INTEGRALS INVOLVING LEGENDRE FUNCTION

by

BHAGIRATH LAL SHARMA

1.1. In this paper we prove a theorem in operational calculus and use it to evaluate a few infinite integrals involving LEGENDRE, Bessel and Appells's function F_4. The results (3), (11), (14), (17), (20), (23) and (26) are believed to be new.

As usual $\Phi(p) \doteqdot f(x)$ will mean

$$\Phi(p) = p \int_0^\infty e^{-px} f(x) \, dx. \quad \dots (1)$$

1.2. Now we shall derive the LAPLACE transform of the product of the Bessel functions, which will be used in our investigation.

LAURICELA had defined the hypergeometric function of three variables F_c as follow.

$$F_c(\alpha, \beta; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta)_{m+n+p}}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p L^m L^n L^p} x^m y^n z^p.$$

where $(|\sqrt{x}| + |\sqrt{y}| + |\sqrt{z}|) < 1$. $\quad \dots (2)$

From (2), we get

$$\int_0^\infty x^{\lambda-1} k_v(px) I_\varrho(\alpha x) I_\mu(\beta x) I_\delta(\gamma x) \, dx =$$

$$= \frac{\alpha^\varrho \beta^\mu \gamma^\delta 2^{\lambda-2} \Gamma\left(\frac{\lambda + \varrho + \mu + \delta \pm v}{2}\right)}{p^{\lambda+\mu+\varrho+\delta} \Gamma(\varrho+1) \Gamma(\mu+1) \Gamma(\delta+1)} \times$$

$$F_c\left[\frac{\lambda+\varrho+\mu+\delta-v}{2}, \frac{\lambda+\varrho+\mu+\delta+v}{2}; \varrho+1, \mu+1, \delta+1; \frac{\alpha^2}{p^2}, \frac{\beta^2}{p^2}, \frac{\gamma^2}{p^2}\right]$$

valid for $R(\lambda\pm v+\varrho+\mu+\delta)>0$, $R(p)> |R(\alpha)|+|R(\beta)|+|R(\gamma)|$. ... (3)

Recently [3] has evaluated the particular cases of (3). From (3), we obtain

$$x^{\lambda-1} K_\varrho(\alpha x) I_\mu(\beta x) I_\delta(\gamma x) \doteqdot$$

$$\sum_{\varrho, -\varrho} \frac{\Gamma(-\varrho)\Gamma(\lambda+\mu+\delta+\varrho) 2^{-1-\mu-\delta-\varrho}}{\Gamma(\mu+1)\Gamma(\delta+1) p^{\mu+\delta+\lambda+\varrho-\frac{3}{2}}} \beta^\mu \gamma^\delta \alpha^\varrho$$

$$F_c\left[\frac{\lambda+\mu+\delta+\varrho}{2}, \frac{\lambda+\mu+\delta+\varrho+1}{2}; \varrho+1, \mu+1, \delta+1; \frac{\alpha^2}{p^2}, \frac{\beta^2}{p^2}, \frac{\gamma^2}{p^2}\right]$$

valid for $R(\lambda\pm\varrho+\mu+\delta)>0$, $R(\lambda\pm\varrho+\mu+\delta+1)>0$

$$R(\varrho+\alpha) > |R(\beta)| + |R(\gamma)|. \qquad \text{... (4)}$$

1.3. THEOREM:

If

$$\Psi(p) \doteqdot f(x)$$

and

$$\Phi(p) \doteqdot x^{\mu-\frac{1}{2}} K_v(\alpha x) f(x)$$

then

$$\Phi(p) = \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} p \int_0^\infty (x^2+2\alpha x)^{-\frac{1}{2}\mu} (x+\alpha+p)^{-1} P^\mu_{v-\frac{1}{2}}(1+\tfrac{x}{\alpha}) \Psi(x+\alpha+p)\, dx$$

... (5)

provided that the integral is convergent, $R(p+\alpha)>$ and $R(\mu)<1$.

Proof: We know that

$$\Psi(\varrho) \doteqdot f(x)$$

then [1, p. 129]

$$p(p+\alpha+\beta)^{-1} \Psi(p+\alpha+\beta) \doteqdot e^{-(\alpha+\beta)x} f(x). \qquad \text{... (6)}$$

Also [1. p. 278]

$$p^{\mu+\frac{1}{2}} e^{\alpha p} k_v(\alpha p) \doteqdot \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} (x^2 + 2\alpha x)^{-\frac{1}{2}\mu} P^{\mu}_{v-\frac{1}{2}}\left(1 + \frac{x}{\alpha}\right). \quad \ldots (7)$$

using the relation (6) and (7) in GOLDSTIENS' result [4] that if $h_1(p) \doteqdot g_1(x)$, $h_2(p) \doteqdot g_2(x)$

then

$$\int_0^{\infty} x^{-1} h_1(x)\, g_2(x)\, dx = \int_0^{\infty} x^{-1} h_2(x)\, g_1(x)\, dx \quad \ldots (8)$$

and replacing β by p, we get (5).

