EXISTENCE OF POLYNOMIAL SOLUTIONS OF A CLASS OF RICCATI-TYPE DIFFERENTIAL EQUATIONS *

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In a previous paper [1] we discussed the degrees of polynomial solutions of a class of Riccati-type equations, and discovered that they belong to well defined classes of numbers, which are functions of the degrees of the coefficient polynomials. In this paper we shall exhibit the existence of polynomial solutions of the following equation

$$Az' = \overline{B}_0 + \overline{B}_1 z + \overline{B}_2 z^2 + \dots + \overline{B}_{n-1} z^{n-1} + z^n \quad (n > 1)$$

where A, $\overline{B}_0$, $\overline{B}_1$ ..., $\overline{B}_{n-1}$ are polynomials in x.

Since every equation of this type can be reduced to the following type of equation

$$Ay' = B_0 + B_1 y + B_2 y^2 + \dots + B_{n-2} y^{n-2} + y^n \qquad (*)$$

by the transformation $z = y - (\overline{B}_{n-1})/n$, it is sufficient to consider equation (*), in which the coefficients

B_i $(i = 0, 1, \dots, n-2)$ are still polynomials in x.

The results obtained for (*) can be modified to apply to a class of higher order differential equations, as we shall point out.

Before dealing with equation (*) we shall consider some special cases, viz. $n = 2$ (Riccati's equation) and $n = 3$ (Abel's equation), which will be helpful in following the lines along which we proceed in the general case. We first give the following definition and Lemma.

DEFINITION. If P is a polynomial such that the degree $d(P)$ of P is a multiple of $n\,(> 1)$, define $[\sqrt[n]{P}]$ as the polynomial part of

* Adapted from a chapter of the first author's doctoral dissertation.

the expansion of $(P)^{1/n}$ in descending powers of x. (For uniqueness, among the n^{th} roots of a number take the one with smallest non-negative amplitude.)

Now let $[\sqrt[n]{P}] = T$. Then there exists a polynomial Q such that $0 \leq d(Q) < d(T^n)$ and

$$P = T^n + Q.$$

LEMMA. — The degree of Q is less than the degree of T^{n-1}.

Proof: Since $P = T^n + Q$

$$[\sqrt[n]{P}] = [\sqrt[n]{T^n + Q}]$$

$$= \text{the polynomial part of } T\left(1 + \frac{Q}{T^n}\right)^{1/n}$$

$$= \text{the polynomial part of } T + \left\{\frac{1}{n}\frac{Q}{T^{n-1}} + \dots\right\}$$

Now suppose that the degree of $Q \geq$ degree of T^{n-1}.
Then we would have non-zero polynomial terms in the expression $\{Q/(nT^{n-1}) + \dots\}$ which we call $R(x)$. Thus

$$[n\sqrt[n]{P}] = T + R(x)$$

which is a contradiction. Hence $R(x) = 0$; that is, the degree of $Q <$ the degree of T^{n-1}.

Q.E.D.

In the following discussion $S = [\sqrt[n]{-B_0}]$ and Q is a polynomial such that $B_0 = -S^n - Q$.

Consider now the equation:

$$Ay' = B_0 + y^2 \tag{1}$$

For the case $A = 1$, the following result was obtained by RAINVILLE [2]:
« If in the equation

$$y' = B_0 + y^2$$

the degree of B_0 is even, then no polynomial other than $[\pm\sqrt[n]{-B}]$ can be a solution. If the degree of B_0 is odd, there are no polynomial solutions. »

We may remark here that the above result is true not only for $A = 1$, but also if A is any constant. This will be clear when we obtain a similar result for equation (*).

Consider now the equation

$$Ay' = B_0 + B_1 y + y^3 \tag{2}$$

which is equation (*) for $n = 3$. It is not immediately obvious that solutions of equation (2) can be obtained by applying a procedure similar to that applied to equation (1) by RAINVILLE, although it may seem plausible to obtain similar results for the equation $y' = B_0 + y^3$, and in general for $y' = B_0 + y^n$. However, it turns out that by suitable restrictions on the coefficient polynomials, we can even obtain all possible polynomial solutions of equation (*), and in particular for equation (2), for which our result is as follows : (a, b_0, b_1 are the degrees of A, B_0, B_1 ; α, β, are the coefficients of the highest power in A, B_1).

THEOREM 1. Let b_0 be a multiple of 3 in the equation

$$Ay' = B_0 + B_1 y + y^3 \tag{2}$$

and let (i) $a - 1 < b_0/3$, $b_1 < b_0/3$

(ii) $\beta_1/\alpha < 2b_0/3$, if β_1/α is an integer.

