(u_3, \ldots, u_{n-1})\, f_{nk-2}(u_2, \ldots, u_{n-1}) -$$
$$= -f_{n-1}(u_2, \ldots, u_n)\, f_{nk-3}(u_3, \ldots, u_{n-1})$$
$$= -\ u_1\, u_2 \ldots u_n\, f_{d(k-1)-1}(u_3, \ldots, u_{n-1}) \quad (k > 1),$$

$$(5.4) \qquad f_{n-2}(u_3, \ldots, u_n)\ f_{nk-2}(u_2, \ldots, u_{n-1}) -$$
$$= -f_{n-3}(u_3, \ldots, u_{n-1})\ f_{nk-1}(u_2, \ldots, u_n)$$
$$= u_2\, u_3 \ldots u_n\, f_{n(k-1)-1}(u_2, \ldots, u_n) \quad (k > 1).$$

In these formulas it is understood that, for example,

$$f_{nk-1}(u_2, \ldots, u_n) = f_{nk-1}(u_2, \ldots, u_n, u_1, \ldots, u_n, \ldots, u_1, \ldots, u_n).$$

To prove (5.3) and (5.4) we make use of (2.6). Thus the left side of (5.3) is equal to

$$f_{n-2}(u_2, \ldots, u_{n-1})\ \{f_{n-1}(u_2, \ldots, u_n)\ f_{n(k-1)-2}(u_3, \ldots, u_n)$$

$$- u_1 f_{n-2}(u_3, \ldots, u_n) f_{n(k-1)-3}(u_3, \ldots, u_{n-1})\}$$

$$- f_{n-1}(u_2, \ldots, u_n) \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{n(k-1)-2}(u_3, \ldots, u_n)$$

$$- u_1 f_{n-3}(u_3, \ldots, u_{n-1}) f_{n(k-1)-3}(u_3, \ldots, u_{n-1})\}$$

$$= - u_1 \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{n-2}(u_3, \ldots, u_n)$$

$$- f_{n-1}(u_2, \ldots, u_n) f_{n-3}(u_3, \ldots, u_{n-1})\} f_{n(k-1)-3}(u_3, \ldots, u_{n-1})$$

$$= - u_1 u_2 \ldots u_n f_{n)k-1)-7}(n_3, \ldots, u_{n-1}),$$

where at the step before the last we have used (3.2).

Similarly the left member of (5.4) is equal to

$$f_{n-2}(u_3, \ldots, u_n) \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{n(k-1)-1}(u_2, \ldots, u_n)$$

$$- u_1 f_{n-3}(u_3, \ldots, u_{n-1}) f_{n(k-1)-2}(u_2, \ldots, u_{n-1})\}$$

$$- f_{n-3}(u_3, \ldots, u_{n-1}) \{f_{n-1}(u_2, \ldots, u_n) f_{n(k-1)-1}(u_2, \ldots, u_n)$$

$$- u_1 f_{n-2}(u_3, \ldots, u_n) f_{n(k-1)-2}(u_2, \ldots, u_{n-1})\}$$

$$= \{f_{n-2}(u_3, \ldots, u_n) f_{n-2}(u_2, \ldots, u_{n-1})$$

$$- f_{n-3}(u_3, \ldots, u_{n-1}) f_{n-1}(u_2, \ldots, u_n)\} f_{n(k-1)-1}(u_2, \ldots, u_n)$$

$$= u_2 u_3 \ldots u_n f_{n(k-1)-1}(u_2, \ldots, u_n).$$

We now prove (5.2). By (1.8) and (3.6) we have

$$U_{n,k+1} - U_n U_{n,k}$$

$$= f_{n(k+1)-1}(u_2, \ldots, u_n) - u_1 f_{n(k+1)-3}(u_3, \ldots, u_{n-1})$$

$$- \{f_{n-1}(u_2, \ldots, u_n) - u_1 f_{n-3}(u_3, \ldots, u_{n-1}) \{f_{nk-1}(u_2, \ldots, u_n) -$$

$$- u_1 f_{nk-3}(u_3, \ldots, u_{n-1})\}$$

$$= f_{n-1}(u_2, \ldots, u_n) f_{nk-1}(u_2, \ldots, u_n) - u_1 f_{n-2}(u_3, \ldots, u_n) f_{nk-2}(u_2, \ldots, u_{n-1})$$