1.4. We now proceed to evaluate a few infinite integrals by applying the above theorem In what follows we have used MAC ROBERTS' definition of $Q^m_n(x)$.

(*i*) From [1, p. 198], we have

$$f(x) = x^{-\mu} k_n(x)$$

$$\doteqdot \sqrt{\frac{\pi}{2}}\, \Gamma(1 - \mu \pm n)\, p\,(p^2 - 1)^{\frac{\mu}{2} - \frac{1}{4}} P^{\mu-\frac{1}{2}}_{n-\frac{1}{2}}(p) = \Psi(p)$$

valid for $R(1 - \mu \pm n) > 0$, $R(p + 1) > 0$. ... (9)

Also, from [1, p. 198], we get

$$x^{\mu-\frac{1}{2}} k_v(\alpha x)\, f(x) = x^{-\frac{1}{2}} k_n(x)\, k_v(\alpha x)$$

$$\doteqdot \frac{\Gamma(\frac{1}{2} - n + v)\, \sqrt{d}\, \cos n\pi \cos v\pi}{\cos(n + v)\pi \cos(n - v)\pi}\, p \times$$

$$Q^{-n}_{v-\frac{1}{2}}(\cosh \gamma)\, Q^{-v}_{n-\frac{1}{2}}(\cosh \delta) = \Phi(p) \quad \ldots (10)$$

where $\sinh \gamma = d$, $\sinh \delta = \alpha d$, $\cosh \gamma \cosh \delta = p\, d$, $|\varrho m\, \gamma|$, $|\varrho m\, \delta| <$

$$< \frac{\pi}{2},\ R(p + \alpha + 1) > 0,\ |R(n)| + |R(v)| < \frac{1}{2}.$$

Using (9) and (10) in (5), we get

$$\int_0^{\infty} [(x + \alpha + p)^2 - 1]^{\frac{\mu}{2} - \frac{1}{4}} (x^2 + 2\alpha x)^{-\frac{\mu}{2}} P^{\mu}_{v-\frac{1}{2}}\left(1 + \frac{x}{\alpha}\right) P^{\mu-\frac{1}{2}}_{n-\frac{1}{2}}(\alpha + x + p)\, dx$$

$$= \frac{2}{\pi}\sqrt{\alpha d}\,\frac{\Gamma(\frac{1}{2}-n+v)\cos n\pi\cos v\pi}{\Gamma(1-\mu\pm n)\cos(n+v)\pi\cos(n-v)\pi}\,Q_{v-\frac{1}{2}}^{-n}(\cosh\gamma)\,Q_{n-\frac{1}{2}}^{-v}(\cosh\delta).$$

where $\sinh\gamma = d$, $\sinh\delta = \alpha d$, $\cosh\gamma\cosh\delta = pd$, $|\varrho m\,\gamma|$, $|\varrho m\,\delta| <$

$$< \frac{\pi}{2},\ R(\mu) < \frac{2}{3},\ R(\tfrac{1}{2}\pm n\pm v) > 0.\ p,\ \alpha > 0,\ p > \alpha + 1. \quad \ldots (11)$$

(*ii*) Starting [1, p. 196], we get

$$f(x) = x^{-\mu}\, I_n(x)$$

$$\doteqdot \sqrt{\frac{2}{\pi}}\,p\,(p^2-1)^{\frac{\mu}{2}-\frac{1}{4}}\,Q_{n-\frac{1}{2}}^{\frac{1}{2}-\mu}(p)$$

$$= \Psi(p) \quad \ldots (12)$$

valid for $R(1-\mu+n) > 0$, $R(p) > 1$.

Also [1, p. 198], we have

$$x^{\mu-\frac{1}{2}}\,k_v(\alpha x)\ f(x) = x^{-\frac{1}{2}}\ I_n(x)\ k_v(\alpha x)$$

$$\doteqdot \frac{\sqrt{c}\ \Gamma(n-v+\frac{1}{2})\cos n\pi}{\cos(n+v)\pi}\,p$$

$$P_{v-\frac{1}{2}}^{-n}(\cosh\gamma)\,Q_{n-\frac{1}{2}}^{-v}(\cosh\delta).$$

$$= \Phi(p) \quad \ldots (13)$$

valid for $R(n\pm v) > -\frac{1}{2}$, $R(p+\alpha\pm 1) > 0$, $\sinh\ \gamma = c$.