Then there exists a polynomial solution of equation (2) if and only if for some $1 \leq i \leq 3$

$$A\omega_i S' = -Q + B_1 \omega_i S$$

(where ω_i are the cube roots of unity), and in that case $\omega_i S$ is the solution.

If b_0 is not a multiple of 3 and hypotheses (i) and (ii) are satisfied, there are no polynomial solutions.

Proof: The proof of this theorem is covered in the proof of the general case, given later.

It may be pointed out again that the ABEL equation

$$Az' = \overline{B}_0 + \overline{B}_1 z + \overline{B}_2 z^2 + z^3$$

can be transformed into equation (2) by putting $z = y - \overline{B}_2/3$, where

$$B_0 = \overline{B}_0 - \overline{B}_1 \overline{B}_2/3 + (2/27)\overline{B}_2^3 + A\overline{B}'_2/3$$

and

$$B_1 = \overline{B}_1 - \overline{B}_2{}^2/3.$$

Thus to be able to apply Theorem 1 to the above equation, all we need to know is B_0 and B_1 in terms of the given coefficients. To see the application of Theorem 1, let us consider the following example :

Example. — In the equation

$$(x^3 + 2x + 1)\, z' = -\, x^9 + x^6 - 5x^4 + 11x^2 - 2x - 20$$
$$+\, 6\,(-\, x^2 + 2)\;\; z + 3\,(x^2 - 2)z^2 + z^3$$

using the transformation

$$z = y - (x^2 - 2),$$

we get

$$(x^3 + 2x + 1)\;\; y' = -\, x^9 + 3x^2 + 3\,(x^2 + 2)\;\; y + y^3.$$

This has the same form as equation (2) and the hypotheses (i) and (ii) of Theorem 1 are satisfied. Also

$$S = [\sqrt[3]{x^9 - 3x^2}] = x^3$$

satisfies the equation

$$A\omega_i\, S' = -\, Q + B_1\, \omega_i\, S \;\text{ for }\; \omega_i = 1,$$

that is,

$$(x^3 + 2x + 1)\; 3x^2 = 3x^2 + 3\,(x^2 + 2)\, x^3.$$

Hence $S = x^3$ is a solution of the above equation in y.

Therefore $z = x^3 - (x^2 - 2)$ is a solution of the given Abel equation.

Thus for a given Abel equation with coefficient of y^3 equal to 1, we can easily find whether a polynomial solution exists or not by using Theorem 1. We shall now extend this result to equation (*).

Let ω_i $(i = 1, 2, ..., n)$ be the n^{th} roots of unity. We then have the following existence theorem for equation (*) :

Theorem 2. — Let b_0 be a multiple of n in the equation

$$Ay' = B_0 + B_1\, y + B_2\, y^2 + \ldots + B_{n-2}\, y^{n-2} + y^n \qquad (*)$$

and let (i) $a - 1 < \frac{n-2}{n} b_0$, $b_i < \frac{n-(i+1)}{n} b_0$, $i = 1, 2, \ldots, n-2$

(ii) $b_2 < \frac{n-2}{n-1}(a-1)$, $b_{i+1} < \frac{n-(i+1)}{n-i} b_1$, $i = 1, 2, \ldots, n-3$

(iii) $\beta_1/\alpha < (2/n)\, b_0$, if β_1/α is an integer.

(a, b_i are the degrees of A, B_i ; α, β_i are the coefficients of the highest power in A, B_i)

Then there exists a polynomial solution of equation (*) if and only if for some $1 \leq i \leq n$

$$A\omega_i S' = -Q + B_1 \omega_i S + B_2 \omega_i{}^2 S^2 + \ldots + B_{n-2}\, \omega_i^{n-2} S^{n-2}$$

and in that case $\omega_i S$ is the solution.

If b_0 is not a multiple of n, and hypotheses (i), (ii) and (iii) are satisfied, there are no polynomial solutions.

Proof: Let m denote the degree of a polynomial solution of (*). We shall show that under the hypotheses (i), (ii) and (iii) the only value m can have is b_0/n, which is from class IV_1 of Theorem 1 [1]. To show this, it suffices to show that m cannot have values from Classes III, IV_1, IV_2, ..., IV_{n-2}, except the value b_0/n from Class IV_1. Since $a - 1$ and b_1 satisfy the same conditions by hypothesis, proofs for values of m in Classes I and II will be similar to those for Class III, and can thus be omitted. All possible degrees for polynomial solutions except b_0/n, given by Classes III, IV_1, ..., IV_{n-2} can be listed as follows:

(*a*) $m = \beta_1/\alpha$, if β_1/α is an integer.