$$- u_1 f_{n-2}(u_2, \ldots, u_{n-1}) f_{nk-2}(u_3, \ldots, u_n) +$$

$$+ u_1^2 f_{n-3}(u_3, \ldots, u_{n-1}) f_{nk-3}(u_3, \ldots, u_{n-1})$$

$$- \{f_{n-1}(u_2, \ldots, u_n) - u_1 f_{n-3}(u_3, \ldots, u_{n-1})\}$$

$$- \{f_{nk-1}(u_2, \ldots, u_n) - u_1 f_{nk-3}(u_3, \ldots, u_{n-1})\}$$

$$= - u_1 \{f_{n-2}(u_2, \ldots, u_{n-1}) f_{nk-2}(u_3, \ldots, u_n) -$$

$$- f_{n-1}(u_2, \ldots, u_n) f_{nk-3}(u_3, \ldots, u_{n-1})\}$$

$$- u_1 \{f_{n-2}(u_3, \ldots, u_n) f_{nk-2}(u_2, \ldots u_{n-1}) -$$

$$- f_{n-3}(u_3, \ldots, u_{n-1}) f_{nk-1}(u_2, \ldots, u_n)\}$$

$$= u_1 u_2 \ldots u_n \{u_1 f_{n(k-1)-1}(u_3, \ldots, u_{n-1}) - f_{n(k-1)-1}(u_2, \ldots, u_n)\}$$

$$= - u_1 u_2 \ldots u_n U_{n(k-1)},$$

by (5.3) and (5.4). This evidently completes the proof of (5.2).

6. We have seen that the polynominal $U_{n,k}$ defined by (5.1) satisfies the following relations.

$$U_{n,1} = U_n, \tag{6.1}$$

$$U_{n,2} = U^2_n - 2u_1 u_2 \ldots u_n, \tag{6.2}$$

$$U_{n,k+1} = U_n U_{n,k} - u_1 u_2 \ldots u_n U_{n,k-1} \quad (k \geq 2). \tag{6.3}$$

By means of these formulas we can express $U_{n,k}$ in terms of U_n and $u_1 u_2 \ldots u_n$. Indeed we show that

$$U_{n,k} = \sum_{2j \leq k} (-1)^j \frac{k}{k-j} \binom{k-j}{j} U_n^{k-2j} (u_1 u_2 \ldots u_n)^j \ (k \geq 1), \tag{6.4}$$

For $k = 1, 2$, (6.4) is evidently in agreement with (6.1) and (6.2). Assuming that (6.4) holds up to and including the value k, we get

$$U_{n,k+1} = U_n \sum_{2j \leq k} (-1)^j \frac{k}{k-j} \binom{k-j}{j} U_n^{k-2j} (u_1 u_2 \ldots u_n)^j$$

$$- u_1 u_2 \ldots u_n \sum_{2j<k} (-1)^j \frac{k-1}{k-j-1} \binom{k-j-1}{j} U_n^{k-2j-1} (u_1 u_2 \ldots u_n)^j$$

$$= \sum_j (-1)^j \left\{ \frac{k}{k-j} \binom{k-1}{k-j} + \frac{k-1}{k-j} \binom{k-j}{j-1} \right\} U_n^{k-2j+1} (u_1 u_2 \ldots u_n)^j$$

$$= \sum_{2j \leq k+1} (-1)^j \frac{k+1}{k-j+1} \binom{k-j+1}{j} U_n^{k-2j+1} (u_1 u_2 \ldots u_n)^j.$$

This completes the proof of (6.4).

If we prefer $U_{n,k}$ can be written in terms of the Chebyshev polynominal [3, p. 223]

$$T_k(x) = \cos(k \arc\cos x).$$

In this connection we note that if we put

(6.5) $$U_{n,k} = \phi_k(U_n, u_1 u_2 \ldots u_n),$$

so that

$$\phi_k(u, v) = \sum_{2j \leq k} (-1)^j \frac{k}{k-j} \binom{k-j}{j} u^{k-2j} v^j,$$

then

(6.6) $$x^k + y^k = \phi_k(x + y, xy).$$

If we put

(6.7) $$x + y = U_n, \; xy = u_1 u_2 \ldots u_n,$$

then by (6.6) we have

(6.8) $$U_{n,k} = x^k + y^k.$$

Note that (6.7) implies

(6.9) $$x - y = (U_n^2 - 4u_1 u_2 \ldots u_n)^{\frac{1}{2}}.$$

It is of interest to evaluate

$$W_{n,k} = \{U_n^2 - 4(u_1 u_2 \ldots u_n)^k\}^{\frac{1}{2}}.$$

It follows from (6.7) and (6.8) that

(6.10) $$W_{n,k} = \{(x^k + y^k)^2 - 4(xy)^k\}^{\frac{1}{2}} = (x^k - y^k).$$

Put

$$\Psi_k(u, v) = \sum_{2j \leq k} (-1)^j \binom{k-j}{j} u^{k-2j} v^j.$$

It is not difficult to show that

(6.11) $$\frac{x^{k+1} - y^{k+1}}{x - y} = \Psi_k(x + y, xy),$$

so that (6.10) becomes, in view of (6.9)