$\sinh\delta = \alpha c$. $\text{Cosh}\,\gamma\cosh\delta = pc$, $|\varrho m\gamma|$, $|\varrho m\ \delta| < \frac{\pi}{2}$.

Using (12) and (13) in (5), we get

$$\int_0^\infty [(x+\alpha+p)^2-1]^{\frac{\mu}{2}-\frac{1}{4}}(x^2+2\alpha x)^{-\frac{\mu}{2}}\,P_{v-\frac{1}{2}}^{\mu}(1+\frac{x}{\alpha})\,Q_{n-\frac{1}{2}}^{\frac{1}{2}-\mu}(x+\alpha+p)\,dx$$

$$= \sqrt{\alpha c}\,\frac{\Gamma(\frac{1}{2}+n-v)\cos n\pi}{\cos(n+v)\pi}\,P_{v-\frac{1}{2}}^{-n}(\cosh\gamma)\,Q_{n-\frac{1}{2}}^{-v}(\cosh\delta),$$

where

$\sinh\gamma = c$, $\sinh\delta = \alpha c$; $\cosh\gamma\cosh\delta = pc$, $|\varrho m\,\gamma|$, $|\varrho m\;\delta| <$

$$< \frac{\pi}{2}.\ R(\tfrac{1}{2}+n\pm v)>0,\ R(\mu)<\frac{2}{3},\ p>0,\ \alpha>0,\ p>\alpha+1. \quad \ldots(14)$$

(*iii*) TAKING [1, p. 146], we get

$$f(x) = x^{-\mu-\frac{1}{2}}\, e^{-\frac{\beta}{2\alpha^2 x}}$$

$$\doteqdot (2p)^{\frac{\mu}{2}+\frac{3}{4}} \left(\frac{\beta}{\alpha^2}\right)^{\frac{1}{4}-\frac{\mu}{2}} k_{\mu-\frac{1}{2}} \left(\sqrt{\frac{2\beta p}{\alpha^2}}\right).$$

$$= \Psi(p).$$

valid for $R\left(\frac{\beta}{\alpha^2}\right) > 0,\ R(p) > 0.$ ... (15)

Also from [1, p. 198], we get

$$x^{\mu-\frac{1}{2}}\, k_v(\alpha x)\, f(x) = x^{-1}\, e^{-\frac{\beta}{2\alpha^2 x}}\, k_v(\alpha x)$$

$$\doteqdot 2p\ k_v\left(\frac{\sqrt{\beta S}}{\alpha}\right) k_v\left(\sqrt{\frac{\beta}{S}}\right)$$

$$= \Phi(p)$$

where $S = p + \left(p^2 - \frac{\alpha^3}{\beta}\right)^{\frac{1}{2}}$, $R\left(\frac{\beta}{\alpha^2}\right) > 0$, $R(p) > -R(\alpha)$. ... (16)

Using (15) and (16) in (5), we have

$$\int_0^\infty (x^2+2\alpha x)^{-\frac{\mu}{2}} (x+\alpha+p)^{\frac{\mu}{2}-\frac{1}{4}} P^{\mu}_{v-\frac{1}{2}}\left(1+\frac{x}{\alpha}\right) k_{\mu-\frac{1}{2}}\left[\frac{\sqrt{2\beta}}{\alpha}(x+\alpha+p)^{\frac{1}{2}}\right] dx$$

$$= \frac{2^{\frac{3}{4}-\frac{\mu}{2}}(\alpha)^{1-\mu}(\beta)^{\frac{\mu}{2}-\frac{1}{4}}}{\sqrt{\pi}}\, k_v\left(\sqrt{\frac{\beta S}{\alpha^2}}\right) k_v\left(\sqrt{\frac{\beta}{S}}\right).$$

where $S = p + \left(p^2 - \frac{\alpha^3}{\beta}\right)^{\frac{1}{2}}$, $-1 < R(\mu) < \frac{2}{3}$, $R\left(\frac{\beta}{\alpha^2}\right) > 0$, $R(p) > 0$. ... (17)

(*iv*) TAKING [1, p. 198], we have

$$f(x) = x^{-m-\frac{1}{2}} k_n(\beta x)$$

$$\doteq \sqrt{\frac{\pi}{2\beta}}\, \Gamma(\tfrac{1}{2} - m \pm n)\, p\,(p^2 - \beta^2)^{\frac{m}{2}}\, P^m_{n-\frac{1}{2}}\left(\frac{p}{\beta}\right)$$

$$= \Psi(p)$$

valid for $R(\frac{1}{2} - m \pm n) > 0,\ R(p + \beta) > 0.$ ... (18)