(*b*) $m = \frac{b_i - b_k}{k - i}$ $i = 0, 1, 2, \ldots, n-3$, $k = i+1, \ldots, n-2$

(*c*) $m = \frac{b_i}{n-i}$, $i = 1, 2, \ldots, n-2$.

Considering (*a*), if β_1/α is an integer and $m = \beta_1/\alpha$ then by Theorem 1 [1] Ay' and $B_1 y$ have the highest degree in equation (*). But

$$\begin{aligned} d(B_1 y) &= b_1 + \beta_1/\alpha \\ &< \frac{n-2}{n} b_0 + \frac{2}{n} b_0 \text{ by hypotheses (i) and (iii)} \\ &= b_0 = d(B_0) \end{aligned}$$

$\therefore d(B_1 y) < d(B_0)$ which is a contradiction.
Hence $m \neq \beta_1/\alpha$.

Consider now the degrees given by (b). If $i = 0$, m can have values $(b_0 - b_k)/k$ ($k = 1, 2, \ldots, n-2$), and in this case B_0 and $B_k y^k$ must have the highest degree in equation (*). But

$$\begin{aligned} d(y^n) &= n\frac{b_0 - b_k}{k}, \quad k = 1, 2, \ldots, n-2 \\ &> \frac{n}{k}\left(b_0 - \frac{n-(k+1)}{n} b_0\right) \text{ by hypothesis (i)} \\ &= b_0 + \frac{b_0}{k} \\ &> d(B_0) \text{ which is a contradiction.} \end{aligned}$$

Thus $m \neq (b_0 - b_k)/k$, $k = 1, 2, \ldots, n-2$.

To show that $m \neq (b_i - b_k)/(k-i)$, $i = 1, 2, \ldots, n-3$, $k = i+1, \ldots, n-2$, we first note that for any $k > i$,

$$b_k < \frac{n-k}{n-i} b_i \tag{α}$$

which is easily proved by induction.

Now if $m = (b_i - b_k)/(k-i)$, $i = 1, 2, \ldots, n-3$, $k = i+1, \ldots, n-2$, then as stated in [1] the terms $B_i y^i$ and $B_k y^k$ have the highest degree in equation (*), and

$$d(B_i y^i) = d(B_k y^k) = b_i + im = b_k + km = \frac{kb_i - ib_k}{k-i}.$$

But

$$\begin{aligned} d(y^n) &= n\frac{b_i - b_k}{k-i} \\ &= \frac{kb_i - ib_k}{k-i} + \frac{1}{k-i}\left[(n-k)b_i - (n-i)b_k\right] \\ &= d(B_i y^i) + b \text{ where } b > 0 \text{ by } (\alpha) \\ &> d(B_i y^i) \text{ which is a contradiction.} \end{aligned}$$

Thus the possibilities of degrees given by (b) cannot occur.

Again, considering (c), if $m = b_i/(n-i)$, $i = 1, 2, \ldots, n-2$ then $B_i y^i$ and y^n are highest degree terms, and

$$d(B_i y^i) = d(y^n) = n\frac{b_i}{n-i}$$

But $d(B_0) = b_0 > \dfrac{nb_i}{n-(i+1)}$ $i = 1, 2, \ldots, n-2$ by hypothesis (i)

$$> \frac{nb_i}{n-i}$$

$$= d(B_i y^i).$$

Hence $d(B_0) > d(B_i y^i)$ which is a contradiction.

Thus the only possible value m can have is b_0/n, provided b_0 is a multiple of n. If b_0 is not a multiple of n, then there can be no polynomial solutions. This proves the last part of the theorem.

If however, b_0 is a multiple of n, and if for some $1 \leq i \leq n$,

$$A\,\omega_i S' = -Q + B_1 \omega_i S + B_2 \omega_i^2 S^2 + \ldots + B_{n-2} \omega_i^{n-2} S^{n-2}$$

then

$$A(\omega_i S)' = -(\omega_i S)^n - Q + B_1 \omega_i S + B_2 (\omega_i S)^2 + \ldots$$
$$+ B_{n-1} (\omega_i S)^{n-2} + (\omega_i S)^n$$

or

$$A(\omega_i S)' = B_0 + B_1(\omega_i S) + B_2(\omega_i S)^2 + \ldots + B_{n-2}(\omega_i S)^{n-2} + (\omega_i S)^n$$

since $-(\omega_i S)^n - Q = -S^n - Q = B_0$, which shows that $y = \omega_i S$ is a solution of (*).

To show the converse, that is, to show that if equation (*) has a solution then

$A\,\omega_i S' = -Q + B_1 \omega_i S + \ldots + B_{n-2} \omega_i^{n-2} S^{n-2}$ for some $1 \leq i \leq n$, it is sufficient to show that the only possible polynomial solutions of (*) are $\omega_1 S, \omega_2 S, \ldots, \omega_n S$. For, then the existence of a solution will imply that $\omega_i S$ is a solution for some i, and substituting $\omega_i S$ in equation (*) will give the desired result.