(6.12) $$W_{n,k} = \Psi_{k-1}(U_n, u_1 u_2, \ldots, u_n)(U_n^2 - 4u_1 u_2 \ldots u_n)^{\frac{1}{2}}.$$

A special case of some interest in $n = 1$. Them $U_1 = 1$ and

$$U_{1,k} = U_k(u_1, u_1, \ldots, u_1).$$

Replacing u_1 by u, (6.4) becomes

$$U_k(u, \ldots, u) = \sum_{2j\leq k} (-1)^j \frac{j}{k-j} \binom{k-j}{j} u^j, \tag{6.13}$$

while (6.12) reduces to

$$W_{1,k} = \sum_{2j<k} (-1)^j \binom{k-j}{j} u^j. \ (1-4u)^{\frac{1}{2}}. \tag{6.14}$$

To sum up, we state the following

Theorem 2. *Let $n \geq 1$, $k \geq 1$. The polynomial*

$$U_{n,k} = U_{n,k}(u_1, \ldots, u_1) = U_{nk}(u_1, \ldots, u_n, \ldots, u_1, \ldots, u_n)$$

can be exhibited as a polynomial in $U_n(u_1, \ldots, u_n)$ and $u_1 u_2 \ldots u_n$:

$$U_{n,k} = \sum_{2j\leq k} (-1)^j \frac{k}{k-j} \binom{k-j}{j} U_n^{k-2j}(u_1, \ldots, u_n)(u_1 u_2 \ldots u_n)^j.$$

Alternatively we have

$$U_{n,k} = x^k + y^k,$$

where

$$x + y = U_n(u_1, \ldots, u_n), \quad xy = u_1 u_2 \ldots u_n.$$

In particular, when $n = 1$, $U_n(u, \ldots, u)$ is evaluated by (6.13).

7. Applications. Put

$$C(k_1, k_2, \ldots, k_1) = \binom{k_1+k_2}{k_2}\binom{k_1+k_3}{k_3}\cdots\binom{k_n+k_1}{k_1}. \tag{7.1}$$

Then $H(k_1, k_2, \ldots, k_n)$ as defined by (1.4) satisfies

$$H(k_1, k_2, \ldots, k_n) = \sum C(j_1, j_2, \ldots, j_{2n}), \tag{7.2}$$

where the summation is over all nonnegative

$$j_s + j_{n+s} = k_s \ (s = 1, 2, \ldots, n).$$

By (3.3) and (4.3) we have

$$(7.3) \qquad U_n(u_1, \ldots, u_n)\, G(u_1, \ldots, u_n) = F(u_1, \ldots, u_n).$$

In view of Theorem 2, a certain linear combination of the $H(k_1, \ldots, k_n)$ is equal to $C(k_1, \ldots, k_n)$. For $n = 2$ we have

$$(7.4) \qquad \{(1-u_1-u_2)^2 - 4u_1 u_2\}^{-\frac{1}{2}} = \sum_{k_1,k_2=0}^{\infty} \binom{k_1+k_2}{k_2}^2 u_1^{k1} u_2^{k2};$$

in this case (7.3) implies (1.1). For $n = 3$, since

$$U_3 = 1 - u_1 - u_2 - u_3,$$

we get

$$(7.5) \quad H(k_1, k_6, k_3) - H(k_1-1, k_2, k_3) - H(k_1, k_2-1, k_3) - H(k_1, k_2, k_3-1) = C(k_1, k_2, k_3).$$

For $n = 4$ we have

$$U_4 = 1 - u_1 - u_2 - u_3 - u_4 + u_1 u_7 + u_2 u_4,$$

which gives

$$(7.6) \quad H(k_1, k_2, k_3, k_4) - H(k_1-1, k_2, k_{,3} k_4) - H(k_1, k_2-1, k_3, k_4) - H(k_1, k_2, k_3-1, k_4) - H(k_1, k_2, k_3, k_4-1) + H(k_1-1, k_2, k_3-1, k_4) + (k_1, k_2-1, k^3, k_4-1) = C(k_1, k_2, k_3, k_4).$$

Both (7.5) and (7.6) have appeared in [1].