Also [2, p. 373], we get

$$x^{\mu-\frac{1}{2}} k_v(\alpha x)\, f(x) = x^{\mu-m-1} k_v(\alpha x)\, k_n(\beta x)$$

$$\doteq \sum_{v,-v} \sum_{n,-n} \frac{\Gamma(-v)\,\Gamma(-n)\,\Gamma(\mu + v + n - m)}{2^{v+n+2}\, p^{\mu+v+n-m-1}}\, \alpha^v \beta^n$$

$$F_4\left[\frac{\mu-m+v+n}{2}, \frac{\mu-m+v+n+1}{2};\ v+1, n+1;\ \frac{\alpha^2}{p^2}, \frac{\beta^2}{p^2}\right] = \Phi(p)$$

valid for $R(\mu - m \pm v \pm n) > 0,\ R(p + \alpha + \beta) > 0.$ ... (19)

Using (18) and (19) in the theorem, we have

$$\int_0^\infty [(x + \alpha + p)^2 - \beta^2]^{\frac{m}{2}}\, (x^2 + 2\alpha x)^{\frac{\mu}{2}}\, P^m_{n-\frac{1}{2}}\left(\frac{x + \alpha + p}{\beta}\right) P^\mu_{v-\frac{1}{2}}\left(1 + \frac{x}{\alpha}\right) dx$$

$$= \frac{1}{\sqrt{\pi}} \sum_{v,-v} \sum_{n,-n} \frac{\Gamma(-v)\,\Gamma(-n)\,\Gamma(\mu + v + n - m)\, \alpha^{v+\frac{1}{2}}\, \beta^{n+\frac{1}{2}}}{2^{v+n+1}\, p^{\mu+v+n-m}\, \Gamma(\frac{1}{2} - m \pm n)}$$

$$F_4\left[\frac{\mu - m + v + n}{2}, \frac{\mu - m + v + n + 1}{2};\ v+1,\ n+1;\ \frac{\alpha^2}{p^2}, \frac{\beta^2}{p^2}\right].$$

valid for $R(\mu) < \frac{2}{3}$. $R(\mu - m \pm v \pm n) > 0, p, \alpha, \beta > 0, p > \alpha + \beta.$ ... (20)

(*v*) TAKING [1, p. 196]

$$f(x) = x^{m-\frac{1}{2}} I_n(\beta x)$$

$$\doteq \sqrt{\frac{2}{\beta\pi}}\, p\,(p^2 - \beta^2)^{-\frac{m}{2}}\, Q^m_{n-\frac{1}{2}}\left(\frac{p}{\beta}\right)$$

$$= \Psi(p)$$ ... (21)

Valid for $R(\frac{1}{2} + m + n) > 0$. $R(p) > |R(\beta)|$.

Also [2, p. 373]

$$x^{\mu-\frac{1}{2}} k_v(\alpha x) f(x) = x^{m+\mu-1} I_n(x\beta) k_v(\alpha x)$$

$$\doteqdot \sum_{v,-v} \frac{\Gamma(-v)\,\Gamma(m+\mu+n+v)\,\alpha^v \beta^n}{2^{v+n+1}\,\Gamma(n+1)\,p^{m+\mu+v+n-1}} \times$$

$$F_4\left[\frac{m+n+\mu+v}{2}, \frac{m+n+\mu+v+1}{2}; n+1, v+1; \frac{\beta^2}{p^2}, \frac{\alpha^2}{p^2}\right]$$

$$= \Phi(p)$$

valid for $R(m+\mu+n \pm v) > 0$, $R(p+\alpha) > |R(\beta)|$. ... (22)

Using (21) and (22) in (5), we have

$$\int_0^\infty [x+\alpha+p)^2 - \beta^2]^{-\frac{m}{2}} (x^2+2\alpha x)^{-\frac{\mu}{2}} P^{\mu}_{v-\frac{1}{2}}\left(\frac{x}{\alpha}+1\right) Q^{m}_{n-\frac{1}{2}}\left(\frac{x+\alpha+p}{\beta}\right)$$

$$dx = \sum_{v,-v} \frac{\Gamma(-v)\,\Gamma(m+\mu+n+v)}{\Gamma(n+1)\,2^{n+v+1}\,p^{m+n+\mu+v}}\,\alpha^{v+\frac{1}{2}}\,\beta^{n+\frac{1}{2}}$$

$$F_4\left[\frac{m+n+\mu+v}{2}, \frac{m+n+\mu+v+1}{2}; n+1, v+1; \frac{\beta^2}{p^2}, \frac{\alpha^2}{p^2}\right]$$

valid for $R(m+\mu+n \pm v) > 0$, $R(\mu) < \frac{2}{3}$, $p, \alpha, \beta > 0$, $p > \alpha + \beta$. ... (23)

