ABELIAN THEOREM FOR A GENERALIZATION OF LAPLACE TRANSFORM

by

J. M. C. JOSHI.

INTRODUCTION

The well known LAPLACE-STIELTJES integral in it's classical form is

$$F(x) = \int_0^{\infty} e^{-xy} \, d.\psi(y) \tag{1.1}$$

where $\psi(t)$ is a function of bounded variation in $0 \leq t \leq R$, for every positive R.
In some recent papers [1, 2, 3] I have discussed some properties of a generalization of (1.1) given by

(1.2)

$$F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^{\infty} (xy)^{\beta} \, {}_1F_1(\beta+\eta+1, \alpha+\beta+\eta+1 - xy) \, d\psi(y)$$

which arises if we apply KOBER'S operators of fractional integration [4] to the function $x_e^{\beta-x}$ [5].
In this paper we give an ABELIAN Theorem for the transform (1.2).
In what follows we assume $\psi(t)$ to be a normalized function of bounded variation and x to be real and positive.

ABELIAN THEOREM

Theorem 2.1. If (1.2) converges for $x > 0$ then for any constant A,

$$\overline{\mathop{\mathrm{Lim}}_{\substack{x \to \infty \\ x \to 0}}} \left| x^{\nu+\alpha-\beta} F(x) - A \right| \leq \overline{\mathop{\mathrm{Lim}}_{\substack{y \to 0 \\ y \to \infty}}} \left| \frac{\psi(y)}{cy \;\; \nu + \alpha - \beta} - A \right| \tag{2.1}$$

provided that $Re\ (\beta+\eta+1)>0,\ 0<Re(\nu+\alpha)<Re(\beta+\eta+2)$

$$\text{(2.2)}\qquad \text{and } C=\frac{(\Gamma(\beta+\eta+1-\nu)}{(\nu+\alpha-\beta)\ \Gamma(\beta+\eta+1-\nu-\alpha)\ \Gamma(\nu+\alpha)}$$

Proof. Integrating by parts we have [1]

$$F(x)=-\frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\int_0^\infty\frac{d}{dy}\left[(xy)\,\beta_1F_1\,(\beta+\eta+1,\alpha+\beta+\eta+1,-xy)\right]\psi\,(y)\,dy$$

$$=-\frac{\Gamma(a)}{\Gamma(b)}\int_0^\infty[\alpha/dy]\ [(xy)^\beta\ {}_1F_1\ (a,\ b,\ -xy)]\ \psi\,(y)\,dy$$

where for convenience we write

$$a=\beta+\eta+1,\ b=a+\alpha$$

Now

$$-\frac{\Gamma(a)}{\Gamma(b)}\int_0^\infty(xy)^{\nu+\alpha-\beta}\frac{d}{dy}\left[[xy)^\beta\ {}_1F_1\ (a,\ b,\ -xy)\right]dy={}^1/_c$$

provided that

$$0<Re\,(\nu+\alpha)<Re\,(\beta+\eta+2)\text{ and }Re\,(\beta+\eta+1)>0$$

Since [6]

$$\frac{d}{dx}\left[F,\ (a,b,x)]\right]=\frac{a}{b}\ {}_1F_1\ (a+1,\ b+1,\ x)$$

and

$$\int_0^\infty x^{l-1},\ {}_1F_1\ (a,b,x)\ dx=\frac{\Gamma(l)\ \Gamma(a-b)\ \Gamma(b)}{\Gamma(b-l)\ \Gamma(a)}\text{ where }0<Re\,(b-1)<Re\,(a)$$

If c be defined by (2.2), then

$$J=x^{\nu+\alpha-\beta}-A$$

$$=-\frac{\Gamma(a)}{\Gamma(b)}\int_0^\infty\frac{d}{dy}\left[(x\ y),^\beta\ {}_1F_1\ (a,\ b,\ -xy)\right]\ x^{\nu+\alpha-\beta}\ \{\psi\,(y)-Acy^{\nu+\alpha-\beta}\}\ dy$$

$$\text{Or}J\le\underset{0<y\le Y}{l.u.b}\left|\frac{\psi\,(y)}{cy^{\nu+\alpha-\beta}}-A\right|\left|\frac{\Gamma(a)}{\Gamma(b)}\int_0^Y c\,(xy)^{\nu+\alpha-\beta}\frac{d}{dy}\,[(xy)_\beta\ {}_1F_1\,(a,\ b,\ -xy)\ dy\right|$$