The case $n = 5$ leads to a more complicated formula, namely

$$(7.7) \quad H(k_1, k_2, k_3, k_4, k_5) - \sum H(k_1-1, k_2, k_3, k_4, k_5) + \sum H(k_1-1, k_2, k_3-1, k_4, k_5) = C(k_1, k_2, k_3, k_4, k_5),$$

where the summations on the left are in accordance with Theorem 1. For the general case we have

$$(7.8) \quad H(k_1, k_2, \ldots, k_n) - \sum H(k_1-1, k_2, \ldots, k_n) + \sum H(k_1-1, k_2, k_3-1, k_4, \ldots, k_n) - \sum H(k_1-1, k_2, k_3-1, k_4, k_5-1, k_6, \ldots, k_n) + \ldots = C(k_1, k_2, \ldots, k_n).$$

By making use the results of § 6 we may state more general results. For fixed $n \geq 1$, $k \geq 1$ we define

$$(7.9) \qquad H_n^{(k)}(m_1, \ldots, m_n) = \sum C(j_1, j_2, \ldots, j_{nk}),$$

where the summation is over all nonnegative j_s such that

$$j_s + j_{s+n} + \ldots + j_{s+n(k-1)} = m_s \quad (s = 1, 2, \ldots, n).$$

It follows from (7.9) that

$$(7.10) \quad G_n^{(k)}(u_1, \ldots, u_1) = \sum_{m_1, \ldots, u_n = 0}^{\infty} H_n^{(k)}(m_1, \ldots, m_1)\, u_1^{m1} \ldots u_n^{mn}$$

$$= F_{nk}(u_1, \ldots, u_n, u_1, \ldots, u_n, \ldots, u_1, \ldots, u_n),$$

where

$$F_n(u_1, \ldots, u_n) = \sum_{m_1, \ldots, m_n = 0} C(m_1, \ldots, m_n)\, u_1^{m_1} \ldots u_u^{m_n}.$$

Thus

$$G_n^{(k)}(u_1, \ldots, u_n) = \{U^2_{n,k}(u_1, \ldots, u_n) - 4(u_1 \ldots u_n)^k\}^{-\frac{1}{2}}$$

$$= \Psi_{k-1}(U_n, u_1 \ldots u_n)(U_n^2 - 4u_1 \ldots u_n)^{-\frac{1}{2}},$$

by (6.12). This implies

$$(7.11) \quad \Psi_{k-1}(U_n, u_1 \ldots u_n)\, G_n^{(k)}(u_1, \ldots, u_n) = F_n(u_1, \ldots, u_n).$$

We can therefore assert that a certain linear combination of the $H_n^{(k)}(m_1, \ldots, m_n)$ is equal to $C(m_1, \ldots, m_n)$. The general case is very complicated. To illustrate we take $n = 2$, $k = 3$. Then since

$$\Psi_2(U_2, u_1 u_2) = (1 - u_1 - u_2)^2 - u_1 u_2$$

$$= 1 - 2u_1 - 2u_2 + u_1^2 + u_1 u_2 + u_2^2,$$

we get

$$(7.12) \quad H_2^{(3)}(m_1, m_2) - 2H_2^{(3)}(m_1 - 1, m_2) - 2H_2^{(3)}(m_1, m_2 - 1) +$$

$$+ H_2^{(3)}(m_1 - 2, m_2) + H_2^{(3)}(m_1 - 1, m_2 - 1) +$$

$$+ H_2^{(3)}(m_1, m_2 - 1) = \binom{m_1 + m_2}{m_1}^2.$$

For $n = 3$, $k = 3$ we have

$$\Psi_2 (U_3, u_1 u_2 u_3) = (1 - u_1 - u_2 - u_3)^2 - u_1 u_2 u_3$$

and the final result can be stated witthout any difficulty.

When $n = 1$, however, we can give the final result an explicit statement. It follows from (6.14) that

$$\sum_{2j<k} (-1)^j \binom{k-j}{j} u^j \sum_{m=0}^{\infty} H_1^{(k)} (m) u^m = \sum_{m=0}^{\infty} \binom{2m}{m} u^m ,$$

so that

$$\sum_{2j<k} (-1)^j \binom{k-j}{j} H_1^{(k)} (m-j) = \binom{2m}{m} . \tag{7.13}$$

We note that (7.9) reduces to

$$H_1^{(k)} (m) = \sum \binom{j_1 + j_2}{j_2} \binom{j_2 + j_3}{j_7} \cdots \binom{j_k + j_1}{j_1} ,$$

where the summation is over all nonnegative j_s such that

$$j_1 + j_2 + \ \ldots \ + j_k = m.$$

REFERENCES

[1] L. CARLITZ, *A binomial identity arising from a sorting problem, SIAM Review*, vol. 6(1964), pp. 20-30.

[2] G. CHRYSTAL, *Algebra*, part II, *wew* York, 1959.

[3] J. RIORDAN, *An Introduction to Combinatorial Analysis*, New York, 1958.