We can write equation (*) as follows:

$$Ay' = -S^n - Q + B_1 y + B_2 y^2 + \ldots + B_{n-2} y^{n-2} + y^n \quad (n > 1)$$

Since the degree of a polynomial solution here can only be b_0/n, for $m = b_0/n = d(S)$ it can be easily verified by hypothesis (i) that

$$d(Ay') < d(S^{n-1})$$
$$d(B_i y^i) < d(S^{n-1}) \quad i = 1, 2, \ldots, n-2.$$

Also by the Lemma $d(Q) < d(S^{n-1})$.

Hence S^n and y^n are not only of highest degree but their terms at least up to $x^{(n-1)m}$ have higher degree than the remaining terms in the equation, and therefore must balance each other. Let a polynomial solution be

$$y = c_m x^m + c_{m-1} x^{m-1} + \dots + c_0 \text{ where } m = b_0/n$$

and let $S = s_m x^m + s_{m-1} x^{m-1} + \dots + s_0$

then

$$y^n = c_m^n x^{nm} + nc_m^{n-1} c_{m-1} x^{nm-1} + (nc_m^{n-1} c_{m-2} + \frac{n(n-1)}{1.2} c_m^{n-2} c_{m-1}^2) x^{nm-2}$$

$$+ (nc_m^{n-1} c_{m-3} + \dots) x^{nm-3} + \dots + (nc_m^{n-1} c_0 + \dots) x^{nm-m} + \dots$$

$$S^n = s_m^n x^{nm} + ns_m^{n-1} s_{m-1} x^{nm-1} + (ns_m^{n-1} s_{m-2} + \frac{n(n-1)}{1.2} s_m^{n-2} s_{m-1}^2) x^{nm-2}$$

$$+ (ns_m^{n-1} s_{m-3} + \dots) x^{nm-3} + \dots + (ns_m^{n-1} s_0 + \dots) x^{nm-m} + \dots$$

Equating the first terms we get $c_m = \omega_i s_m$, and equating the remaining terms up to the yerms involving x^{nm-m}, we see that $c_j = \omega_i s_j, j = 0, 1, \dots, m-1, i = 1, 2, \dots, n$. Hence if y is a polynomial solution then $y = \omega_i S, i = 1, 2, \dots, n$, are its only possible values.

This completes the proof.

REMARK 1. — We see now that the theorem applies to the RICCATI equation if and only if A is a constant, because by hypothesis (i) $a - 1 < [(n-2)/n]b_0$, that is $a - 1 < 0$. This means that the only value a can have is zero, which implies that A is a polynomial of zero degree. Setting $A = c$ (constant), we can now restate RAINVILLE's theorem in the following way:

«If in the equation $cy' = B_0 + y^2$, B_0 is of even degree, then there exists a polynomial solution of this equation if and only if $c\,\omega_i S' = -Q$ for $i = 1$ or $i = 2$ (where $\omega_1 = 1$, $\omega_2 = -1$, and $S = \sqrt{-B_0}$ as before). If the degree of B_0 is odd there are no polynomial solutions.»

REMARK 2. — It must be pointed out that the above theorem can be modified for the equation

$$A_k y^{(k)} + A_{k-1} y^{(k-1)} + \dots + A_2 y'' + Ay' =$$

$$B_0 + B_1 y + \dots + B_{n-2} y^{n-2} + y^n$$

where the A_i $(i = 2, \ldots, k)$ are suitable polynomials. For example, if a_i $(i = 2, \ldots, k)$ are the degrees of A_i $(i = 2, \ldots, k)$ such that $a_i \leq a$, and the hypotheses of Theorem 2 are satisfied then the above equation has a polynomial solution if and only if for some $1 \leq i \leq n$

$$A_k\, \omega_i\, S^{(k)} + A_{k-1}\, \omega_i\, S^{(k-1)} + \ldots + A_2\, \omega_i\, S'' + A\, \omega_i\, S' =$$
$$- Q + B_1\, \omega_i\, S + \ldots + B_{n-2}\, \omega_i^{n-2}\, S^{n-2}.$$

If b_0 is not a multiple of n and hypotheses (i), (ii) and (iii) are satisfied, there are no polynomial solutions.

In general, Theorem 2 is applicable to the above equation if the polynomials A_i $(i = 2, \ldots, k)$ are such that the degree of $A_i\, y^{(i)}$ $(i = 2, \ldots$ $k)$ is less than the degree of any of the remaining terms.

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