(*vi*) TAKING [2, p. 373]

$$f(x) = x^{\varrho-1} I_m(\gamma x)\, I_n(\delta x)$$

$$\doteqdot \frac{\gamma^m \delta^n\,\Gamma(\varrho+m+n)}{2^{m+n}\,p^{\varrho+m+n+1}\,\Gamma(m+1)\,\Gamma(n+1)}$$

$$F_4\left[\frac{\varrho+m+n}{2}, \frac{\varrho+m+n+1}{2}; m+1, n+1; \frac{\gamma^2}{p^2}, \frac{\delta^2}{p^2}\right]$$

$$= \Psi(p)$$

valid for $R(\varrho+m+n) > 0$, $R(p) > |R(\gamma)| + |R(\delta)|$. ... (24)

Also from (4), we get

$$x^{\mu-\frac{1}{2}}\, k_v(\alpha x)\; f(x) = x^{\varrho+\mu-\frac{3}{2}}\, I_m(\gamma x)\; I_n(\delta x)\; k_v(\alpha x)$$

$$\doteqdot \sum_{v,-v} \frac{\Gamma(-v)\;\Gamma(\varrho+\mu+m+n+v-\frac{1}{2})\;\gamma^m\;\delta^n\;\alpha^v}{\Gamma(m+1)\;\Gamma(n+1)\;p^{\varrho+\mu+m+n+v-1}\;2^{m+n+v+1}}$$

$$F_c\left[\frac{\varrho+\mu+m+n+v}{2}-\frac{1}{4},\ \frac{\varrho+\mu+m+n+v}{2}+\frac{1}{4};\ v+1,\right.$$

$$\left. m+1,\ n+1;\ \frac{\alpha^2}{p^2},\ \frac{\gamma^2}{p^2},\ \frac{\delta^2}{p^2}\right]$$

$$= \Phi(p)$$

valid for $R(p+\alpha)>|R(\gamma)|+|R(\delta)|,\ R(\varrho+\mu+m+n\pm v\pm\frac{1}{2})>0.$ ... (25)

Using (24) and (25) in (5) we obtain

$$\int_0^\infty (x^2+2\alpha x)^{-\frac{\mu}{2}}\,(x+\alpha+p)^{-\varrho-m-n}\,P^{\mu}_{v-\frac{1}{2}}\left(1+\frac{x}{\alpha}\right)\cdot$$

$$F_4\left[\frac{\varrho+m+n}{2},\ \frac{\varrho+m+n+1}{2};\ m+1,\ n+1:\ \frac{\gamma^2}{(x+\alpha+p)^2},\ \frac{\delta^2}{(x+\alpha+p)^2}\right]$$

$$dx = \frac{1}{\sqrt{2\pi}} \sum_{v,-v} \frac{\Gamma(-v)\cdot\Gamma(\varrho+\mu+m+n+v-\frac{1}{2})\;\alpha^{v+\frac{1}{2}}}{\Gamma(\varrho+m+n)\;p^{\varrho+\mu+m+n+v-1}} \times$$

$$F_c\left[\frac{\varrho+\mu+m+n+v}{2}-\frac{1}{4},\ \frac{\varrho+\mu+m+n+v}{2}+\frac{1}{4};\ v+1,\ m+1,\right.$$

$$\left. n+1;\ \frac{\alpha^2}{p^2},\ \frac{\gamma^2}{p^2},\ \frac{\delta^2}{p^2}\right].$$

valid for $R(\mu)<\frac{2}{3},\ R(\varrho+\mu+m+n+v-\frac{1}{2})>0,\ p,\alpha,\gamma,\delta>0,$

$p>\alpha+\gamma+\delta.$ ... (26)

REFERENCES

1. ERDÉLYI, A. et AL. — *Tables of Integral Transforms.* Vol. I (New York, 1954).

2. ERDÉLYI. A. et AL. — *Tables of Integral Transforms.* Vol. II (New York, 1954).

3. SAXENA, R. K. — *Integrals involving product of Bessel functions.* Proc. Glasgow. Math. Assoc. Vol. 6, (1964).

4. GOLDSTIEN, S. — *Operational representation of Whittaker's confluent hypergeometric function and Webers' parabolic cylinder functions.* Proc. Lond. Math. Soc. 34 (1932).

Department of Applied Sciences
Punyali Engineering College
CHANDIGARH (India).