$$+\left| x^{\nu+\alpha-\beta} \int_{Y}^{\infty} - \frac{\Gamma(a)}{\Gamma(b)} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1\,(a,\, b,\, -xy) \right] \psi\,(y)\, dy) \right|$$

$$+ A \left| c \int_{Y}^{\infty} - \frac{\Gamma(a)}{\Gamma(b)} (xy)^{\nu+\alpha-\beta} \frac{d}{dy} \left\{ (xy)^{\beta},\, {}_1F_1\,(a,\, b,\, -xy) \right\} dy \right|$$

$$\leq \underset{0 \leq y \leq Y}{l.\,u.\,b.} \left| \frac{\psi(y)}{cy^{\nu+\alpha-\beta}} - A \right| + \left| x^{\nu+\alpha-\beta} \int_{Y}^{\infty} - \frac{\Gamma(a)}{\Gamma(b)} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1\,(a,\, b,\, -xy) \right] \psi\,(y)\, dy \right|$$

$$+ A \left| c \int_{xY}^{\infty} y v^{\nu+\alpha-\beta} \left[v^{\beta} \frac{\Gamma(a+1)}{\Gamma(b+1)}\, {}_1F_1\,(a+1,\, b+1,\, -v) - \frac{\Gamma(a)}{\Gamma(b)} \times \right. \right.$$

$$\left. \beta v^{\beta-1}\, {}_1F_1\, [a,\, b,\, -v) \right] dv \,|$$

Now the integral

$$\int_{xy}^{\infty} v^{\nu+\alpha-\beta} \left[v^{\beta} \frac{\Gamma(a+1)}{\Gamma(b+1)}\, {}_1F_1\,(a+1,\, b+1,\, -v) + \beta v^{\beta-1}\, {}_1F_1\,(ab - v) \right] dv$$

Clearly tends to zero as $x \to \infty$

$$\text{Also [6], } {}_1F_1\,(a,\, b,\, -x) = 0\,(x^{-a})\, \frac{\Gamma(b)}{\Gamma(b-a)}\, (x \to \infty) \tag{1}$$

Let us set

$$I = -\, x^{\nu+\alpha-\beta} \int_{Y}^{\infty} \frac{\Gamma(a)}{\Gamma(b)} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1\,(a,\, b,\, -xy) \right] \psi\,(y)\, dy$$

Then

$$\mathrm{I} = |\, -\, x^{\nu+\alpha-\beta} \int_{Y}^{\infty} \left[\frac{\Gamma(a)}{\Gamma(b)} x^{\beta}\, \beta y^{\beta-1}\, {}_1F_1\,(a,\, b - xy) \right] \psi\,(y)\, dy$$

$$+ \int_{Y}^{\infty} x^{\nu+\alpha-\beta} \left[\frac{\Gamma(a+1)}{\Gamma(b+1)}\, xy\, {}_1F_1\,(a+1,\, b+1,\, -xy) \right] \psi\,(y)\, dy \,|$$

$$\sim \left| -\, x^{\nu+\alpha} \int_{Y}^{\infty} \frac{\Gamma(a)}{\Gamma(b-a)} \beta y^{\beta-1}\, (xy)^{-(\beta+\eta+1)}\, \psi\,(y)\, dy \right| +$$

$$\left| \int_{Y}^{\infty} x^{\nu+\alpha+1} \frac{\Gamma(a+1)}{\Gamma(b-a)}\, y^{\beta}\, (xy)^{-(\beta+\eta+2)}\, \psi\,(y)\, dy \right| \text{ by} \tag{1}$$

$$\leq \left| x^{\nu+\alpha-\beta-\eta-1} \beta \int_{Y}^{\infty} y^{-(\eta+2)} \psi(y)\, dy \right| +$$

$$\left| a x^{\nu+\alpha-\beta-\eta-1} \int_{Y}^{\infty} y^{-(\eta+2)} \psi(y)\, dy \right|$$

But under conditions of the theorem $\alpha(y) = 0(y^{\eta+1})$. Therefore $I = 0(1)$ $(x \to \infty)$ if $Re\,(\beta + \eta + 1 - \nu + \alpha) > 0$ [1]

Therefore $\overline{\text{Lim}}_{x \to \infty} \left| y^{\nu+\alpha-\beta} F(x) - A \right| \leq \overline{l.\,u.\,b}_{0 < y \leq Y} \left| \dfrac{\psi(y)}{c y^{\nu+\alpha-\beta}} - A \right|$

The L.H.S being independent of Y we allow Y to approach zero and obtain,

$$\overline{\text{Lim}}_{x \to \infty} \left| x^{\nu+\alpha-\beta} F(x) - A \right| \leq \overline{\lim}_{y \to 0} \left| \frac{\psi(y)}{c y^{\nu+\alpha-\beta}} - A \right|$$

Also as before

$$\left| x^{\nu+\alpha-\beta} F(x) - A \right| \leq \frac{l.\,u.\,b}{Y \leq y < \infty} \left| \frac{\psi(y)}{c' y^{\nu+\alpha-\beta}} - A \right| \times$$

$$\frac{\Gamma(a)}{\Gamma(b)} \int_{Y}^{\infty} c\,(xy)^{\nu+\alpha-\beta} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1(a, b, -xy) \right] \psi(y)\, dy + \left| \frac{\Gamma(a)}{\Gamma(b)} x^{\nu+\alpha-\beta} \times \right.$$

$$\left. \times \int_{0}^{Y} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1(a, b, -y) \right] \psi(y)\, dy \right| +$$

$$+ A \left| c \int_{0}^{xY} v^{\nu+\alpha-\beta} \left[v^{\beta} \frac{\Gamma(a+1)}{\Gamma(b+1)}\, {}_1F_1(a+1, b+1, -\nu) - \right. \right.$$

$$\left. \left. \frac{\Gamma(a)}{\Gamma(b)} \beta v^{\beta-1}\, {}_1F_1(a, b, -v) \right] dv \right.$$

The last integral clearly tends to zero as $x \to 0$ also for given y the integral

$$V = x^{\nu+\alpha-\beta} \int_{0}^{Y} - \frac{\Gamma(a)}{\Gamma(b)} \frac{d}{dy} \left[(xy)^{\beta}\, {}_1F_1(a, b, -xy) \right] \psi(y)\, dy$$

tends to zero if, $0 < Re\,(\nu + \alpha)$

For

$$|v| \sim \left| x^{v+\alpha} \frac{\Gamma(a)}{\Gamma(b)} \beta \int_0^Y y^{\beta-1} \psi(y)\, dy + x^{v+\alpha+1} \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^Y \psi(y)\, dy \right|$$

$$\leq \left| x^{v+\alpha} \beta \int_0^Y y^{\beta-1} \psi(y)\, dy \right| + \left| x^{v+\alpha+1} a \int_0^Y y^{\beta} \psi(y)\, dy \right|$$

Therefore v tends to zero if $\psi(y) = 0\,(y^{-\beta})$ and $Re\,(v + \alpha) > 0$ Allowing Y to approach infinity we have

$$\overline{\lim_{x \to 0}} \left| x^{v+\alpha-\beta} F(x) - A \right| \leq \lim_{y \to \infty} \left| \frac{\psi(y)}{c y^{v+\alpha-\beta}} - A \right|$$

This concludes the proof of the theorem.

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REFERENCES

1. JOSHI, J. M. C. — *Convergence Theorems for a Generalization of Laplace Transform*, Bull. Unione Matematica Italiana (Communicated for publication).

2. JOSHI, J. M. C. — *Real Inversion Theorems for a Generalized Laplace Transform.* — Accepted for publication in Collectanea Mathematica, Vol. XIV, Barcelona.

3. JOSHI, J. M. C. — *A complex inversion Theorem for a Generalized Laplace Transform.* — Collectanea Mathematica, Vol. XV.

4. KOBER, H. — *On Fractional Integral and Derivatives*, Quart. Jour. Math. II (1940), 193-211.

5. ERDELYI, A. — *On some functional Transformations.* — Rend. Del. Semin. Mat. 10 (1950-51) 217-234.

6. SLATER, L. J. — *Confluent Hypergeometric Functions*, Cambridge University Press.

Govt. College Pithoragarh
India

S/O Dr. S. M. JOSHI
Ranikhet